Putting Research into Practice in the Elementary Grades

Readings from Journals of the National Council of Teachers of Mathematics

Edited by
Donald L. Chambers
Wisconsin Center for Education Research
University of Wisconsin—Madison
Madison, Wisconsin

National Council of Teachers of Mathematics
Reston, Virginia

Number Sense and Nonsense

Zvia Markovits, Rina Hershkowitz, and Maxim Bruckheimer

Common sense has many aspects and is developed by a variety of experiences in and out of school. Number sense is one aspect of common sense that we rightly expect schooling to improve. But does it? Given a problem, do students pay any attention to the meaning of the numbers in the data or in the solution they obtain? In study devoted to estimation and reasonableness of results, we found that sixth- and seventh-grade students either do not have a reasonably developed number sense or, if they have it, do not apply it to simple tasks in a mathematical context.

This "number nonsense" occurred in purely numerical tasks as well as in tasks with a contextual setting. The former kinds of tasks are abstract and confined to the classroom, but the latter tasks refer to real situations that should naturally encourage the application of number sense. Consider this classic problem:

> The height of a 10-year-old boy is 1.5 meters. What do you think his height will be when he is twenty?

About a third of the students in our study answered 3 meters! These students apparently invoked a standard algorithm without paying any attention to its relevance or to the meaning of the number obtained. One suspects that their mathematical experience has been confined almost exclusively to applying one standard algorithm at any one time and obtaining one answer. Neither the choice of algorithm nor the meaning of the answer was explicitly questioned. In other words, number sense has not been explicitly and meaningfully developed.

We investigated the performance of sixth and seventh graders on a variety of tasks involving estimation and reasonableness of results, both of which invoke number sense. We also examined the effect of a short unit of activities on these two topics. Although the unit was written specifically for these grades, most of the tasks could be given to students in lower grades with some adjustment where necessary.

Number Sense in Purely Numerical Tasks

Students were asked to locate the decimal point in the answers given in these two exercises:

$$\begin{array}{r} 3.5 \\ \underline{\times 4.5} \\ 1575 \end{array} \qquad\qquad \begin{array}{r} 5.5 \\ \underline{\times 3.2} \\ 176 \end{array}$$

Students with number sense would find these exercises easy, even if they did not know the standard algorithm. But we found that students counted the digits after the decimal point, obtaining 1.76 as the second answer. It never crossed their minds to consider whether the answer made sense; apparently they think that algorithms never lie.

Experiences with whole numbers often carry over into work with fractions and decimals and dominate students' thinking to the extent that number sense never appears to play a role in obtaining an answer. For example, consider another exercise we used:

The result of $426.5 \div 0.469$ is . . .

a) less than 426.5
b) equal to 426.5
c) greater than 426.5

Explain your answer.

The correct answer, *(c)*, was given by 40 percent of the 328 students. The explanation for most wrong answers was that "when divided, a number becomes smaller." Slightly better, but similar, results were obtained for multiplication.

To improve number sense, students throughout their schooling should be given many varied tasks that require little, if any, computation. Here are some examples of tasks that increase number sense:

1. Which is greater—

   13 + 11 or 9 + 8?

   46 − 19 or 46 − 17?

   1/2 + 3/4 or 1 1/2?

   0.0358 or 0.0016 + 0.393?

   Many students will do exact calculations before answering, but the teacher can use various strategies to wean them from this practice, for instance, by giving the questions orally and imposing a time limit.

2. The result of 46 × 91 will be in the—

   a) hundreds

   b) thousands

   c) ten thousands.

Is 46 × 91 more or less than 5000? More or less than 3600?

3. If you could round only one of the numbers in the problem 32 × 83, rounding which factor would produce a product closest to the exact answer?

4. *a*) 127 × □ ends in 3. What can you say about the missing whole number?

   If you also know that the product is less than 4000, what can you then say about it?

   If the product is also greater than 3000, what is the missing number?

   b) The sum of five two-digit numbers is less than 100. For each of the following statements decide whether it is necessarily true, necessarily false, or possibly true. Explain your answers.

   (1) Each number is less than 20.

   (2) One number is greater than 60.

   (3) Four numbers are greater than 20, and one is less than 20.

   (4) If two numbers are less than 20, at least one is greater than 20.

