Place Value as the Key to Teaching Decimal Operations

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Some years ago I examined several middle school students' understanding of numbers (Threadgill-Sowder 1984). The answers that students gave me during that study showed me that their understanding, developed largely through experiences in the elementary grades, was fuzzy and led me to undertake a decade of research on children's number sense in the elementary and middle school grades. I will set the stage for this article by sharing two of the questions I gave the students during that study and some of the responses I received.

Question 1

> The sum of 148.72 and 51.351 is approximately how much?

One student said, "Two hundred point one zero zero. Because the sum of 72 and 35 is about 100, and then 148 and 51 is about 200." (Note: I have used words for numerals where there is confusion about how the students read the numbers.) Another said, "One hundred fifty point four seven zero, because one hundred forty-eight point seven two rounds to one hundred point seven and fifty-one point three five one rounds to fifty point four zero zero. Add those." Fewer than half the students gave 200 as an estimate of this sum. The others saw a decimal number as two numbers separated by a point and considered rounding rules to be inflexible.

Question 2

> 789×0.52 is approximately how much?

One response was "789. I rounded point five two up to 1 and multiplied." A second student said, "Zero. This (789) is a whole number, and this (0.52) is not. It (0.52) is a number, but it is very small. You round 789 to 800, times zero is zero." Only 19 percent of the students rounded 0.52 to 0.5 or 1/2 or 50 percent. Several of them said that answering this question without paper and pencil was impossible and refused to continue. The majority of students had little idea of the size of a decimal fraction and applied standard rounding rules that were inappropriate for this estimation.

Others who have studied elementary school children's understanding of decimal numbers have found that when students are confronted with decimal numbers, many are confused about what the symbols mean. In a study of fourth through seventh graders by Sackur-Grisvard and Léonard (1985), children devised two "rules" to help them compare decimal numbers (p. 161). These rules worked just often enough that students did not recognize that they were in error. (I suspect that many teachers will recognize them.)

> Rule 1: Select as smaller the number whose decimal portion, as a whole number, is the smaller (e.g., 12.4 is smaller than 12.17, because 4 is smaller than 17.
>
> Rule 2: Select as smaller the number whose decimal portion has more digits (e.g., 12.94 and 12.24 are smaller than 12.7, because they each have two digits and 12.7 has only one).

The first rule has its origins in the separation of a decimal number into two numbers; that is, children treat the portions separately, and in this case treat the portion to the right of the decimal point as though it is itself a whole number separate from the number to the left of the decimal point. The second rule is slightly more sophisticated; it is based on the thinking that tenths are larger than hundredths.

More recent research on decimal-number understanding confirms that many students have a weak understanding of decimal numbers. For a summary of this work, see Hiebert (1992). The children in these studies were primarily from classes where the introduction to decimal numbers was brief so that sufficient time would remain for the more difficult work of learning the algorithms for operating on decimal numbers. But time spent on developing students' understanding of the decimal notation is not time wasted. Teachers with whom I have worked claim that much less instructional time is needed later for operating on decimal numbers if students first understand decimal notation and its roots in the decimal-place-value system we use. In the remainder of this article I will discuss decimal notation and how we can help students construct meaning for decimal number.

Giving Meaning to Decimal Symbols

The system of decimal numbers is an extension of the system of whole numbers and, as such, contains the set of whole numbers. For the sake of convenience, this article refers to decimal numbers as those numbers whose numerals contain a decimal point.

Decimal numbers, like whole numbers, are symbolized within a place-value system. Place-value instruction is traditionally limited to the placement of digits. Thus, children are taught that the 7 in 7200 is in the thousands place, the 2 is in the hundreds place, a 0 is in the tens place, and a 0 is in the ones place. But when asked how many $100 bills could be obtained from a bank account with $7200 in it, or how many boxes of 10 golf balls could be packed into a container holding 7200 balls, children almost always do long division, dividing by 100 or 10. They do not read the numbers as 7200 ones or 720 tens, or 72 hundreds, and certainly not as 7.2 thousands. But why not? These names all stand for the same number, and the ability to rename in this way provides a great deal of flexibility and insight when working with the number. (Interestingly, we later expect students to understand such newspaper figures as $3.2 billion.) Therefore, before we begin instruction on decimal numbers, we need to provide more instruction on place value as it is used for whole numbers, by asking such questions as the bank question and the golf-ball question, and we need to practice reading numbers in different ways. Problems that require working with powers and multiples of 10, both mentally and on paper, give students a flexibility useful with whole numbers, and this flexibility makes it easier to extend instruction to decimal numbers.

