General Examination: Part II

Problem 1. Let G be a cyclic group, and let $a, b \in G$ be elements which are not squares. Prove that ab is a square.

Give an example to show that this result is false if the group is not cyclic.

Problem 2. Let a sequence $\{f_n\}$ of Lebesgue-integrable functions on $X \subset \mathbb{R}$ be monotonically increasing. Let $f(x) = \lim_{n\to\infty} f_n(x)$ and integrals $\int_X f_n(x)dx$ be uniformly bounded. Prove that f is finite almost everywhere.

Problem 3.

(a) Prove that the norm $\|x\|_\infty = \sup\limits_{1 \le k \le \infty} |x_k|$ in the space l_∞ of infinite number sequences $\{x_k\}_{k=1}^{\infty}$ is the limit of norms $\|x\|_p = \left(\sum\limits_{k=1}^{\infty} |x_k|^p \right)^{1/p}$ of spaces l_p as $p \to \infty$.

(b) Prove that the space l_∞ is complete.

Problem 4. Let G be an open connected set in $\mathbb{C}$, and suppose that $f : G \to \mathbb{C}$ is analytic and $a \in G$ is such that $|f(a)| \leq f(z)$ for all z in G. Show that either $f(a) = 0$ or $f(z)$ is constant.