General Examination: Part I

Problem 1. An element a of a (commutative with unity) ring R is **nilpotent** if $a^n = 0$ for some $n \in \mathbb{N}$. Prove that if $a, b \in R$ are nilpotent then $a + b$ is also nilpotent.

Problem 2. Let G be a group and $n \in \mathbb{N}$. Assume that the set of subgroups

$$S = \{H \leq G\ ;\ |H| = n\}$$

is not empty. Prove that the subgroup

$$F = \bigcap_{H \in S} H$$

is normal in G. [Hint: if $|H| = n$ then $|g^{-1}Hg| = n$.]

Problem 3. Let

$$f_n(x) = \frac{n \sin x}{1 + n^2 \sin^2 x}, \qquad x \in [0, \pi].$$

Given $\delta > 0$ find the Egorov set E_δ where the sequence f_n is uniformly convergent.

Problem 4. Let $f : [0,1] \mapsto \mathbb{R}$ and $g : [0,1] \mapsto \mathbb{R}$ be measurable functions. Consider the metric

$$d(f,g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

Show that a sequence $\{f_n\}$ converges in measure to f on $[0,1]$ if and only if $d(f_n, f) \to 0$ as $n \to \infty$.

Problem 5. For $0 < \alpha < 4$, evaluate the integral

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x^2)^2}\,dx.$$

Problem 6.

(a) Prove $\sum_{n=0}^{\infty} f_n(z)$ is uniformly convergent if and only if the series is uniformly Cauchy.

(b) Use part (a) to prove the Weierstrass M-test.