General Examination: Part II

Problem 1. Let G be an **abelian** group. A torsion subgroup of G is a set of elements

$$T_G = \{g \in G \mid \exists n > 1,\ \ g^n = e\},$$

i.e., the set of all elements of finite order. We say that G is torsion free if its torsion subgroup T_G is trivial, i.e., T_G contains only the trivial element.

(1) Prove that for every abelian group G, T_G is indeed a subgroup.

(2) Prove that the factor group G/T_G is torsion free, i.e., T_{G/T_G} is trivial.

Problem 2. Let X and Y be metric spaces. Consider a sequence $\{f_n\}$ of continuous functions from X to Y, which converges to a function f uniformly on each compact subset K of X. Prove that f is continuous.

Problem 3. Find the Lebesgue integral over segment $[0, \pi/2]$ of the function

$$g(x) = \begin{cases} \sin x, \text{ if } \cos x \text{ is rational,} \\ \sin^2 x, \text{ if } \cos x \text{ is irrational.} \end{cases}$$

Problem 4. Let $f(x,y) = u(x,y) + i\,v(x,y)$ be analytic in a connected open set D. Suppose there are real constants a, b, and c such that $a^2 + b^2 \neq 0$ and $au + bv = c$ in D. Show that f is a constant in D.