   (5) If all five numbers are different, then their sum is greater than or equal to 60.

5. Without computing, explain why each of the following is incorrect:

a)	*b*)
310	119
520	46
630	137
150	940
+ 470	+ 300
2081	602

c) 27 × 3 = 621 *d*) 36 ÷ 0.5 = 18

e)
0.46
1.93
2.46
0.99
+ 0.87
672

Number Sense in Context

Another question we used in the study was the following:

> A bath has two outlets. The first alone empties the bath in 10 minutes, and the second alone in 4 minutes. Both were opened at the same time. In how many minutes will the bath be empty?

Choose the most appropriate answer from among the following and explain your answer.

a) 40 minutes
b) 14 minutes
c) less than 14 but more than 7
d) 7 minutes
e) 6 minutes
f) 4 minutes
g) about 3 minutes
h) about ½ minute

The students had not been taught to solve such problems using algebra, but number sense is all that is necessary: if one outlet empties the bath in four minutes, then the two outlets will empty the bath in less than four minutes but not in as little as half a minute.

Once again we found that students tended to apply an algorithm and operate on the numbers in the question without thinking. They added, multiplied, computed the mean, and so on. Even if we also allow one-half minute as a reasonable option, since it is less than four minutes, only 36 percent of the students who had not received the special instruction gave a reasonable answer. After receiving the brief special instruction 85 percent of the students gave a reasonable answer to a question in a similar context.

In this task, the algorithm—even if known—is irrelevant, but there is nothing inherently ridiculous in the answers obtained by incorrect algorithms. To conclude that an answer is incorrect, the student must relate it to the data in the question. This ability demands a more sophisticated number sense than that required by the height problem at the beginning of this article—three meters is ridiculous for a man's height.

The general problem of learning to judge the reasonableness of a result in a given context involves number sense. Although students may be encouraged to check their answers, they do so far too often only in purely mathematical situations, where a standard algorithm has been applied, and they check only whether that algorithm has been performed correctly. The numerical work itself has no real-life context. This situation leads students to divorce the mathematical answer completely from its implication in the context; the answer, they think, is mathematically correct even though it makes no sense in the context. This phenomenon is again demonstrated by the following task from our study:

> A bookstore delivered 188 books to 6 libraries. How many books do you think each library received?

This question cannot be answered algorithmically; nevertheless, many students did use the division algorithm. Fractional answers (31 ⅓ books, 31.3 books, etc.) were given by 34 percent of the students before doing our instructional activities, but only 3 percent gave such answers after the instruction. When we asked whether a man can be three meters high or whether a library can receive thirty-one and one-third books, the answer was definitely "no!" But if these answers are given by the algorithms, then students assume that they are correct!

We also found that students know little of various everyday measurements that can serve as reference data for number sense in contextual problems, for example, the speed of a car, the height of an eight-story building, or the number of liters of water in a bucket. When we asked for the speed of a plane flying from London to New York, a reasonable answer (we allowed anything from 500 to 1500 km/h) was given by 29 percent of the students who had no instruction. We also found that if we asked for the measures of various objects of the same kind (e.g., the width of a car, a truck, a main road, etc.), then often the answers were internally inconsistent (e.g., the width of car would be given as greater than that of a road). Therefore, we recommend that students first be asked to order the given objects qualitatively and only then be asked to assign individual measures to them. For example, order the following from the heaviest to the lightest and estimate the weight of each: car, bicycle, truck, dog, chair, elephant, kitten. Correct responses approximately doubled after our instruction, indicating that similar instruction needs to be extended or improved in the classroom. We feel that this sort of number sense is not easy to acquire and probably needs to be developed gradually over a number of years of schooling.

Conclusion

Number sense is not a single curricular topic but has many aspects. The development of number sense is not something that occurs naturally for most students in the regular curriculum, as we have seen in this study. Evidence clearly suggests, however, that number sense can be developed with appropriate activities. The short instruction we gave to sixth and seventh graders improved their number sense, in some aspects very significantly. They also learned to look at all the answers they obtained in a different way: the numbers in the answer had a meaning that needed to be examined critically.

We suggest that activities of the types described here should be integrated into the curriculum, even before sixth and seventh grades. We should begin in the first grade with appropriate tasks that call on number sense and give the children a less mechanical view of mathematics.