General Examination Part II

Problem 7: Let A be the ring of real 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. What are the 2-sided ideals in A? Justify your answer.

Problem 8: Let Ω be a nonempty simply connected subset of $\mathcal{C}$, the complex plane. Show that if $f : \Omega \to \mathcal{C}$ is holomorphic and has no zeros in Ω, then there exists a holomorphic function $g : \Omega \to \mathcal{C}$ such that such that $f(z) = e^{g(z)}$ at each point $z \in \mathcal{C}$.

Problem 9: Let M be a non-empty complete metric space. Let $T : M \to M$ be such that $T \circ T = T^2$ is a strict contraction; that is, the T^2 strictly decreases the distance between points. Prove that T has a unique fixed point in M, i.e., there is a unique point x_0 with $T(x_0) = x_0$.

Problem 10: It is well known from advanced calculus or complex variables that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

It is also a common statement that functions that are Riemann integrable are also Lebesgue integrable. But to be Lebesgue integrable a function must be absolutely integrable. Here one has

$$\int_{-\infty}^{\infty} \left|\frac{\sin x}{x}\right| dx = \infty. \tag{1}$$

a) How this can be?

b) Prove (1).