General Examination: Part I

Problem 1. Give an example or say why there are no examples.

(a) An integral domain which is not a principal ideal domain.

(b) A group of order 15 with four Sylow 3-subgroups.

(c) A 5×3 real matrix A and a 3×5 real matrix B with $A \times B = I_5$, the 5×5 identity matrix.

(d) An element of order 15 in the symmetric group on 9 letters.

Problem 2. Let V be the vector space of polynomials with real coefficients and degree less than or equal to 2. Consider the linear transformation $D : V \to V$ defined by taking the derivative; that is, $D(f) = f'$ where f' is the derivative of the polynomial f.

(a) Find the matrix for D with respect to the basis $\{1, x, x^2\}$.

(b) Determine the eigenvalues and eigenvectors of D.

(c) Is there a basis for which D has a diagonal matrix. Why or why not?

Problem 3. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $\int f d\mu$ be a Lebesgue integral for $(X, \mathfrak{M}, \mu)$ with a measurable function f. Prove that

(a) If $\int_E f d\mu = 0$ for $f : X \to \mathbb{C}$ and any $E \in \mathfrak{M}$ then $f = 0$ almost everywhere.

(b) If $f : X \to [-\infty, \infty]$ is integrable then $\mu(\{x \mid |f(x)| \geq \varepsilon\}) < \infty$ for any $\varepsilon > 0$.

Problem 4. Does the sequence

$$x_n(t) = \frac{t^{2n}}{2n} - \frac{t^{3n}}{3n}, \qquad n \geq 1,$$

converge in the space
(a) $C[0,1]$
(b) $C^1[0,1]$
(c) $L_1(0,1)$

Problem 5. State and prove *the maximum modulus principle* for analytic functions.

Problem 6. Using complex analysis techniques, prove that in the complex plane $\mathbb{C}$, a polynomial of degree $n \geq 1$ (with nonzero leading coefficient) has n zeros.