General Exam Part I

Name: ______________________________

You must attempt to solve all problems in this part of the examination. You have 3 hours.

1. Let $\mathbb{Q}$ be the group of rational numbers under addition and $\mathbb{Z}$ the subgroup of integers. Show that the factor group $\mathbb{Q}/\mathbb{Z}$ is infinite, but every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

2. Let $\mathbb{Q}$ denote the field of rational numbers. Show that the polynomial $P(x) = x^4 - 2$ is irreducible in $\mathbb{Q}[x]$.

   Let F be a field obtained by adjoining a root a of P to $\mathbb{Q}$. Every element of F can be written as $p + qa + ra^2 + sa^3$ for some coefficients $p, q, r, s \in \mathbb{Q}$. Write $1/(1 + a)$ this way.

3. Let $(X, \mathcal{S}, \mu)$ be a measure space. Suppose that f and f_n, $\forall n \geq 1$, are integrable functions $X \longrightarrow \mathbb{R}$. As $n \to +\infty$, does

   $$\int_X |f_n - f| \, d\mu \to 0$$

   imply $f_n \to f$ almost everywhere? Does it imply $f_n \to f$ in measure? Justify your answers with proofs or counterexamples.

4. Does

   $$\sum_{j=0}^{\infty} \frac{1}{\sqrt{j} + \sqrt{j+1}}$$

   converge? If so, prove. If not, find the sum of the first $n + 1$ terms.

5. Suppose that f is an analytic function on a sufficiently large disk D centered at the origin, and write

   $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

   for its Taylor expansion there. If γ is the curve that runs along the unit circle (one time and counterclockwise), calculate

   $$\int_\gamma f\left(\sin\left(\frac{1}{z}\right)\right) dz$$

   in terms of the coefficients a_n.

6. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. Prove that, if there exist a $C > 0$ and an $n \in \mathbb{N}$ such that

$$\sup_{|z|=r} |f(z)| < Cr^n,$$

$\forall r > 10$, then f is a polynomial of degree at most n.

General Exam Part II

Name: ____

You must solve at least two problems in this part of the examination. You have 2 hours.

1. Show that $\sum_{k=1}^{n} \frac{1}{k} - \ln(n)$ converges to a constant as $n \to +\infty$.

2. Prove the fundamental theorem of algebra, i.e., given a polynomial $P(z)$, there exists a $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

3. Prove that the Taylor coefficients at the origin of $f(z) = \frac{z}{e^z - 1}$ are rational.

4. Within the (real) vector space L of all functions $\mathbb{R} \longrightarrow \mathbb{R}$, calculate the dimension of

$$V = \operatorname{span}\{\sin x,\ \sin(x + \pi),\ \sin(x + 1),\ \sin(x + \pi/3),\ \sin(2x)\}.$$