General Examination: Part II

Problem 1. Prove that the polynomial x^4+1 is irreducible over Q but is reducible over F_p, the finite field with p elements, for all primes p.

Problem 2.

(a) State Lebesgue's monotone convergence theorem.

(b) Let $X = \mathbb{Z}^+$ and $\mathfrak{M} = P(\mathbb{Z}^+)$ (the power set of $\mathbb{Z}^+$). Prove that a real-valued or complex-valued function $f : X \to \mathbb{R}, \mathbb{C}$ is integrable iff $\sum_{n \geq 1} f(n)\mu(\{n\})$ is absolutely convergent and $\int f \, d\mu = \sum_{n \geq 1} f(n)\mu(\{n\})$.

Problem 3. Let $(X,\mathfrak{M})$ be a measure space and let $E \in \mathfrak{M}$.
Show $\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$ is a measurable function.

Problem 4. Determine all complex analytic functions f defined on the unit disk D which satisfy

$$f'\left(\frac{1}{\sqrt{n+1}}\right) - 3i\,f\left(\frac{1}{\sqrt{n+1}}\right) = 0, \quad \text{for all } n \in \mathbb{Z}^+.$$