The naming of decimal numbers needs special attention. The place-value name for 0.642 is six hundred forty-two *thousandths*. Compare this form with 642, where we simply say six hundred forty-two, not 642 *ones*. This source of confusion is compounded by the use of the *dths* (thousan*dths*, hundre*dths*) or *nths* (te*nths*) with decimal numbers and the use of *d* (thousan*d*, hundre*d*) or *n* (te*n*) with whole numbers. The additional digits in the whole number with a similar name is another source of confusion. Whereas 0.642 is read 642 *thousandths*, 642 000 is read 642 *thousand*, meaning 642 thousand *ones*.

In a number containing a decimal point, the units place, not the decimal point, is the focal point of the number, as shown in **figure 1.** The decimal point identifies where the units, or ones, place is located; it is the first place to the left of the decimal point. The decimal point also tells us that to the right the unit one is broken up into tenths, hundredths, thousandths, and so on. So really, 0.642 is 642 thousandths of 1. Put another way, 0.6 is six-tenths of 1, whereas 6 is 6 ones and 60 is 6 tens, or 60 ones. But just as 0.6 is six-tenths of 1, 6 is six-tenths of 10, 60 is six-tenths of 100, and so on. These relationships can be more clearly seen in the base-ten-blocks representations shown in **figure 2.**

Similarly, starting with the smaller numbers, 0.006 is six-tenths of 0.01, whereas 0.06 is six-tenths of 0.1.

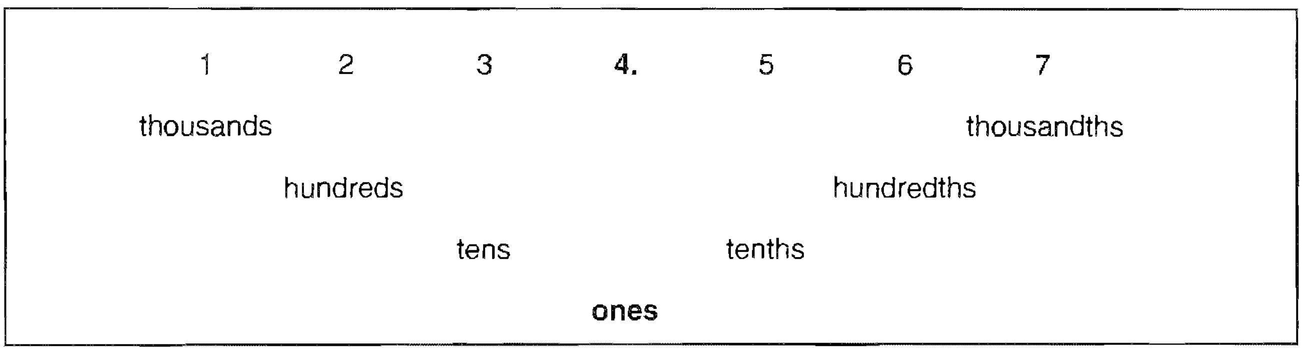


Fig. 1. Ones as the focal point of the decimal system

Moving in the opposite direction, 6000 is 60 hundreds, 600 is 60 tens, 60 is 60 ones, 6 is 60 tenths, 0.6 is 60 hundredths, 0.06 is 60 thousandths, and so on. Although this convention might seem confusing at first, it becomes less so with practice. These issues are discussed more fully in Sowder (1995).

Students who try to make sense of mathematics must become very confused when they are told to "add zeros so the numbers are the same size" when comparing numbers such as 0.45 and 0.6. This strategy does not develop any sense of number size for decimal numbers. Instead of annexing zeros, it would be more appropriate to expect students to recognize that another name for 6 tenths is 60 hundreths, which is more than 45 hundredths, or that 45 hundredths has only 4 tenths and what is left is less than another tenth, so it must be less than 6 tenths. Students do come to think this way when comparing decimal numbers if they have had sufficient opportunities to explore and think about place value, using manipulatives as representations for numbers. Heibert (1992) discusses research showing that if students do not have a sound understanding of place value when they learn to add and subtract decimal numbers, they make many errors that are very difficult to overcome because they are reluctant to relearn how to operate on decimal numbers in a meaningful way.

