General Examination: Real Variables

Problem 1. The following statement is *not* true: Let E be a measurable subset of $\mathbb{R}$, and let μ be the Lebesgue measure on $\mathbb{R}$. If f_n, $n \geq 1$, are measurable functions on E and $f_n(x) \to f(x)$ a.e., then for each $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $\mu(E \backslash F) < \varepsilon$ and $f_n(x) \to f(x)$ uniformly on F.

(a) Add an assumption that makes this statement true. How is this theorem (correct statement) called?

(b) Give an example showing that without this additional assumption the statement is not true.

Problem 2. For each $n = 1, 2, \ldots$ and $k = n, n+1, n+2, \ldots, 2^n - 1$, let $f_{n,k}(x) = 1/x$ for $k\,2^{-n} \le x < (k+1)\,2^{-n}$ and $f_{n,k}(x) = 0$ otherwise. Consider the sequence $f_{1,1}, f_{2,2}, f_{2,3}, f_{3,3}, \ldots, f_{3,7}, f_{4,4}, \ldots, f_{4,15}, \ldots$ Show that

(a) This sequence converges in measure to 0 on the interval $(0,1)$, but does *not* converge at any point of $(0,1)$.

(b) There is no integrable $g(x)$ on $(0,1)$ such that $f_{n,k}(x) \le g(x)$ a.e. on $(0,1)$ for each n and k.

Problem 3. Let S be any subset of a normed linear space X, and let A be a bounded subset of the dual space X^*. Show that A is equicontinuous on S.

Problem 4. For the sequence of functions $f_n(t) = \dfrac{n(e^{\cos \pi t} - 1)}{(t+1/2)^n}$, $t \in [0, 10]$, find

(a) its pointwise limit as $n \to \infty$ if the limit exists.

(b) find a subset of $[0, 10]$ with the Lebesgue measure greater than 9.5 on which the uniform limit of $f_n(t)$ exists.

Problem 5. Let M_n be a sequence of compact sets in a Banach space X such that any intersection of a finite number of these sets is not empty. Is it possible $\bigcap_{n=1}^{\infty} M_n = \emptyset$? Explain your answer.

Problem 6. Let $(x,y) \in [0,1] \times [0,1]$ and

$$f(x,y) = \sum_{n=1}^{\infty} \frac{\sin^2(\pi n^2 (x^2 - y))}{n^2(1+x+y)}.$$

Prove that $f(x,y)$ is a Lebesgue integrable function.