General Examination: Part I

Problem 1. Let V be a vector space of dimension n over $\mathbb{C}$, the field of complex numbers. Is V a vector space over $\mathbb{R}$? If so, what is the dimension of that vector space?

Problem 2.

(a) Define direct product of groups. Show that the direct product of two abelian groups is abelian.

(b) A group G has exponent t, if t is the smallest positive integer such that $g^t = 1$ for all $g \in G$. Show that, if G has exponent 2, then G is abelian.

Problem 3. Let $A, B \subseteq \mathbb{R}^n$ and let the family $\mathscr{C} = \{A, B\}$.

(a) Suppose that $A \cap B = \emptyset$. Find the σ-algebra $\sigma(\mathscr{C})$ generated by the family of sets $\mathscr{C}$.

(b) Find $\sigma(\mathscr{C})$ in the case $A \cap B \neq \emptyset$.

Problem 4. Suppose that $E, \mathscr{E}, \mu$ is a measure space and let f_n, g_n be two sequences of measurable functions such that $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$ on E where $\xrightarrow{m}$ denote convergence in measure.

(a) Show that $f_n + g_n \xrightarrow{m} f + g$ on E.

(b) If $\mu(E) < +\infty$ show that $f_n g_n \xrightarrow{m} fg$ on E.

(c) If $\mu(E) < +\infty$, $g_n \to g$ on E and $g \neq 0$ a.e., then $f_n/g_n \xrightarrow{m} f/g$ on E.

Problem 5. Let C_r denote the positively oriented circle of radius r centered at the origin. For all positive values $r \neq 2$, evaluate

$$\int_{C_r} \frac{z^2 + e^z}{z^2(z-2)}\, dz.$$

Problem 6. Let f be an entire function that has the property $|f(z)| \geq 1$ for all z. Show that f is a constant.