An Instructional Unit on Decimal Numbers

The unit summarized here was developed for a research study (Markovits and Sowder 1994) and resulted in students' performing much better on later decimal topics in their textbook. This unit has also been used by teachers who asked me for a way to teach decimals meaningfully. These teachers later

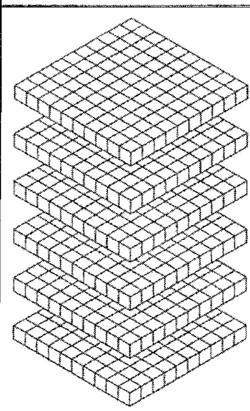
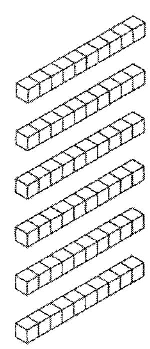
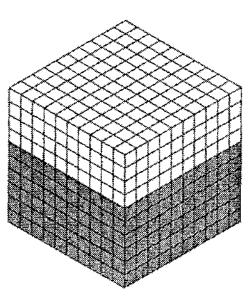
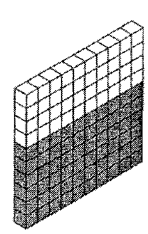
Number	60	6	0.6
Base-ten-number name	six tens	six ones	six tenths
Base-ten-block representation			
Alternative base-ten representation			
Alternative base-ten name	six-tenths of 100	six-tenths of 10	six-tenths of 1

Fig. 2. Alternative number names and representations when a long represents one unit

told me that they thought the students who completed this instructional unit had a much better grasp of decimal numbers than did their students in previous years.

The first lessons focus on gaining familiarity with base-ten materials, which can be ordered from most catalogs of mathematical aids. The materials consist of individual centimeter cubes, long blocks that are marked to look as though ten cubes have been glued in a row; flat blocks that are marked to look as though ten longs have been glued into a ten-by-ten block, and large blocks that are marked to look as though ten flat blocks have been glued to form a ten-by-ten-by-ten cube. Note that we do not call the smallest block a unit as is commonly done, because in this instruction we change the naming of the unit so that other blocks can represent one, that is, one unit.

Students must play with the blocks and learn relationships to answer such questions as the following:

- How many longs are in a flat?
- How many small blocks are in 3 longs?
- Where do you think there will be more longs, in 3 flats or in 1 big block?
- I have 6 longs and 3 small blocks. What do I have to add in order to have a flat?
- Which is bigger, that is, has more wood, 1 block or 10 flats? Four flats or 48 longs?

In the next lesson we begin to use the blocks to represent numbers. The small blocks are used to represent the number 1. Students then are asked what numbers are represented by various sets of blocks: two big blocks, three flats, and two little blocks; one flat and two longs; and so on. They must also represent numbers with blocks; for example, they show 404 with blocks. Two-dimensional drawings can later be used for the blocks, and these drawings can be used on assignments for problems like the one in **figure 3**.

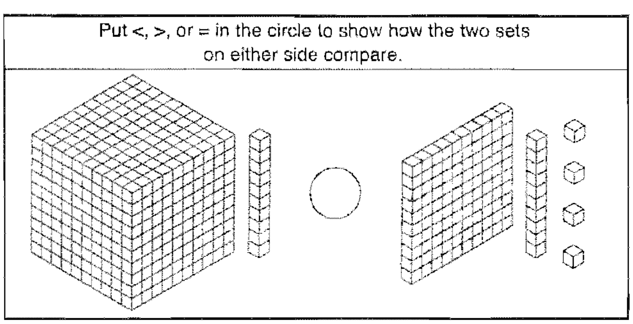


Fig. 3. Substituting a two-dimensional drawing for blocks

Alternatively, students can be asked to show with blocks the larger of 99 and 111, which of the numbers 204 and 258 is closer to 235, and so on. Students should be asked questions that indicate the limitations of block representations of numbers when the small cube represents 1:

- Can you represent 46,321 with the blocks you have? Why or why not?
- Can you represent 8 1/2 with the blocks you have? Why or why not?

The next lessons should focus on changing the unit. First let one long represent one whole, or one unit. Students can then be asked to represent 76. (They would do so with seven flats and six longs.) After many such questions, they can again be asked, "Can you represent 8 1/2 with the blocks you have? Why or why not?" (Yes, with eight longs and five small cubes.) It is then worthwhile to ask a few questions—remaining in the whole-number system—where the flat represents one unit.

It is then natural to begin decimal instruction. If the flat represents one whole, then what does a long represent? It is obviously less than 1. What part of 1 is it? Since ten longs are in a flat, one long represents 0.1. Several questions should follow:

- How would you represent 0.3? 4.3? **(See fig. 4.)**
- How many tenths are in four wholes?
- What do you have to add to 0.9 to have one whole?
- 4.5 is ___ ones and ___ tenths, or ___ tenths.
- Which of the following are equivalent to one flat and four longs: 14? 1.4? 140? 14 longs? 41 longs? 41?

