General Examination: Part I

Problem 1. Suppose that A and B are linear transformations from a finite-dimensional complex vector space V to itself. Prove or disprove the following two statements.

(a) Every eigenvector of AB is also an eigenvector of BA.

(b) Every eigenvalue of AB is also an eigenvalue of BA.

Problem 2. Prove that the ring of polynomials in one variable with real coefficients is a principal ideal domain.

Problem 3. Let $f_n(x) = \sum_{k=1}^{n} \frac{1}{k} e^{-k(x-k)^2}$ for $x \in \mathbb{R}$. Prove or disprove the following statements.

(a) $f_\infty \in L_1$.

(b) $\int f_n \to \int f_\infty$ as $n \to \infty$.

(c) f is continuous.

Problem 4. Let m^* be the Lebesgue outer measure on the real line $\mathbb{R}$. Let E_n be a countable number of sets on $\mathbb{R}$, each of which has zero outer measure. Prove that $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$.

Problem 5. For $a > 0$, evaluate

$$\int_0^\infty \frac{\ln z}{z^2 + a^2}\, dz$$

Problem 6.

(a) Define a curve being rectifiable on $[a, b]$.

(b) Prove that a curve in the complex plane is rectifiable iff its real and imaginary parts are of bounded variation.

(c) Give an example of a continuous nonrectifiable curve on $[0, 1]$.