General Examination: Complex Analysis

Problem 1. State and prove Schwarz's lemma.

Problem 2. Prove or disprove the following assertions (give a relevant theorem or a counterexample):

(a) If $f(z) = u + iv$ is a continuous complex-valued function, and the Cauchy-Riemann equations hold at z_0, then $f(z)$ is analytic at z_0.

(b) Let D be an open connected set in $\mathbb{C}$. If $f : D \to \mathbb{C}$ is analytic and $z_0 \in D$ is such that $|f(z_0)| \leq |f(z)|$ for all $z \in D$, then $f(z_0) = 0$ or f is a constant.

(c) If $f(z)$ is analytic in a bounded domain D and has infinitely many zeros in D, then $f \equiv 0$ in D.

(d) If $f(z)$ is entire and $|f(z)| \geq 1$ for all z, then f is a constant.

Problem 3.

(a) Let $f(z)$ be analytic in $D = \{z \in \mathbb{C} : \ |z| < 2\}$. Evaluate the integral

$$\frac{2}{\pi}\int_0^{2\pi} f\left(e^{it}\right)\cos^2\frac{t}{2}\,dt.$$

(b) Using complex analysis techniques, evaluate the integral

$$I = \int_0^{\infty} \frac{\ln x}{\sqrt{x}\,(x+1)^2}\,dx.$$