General Examination Part I

Problem 1: Using the properties of the Riemann integral, show that if f is a non-negative continuous function on $[0, 1]$, and $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for all $x \in [0, 1]$.

Problem 2: Let A be a subset of the unit interval $[0, 1]$, and let B be the complement of A in $[0, 1]$. Assume that $m^*(A) + m^*(B) = 1$ where m^* is Lebesgue outer measure. Prove that A is Lebesgue measurable.

Poisson's integral formula, analytic continuation, Picard's theorem.

Problem 3: Evaluate the integral

$$I = \frac{1}{2\pi i}\int_C \frac{dz}{(z-2)(1+2z)^2(1-3z)^3}\,,$$

where C is the circle $|z| = 1$ with counterclockwise orientation.

Problem 4: Let $f(z)$ be an entire function whose restriction to the real axis is periodic with period c; that is, $f(x+c) = f(x)$ for all real numbers x. Prove that $f(z+c) = f(z)$ for all complex numbers z.

Problem 5: Suppose V is a real vector space of finite dimension n, and $T : V \to V$ is a linear transformation with no repeated eigenvalues. Show that there exists a vector $v \in V$ such that $\{v, Tv, T^2v, \ldots, T^{n-1}v\}$ is a basis of V.

Problem 6: An element a of a ring is *nilpotent* if $a^n = 0$ for some positive integer n. Prove that in a commutative ring $a + b$ is nilpotent if a and b are. Show that this result need not hold in a noncommutative ring.