Likewise, children can come to understand that a small block in this context represents one-hundredth,

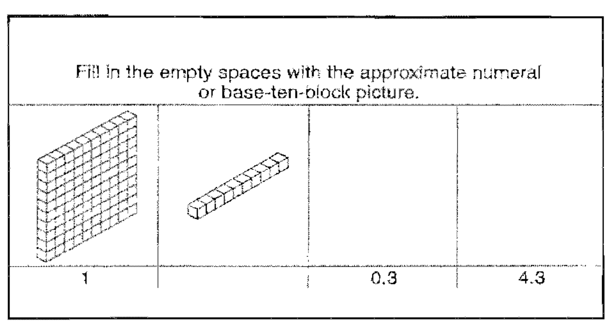


Fig. 4. Representing numbers with base-ten blocks

and many questions similar to the previous questions can be asked. Teachers can also present such problems as the following:

In 6.40 are ___ tens and ___ ones and ___ tenths and hundredths.

In 6.4 are ___ tens and ___ ones and ___ tenths and hundredths.

In 6.04 are ___ tens and ___ ones and ___ tenths and hundredths.

Are any of these numbers the same? Why?

A great deal of practice is needed in each of the lessons described here; the questions indicated are only a small sample. Try using the big block as the unit and going through all the exercises again, this time introducing one-thousandth. Ask students to describe how they could cut up the blocks to represent one ten-thousandth and one hundred-thousandth. When students feel very secure with the blocks, with changing units, and with problems involving decimals, it is time to switch to another representation. A day or two spent with money—dollars and cents—will work well. Finally, a lesson or two should focus directly on decimal numbers without using another representation (although many children will naturally answer in terms of "blocks" or "wood"). Questions like the following can be asked:

- Is 0.1 closer to 0 or to 1?
- Is 1.72 closer to 1 or to 2?
- I am a number. I am bigger than 0.5 and smaller than 0.6. Who am I?
- Are there decimals between 0.3 and 0.4? How many do you think there are?
- Are there decimals between 0.35 and 0.36? How many?
- Are there decimal numbers between 0.357 and 0.358? How many?

Draw baskets and label them "Numbers smaller than 0.5," "Numbers bigger than 0.5 but smaller than 1," "Numbers between 1 and 3," and "Numbers bigger than 3." Then give the students the following numbers and ask them to place each number in the appropriate basket: 0, 0.03, 1.01, 5.03, 2.63, 0.49, 0.93, 0.60, 1.19, and so on. This type of problem can be made more difficult with baskets labeled "Numbers between 0.4 and 0.5," "Numbers between 0.7 and 0.8," and "All other numbers."

If desired, these lessons could be interrupted before decimal numbers are introduced, and addition and subtraction of whole numbers could be introduced using the blocks. But when addition and subtraction of decimal numbers are introduced, lessons with the blocks should come first.

The two questions at the beginning of this article are trivial for students who have had this instruction. The students see the part of the number to the right of the decimal point as a natural extension of the place-value system, and they treat the entire number as one quantity. Such students also develop a good feeling for the sizes of decimal numbers and can compare them with one another. It did not occur to any of the students who received this instruction to round 0.52 to 0 or to 1 when estimating a product—0.52 was simply seen as "about a half."

When students understand what they are doing, they tend to enjoy doing mathematics. It is worth the time needed to build strong foundations. The time will be easily made up in future lessons, and students are much more likely to be successful.

Action Research Ideas

1. For each of the following pairs of decimal numbers, ask students to tell which is smaller. Then analyze their answers to see if any are making the rule-1 or rule-2 errors identified in the Sacker-Grisvard and Léonard (1985) study.

Number Pair	Use of Rule 1	Use of Rule 2	Correct
3.17 or 3.4	3.4	3.17	3.17
14.285 or 14.19	14.19	14.285	14.19
6.43 or 6.721	6.43	6.721	6.43
11.01 or 11.002	11.01	11.002	11.002
9.642 or 9.99	9.99	9.642	9.642
15.134 or 15.12	15.12	15.134	15.12
156.1 or 156.012	156.1	156.012	156.012

If you find evidence of systematic rule-1 or rule-2 errors, use some of the instructional ideas in this article and then reassess your students to determine whether their understanding of place value has improved. In addition to rule-1 or rule-2 errors, look for other systematic errors that students make. What are the misconceptions that underlie these errors?

2. Assess your students' understanding of place value by asking such questions as the following.

   (a) The Sweet Candy Company places 10 pecan clusters in each box they sell. The cook just made 262 pecan clusters. How many boxes can be filled with the fresh pecan clusters?

   (b) There is $2148 in the bank, ready to be used for prizes for the state science fair. If each prize is $100, how many prizes can be given?

Students with good place-value understanding will not need to do any division. Some students may solve *(a)* and *(b)* by using division. Some may not solve them at all. In either case try numbers like 260 or $2100 to see if easier numbers allow them to use their more limited place-value knowledge. If you find some students making large numbers of errors, use some of the instructional ideas in this article. Then reassess them using similar questions to determine whether their knowledge of place value has improved.

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