General Examination: Part II

Problem 1. Let ϕ be an automorphism of a group G that only fixes the identity, i.e., $\phi(x) = x$ implies $x = 1$. Prove that $G = \{x^{-1}\phi(x) \mid x \in G\}$.

Problem 2. Let

$$f_n(x) = \frac{n\left[\cos\left(x + \frac{1}{n}\right) - \cos x\right]}{x^{3/2}} \mathbf{1}_{[\frac{1}{n},1]}(x),$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. Calculate

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx.$$

Problem 3. Let f be a simple measurable function (not necessarily positive), taking values a_j on the sets E_j, with $j = 1, \ldots, N$. Show that

$$(L) \int_E f = \sum_{j=1}^{N} a_j |E_j|,$$

where $(L) \int$ and $|\cdot|$ denote the Lebesque integral and measure, respectively.

Problem 4. Let f be an analytic function in a domain D. If $|f(z)| \equiv K$, where K is a constant, then f is a constant in D.