Problem I. (7pts) The position $\vec{r}(t)$ of a particle at time $t$ is given by the vector function, $\vec{r}(t) = \langle 6 \sin(2t), 6 \cos(2t), 5t \rangle$. What is the length of the curve traveled by the particle over the time interval $0 \leq t \leq \pi$.

Solution:

$$L = \int_{0}^{\pi} |\vec{r}'(t)| \, dt = \int_{0}^{\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt$$

$$= \int_{0}^{\pi} \sqrt{12 \cos(2t)^2 + [-12 \sin(2t)]^2 + 5^2} \, dt$$

$$= \int_{0}^{\pi} \sqrt{169} \, dt = \int_{0}^{\pi} 13 \, dt = 13\pi$$

Problem II. (10pts) A particle has an acceleration vector $\vec{a}(t) = t \vec{i} + t^2 \vec{j} + \cos(2t) \vec{k}$. Find its position vector $\vec{r}(t)$ if the particle has initial velocity $\vec{v}(0) = \vec{i} + \vec{k}$ and initial position $\vec{r}(0) = \vec{0}$.

Solution:

$\vec{a}(t) = \vec{r}''(t) \implies \vec{v}(t) = \vec{r}'(t) = \left\langle \frac{t^2}{2}, \frac{t^3}{3}, \frac{\sin(2t)}{2} \right\rangle + \vec{K}_1$

$\vec{v}(0) = \langle 1, 0, 1 \rangle = \langle 0, 0, 0 \rangle + \vec{K}_1 \implies \vec{K}_1 = \langle 1, 0, 1 \rangle$

$\vec{r}(t) = \left\langle \frac{t^3}{6}, \frac{t^4}{12}, \frac{-\cos(2t)}{4} \right\rangle + \vec{K}_1 t + \vec{K}_2$

$\vec{r}(0) = \langle 0, 0, 0 \rangle = \langle 0, 0, -1/4 \rangle + \vec{0} + \vec{K}_2 \implies \vec{K}_2 = \langle 0, 0, 1/4 \rangle$

$\vec{r}(t) = \left\langle \frac{t^3}{6}, \frac{t^4}{12}, \frac{-\cos(2t)}{4} \right\rangle + \langle 1, 0, 1 \rangle t + \langle 0, 0, 1/4 \rangle$

$\vec{r}(t) = \left( \frac{t^3}{6} \right) \vec{i} + \left( \frac{t^4}{12} \right) \vec{j} + \left( \frac{1 + t - \cos(2t)}{4} \right) \vec{k}$
Problem III. (8pts) For the function \( f(x, y) = e^{xy^2} \), compute the second derivative \( f_{xy} \).

**Solution:**

\[
f_x = y^2 e^{xy^2} \quad \rightarrow \quad f_{xy} = \left(y^2 2xy + 2y\right) e^{xy^2}
\]

\[
f_{xy} = \left(2xy^3 + 2y\right) e^{xy^2} = 2y \left(xy^2 + 1\right) e^{xy^2}
\]

Problem IV. (8pts) Consider the function \( z = f(x, y) = \frac{1}{1 + x^2 + y^2} \).

Determine the domain and range of \( f \). Sketch the level curves \( f(x, y) = k \) for the levels \( k = 1 \), \( k = 1/2 \), and \( k = 1/4 \). These sketches should be reasonably accurate. Clearly identify the coordinates of at least one point on each of the level curves.

**Solution:**

*Domain:* the entire \( x\)-\( y \) plane; \(-\infty < x, y < \infty\)

*Range:* \( 0 < z \leq 1 \)

*Level Curves:* \( k = 1/(1 + x^2 + y^2) \)
\[
\Rightarrow x^2 + y^2 = \left(1/k\right) - 1.
\]

For \( 0 < k < 1 \), the level curves are circles centered at \((0, 0)\) with radius \( r = \sqrt{(1/k) - 1} \).

\[
k = 1/2 \rightarrow r = 1 \quad \text{and} \quad k = 1/4 \rightarrow r = \sqrt{3}.
\]

For \( k = 1 \), the level curve is just the origin \((x, y) = (0, 0)\).

For all other values of \( k \), the set is empty - there are no real-valued solutions.
**Problem V.** (10 pts) Find the linear (tangent plane) approximation of the function 
\[ f(x, y) = \sqrt{x^2 + 4y^2} \] at (3, 2), and use it to approximate \( \sqrt{(2.97)^2 + 4 \cdot (2.02)^2} \).

**Solution:**

\[
\begin{align*}
f_x(x, y) &= \frac{1}{2}(x^2 + 4y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 4y^2}} = \frac{x}{z} \\
\frac{f_y(x, y)}{y} &= \frac{1}{2}(x^2 + 4y^2)^{-1/2}(8y) = \frac{4y}{\sqrt{x^2 + 4y^2}} = \frac{4y}{z}
\end{align*}
\]

\((x, y) = (3, 2) \implies z = 5, \quad f_x(3, 2) = \frac{3}{5}, \quad f_y(3, 2) = \frac{8}{5}\)

\[
f(x, y) \approx 5 + \frac{3(x - 3)}{5} + \frac{8(y - 2)}{5} = 5 + \frac{3(-0.03)}{5} + \frac{8(0.02)}{5}
\]

\[f(x, y) \approx 5 - 0.18 + 0.32 = 5.014\]

**Exact Answer** = 5.01423

**Problem VI.** (7 pts) Determine all values of the constant \( k \) that make the function 
\[ u(t, x) = \sin(x - kt) + \ln(x + kt) \] a solution to the differential equation,

\[
\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}
\]

**Solution:**

\[
\begin{align*}
u_t(x, y) &= -k \cos(x - kt) + \frac{k}{x + kt} \\
u_{tt}(x, y) &= -k^2 \sin(x - kt) - \frac{k^2}{(x + kt)^2} \\
u_x(x, y) &= \cos(x - kt) + \frac{1}{x + kt} \\
u_{xx}(x, y) &= -\sin(x - kt) - \frac{1}{(x + kt)^2}
\end{align*}
\]

\[
u_{tt} - 4u_{xx} = (-k^2 + 4) \sin(x - kt) + \frac{-k^2 + 4}{(x + kt)^2}
\]

\[u_{tt} - 4u_{xx} = 0 \iff \frac{k^2}{4} = 4 \iff k = \pm 2\]