Deterministic Channel Design for Minimum Leakage

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Objective of this work

Study the application of this framework to deterministic systems

Preliminaries

Quantitative Information Flow

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- A system takes the secret value as input and produces an observable behaviour (or simply observable) in $\mathcal{Y} = \{y_1, \ldots, y_m\}$
- An adversary, observing the behaviour of the system, may obtain some information about the secret value

Systems as Channels

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  - $C(x, y) > 0$ $\qquad$ $\sum_y C(x, y) = 1$

C	y_1	y_2	y_3	y_4
x_1	$1/2$	$1/4$	$1/8$	$1/8$
x_2	$1/4$	$1/2$	$1/4$	0
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How is information leaked?

- The adversary knows π and C

π
$1/3$
$1/4$
$1/4$
$1/6$

C	y_1	y_2
x_1	1	0
x_2	0	1
x_3	0	1
x_3	1	0

How is information leaked?

- The adversary knows π and C
- Joint distribution $p(x,y) = \pi(x)C(x,y)$

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	$p_{\mathcal{X}\|y_1}$	$p_{\mathcal{X}\|y_2}$
x_1	$2/3$	0
x_2	0	$1/2$
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x_3	0	$^1/_2$
x_4	$^1/_3$	0

$$p(y_1) = {}^1/_2$$

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- By observing y, the adversary updates the distribution from π to $p_{\mathcal{X}|y}$

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- What is a reasonable entropy?
- How should we model operational constraints?

What is a Reasonable Entropy?

- A entropy H is core-concave if there is η, F such that
  - $H(\pi) = \eta(F(\pi))$
  - F is a real valued, continuous and concave function
  - $\eta : I \to \mathbb{R}$ is continuous and increasing

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- Generalizes most entropy measures in QIF

How Should We Model Operational Constraints?

- Hard constraints: A set $\Omega \subset \mathcal{X} \times \mathcal{Y}$ of which observables can be produced for each secret.
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  - Execution time, difference between real and reported data, ...
- Constraint: $\mathbb{E}[u] = \sum_{x,y} \pi(x) C(x,y) u(x,y) \geq u_{min}$

The general framework for the Channel Design Problem

(Probabilistic) Channel Design Problem (Khouzani and Malacaria, CSF 2017)

Find channel $C : \mathcal{X} \to \mathcal{Y}$ that maximizes $H(\pi, C)$ subject to

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- Solved by convex programming (Karush-Kuhn Tucker conditions)

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(π_u is the uniform distribution, and $\Omega = \{(x, y) \mid x \in y\}$)

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0.15
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Optimal for Shannon entropy

The complete k-hypergraph problem

The Complete k-hypergraph Problem:

- The Complete k-hypergraph Problem: at most k secret values can be mapped to each observable
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  - $\mathcal{Y} = \{\mathcal{A} \subset \mathcal{X} \,|\, |\mathcal{A}| \leq k\}, \qquad \Omega = \{(x, y) \,|\, x \in y\}$.
- **Result:** There is a greedy solution to a subset of core-concave entropies, called leakage-supermodular
  - Includes most entropies used in QIF: Shannon entropy, min-entropy, guessing entropy...

Leakage-Supermodularity: Supermodular Functions

- Lattice on $\mathbb{R}^n_{\geq 0}$: Let $\mathbf{r} = (r_1, \ldots, r_n)$,$\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n_{\geq 0}$
  - Join: $\mathbf{r} \vee \mathbf{s} = (\max(r_1, s_1), \ldots, \max(r_n, s_n))$
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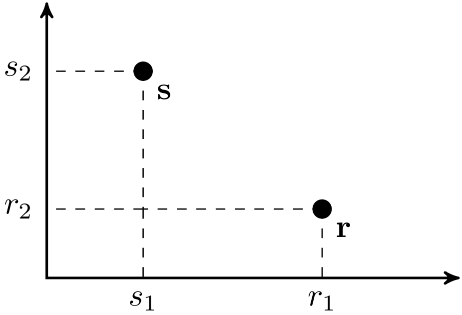
Example ($n = 2$):

$$\begin{aligned} \mathbf{r} &= (r_1, r_2) \\ \mathbf{s} &= (s_1, s_2) \end{aligned}$$



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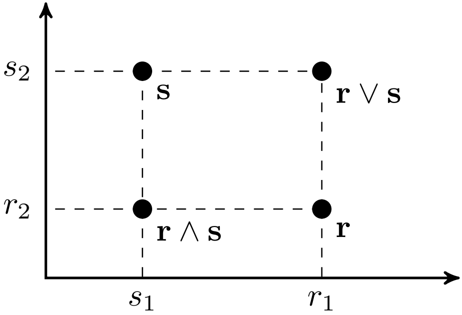

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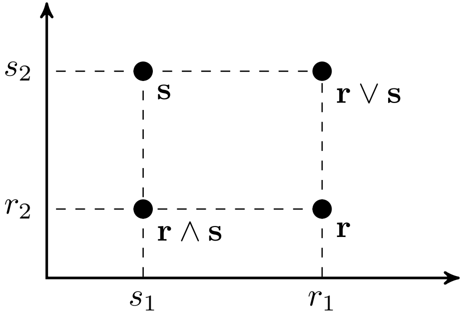

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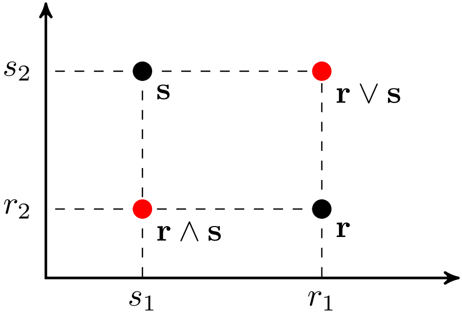

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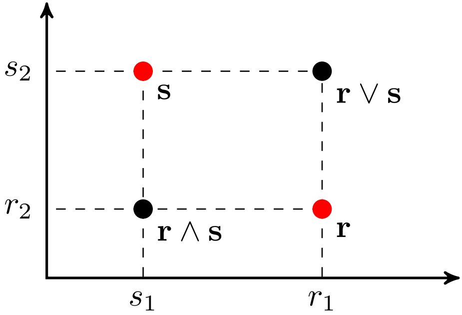
- A function $\phi : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is Supermodular if

$$\phi(\mathbf{r} \vee \mathbf{s}) + \phi(\mathbf{r} \wedge \mathbf{s}) \geq \phi(\mathbf{r}) + \phi(\mathbf{s})$$

Example ($n = 2$):

$$\mathbf{r} = (r_1, r_2)$$
$$\mathbf{s} = (s_1, s_2)$$



Leakage-Supermodular Entropies

- For now on, we restrict our attention to entropies that are
  - Symmetric: $H(\pi_1, \ldots, \pi_n) = H(\pi_{\phi(1)}, \ldots, \pi_{\phi(n)})$ for all permutations ϕ
  - Expansible: $H(\pi_1, \ldots, \pi_n, 0) = H(\pi_1, \ldots, \pi_n)$

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- Given a core-concave H for some η, F, define $G_F : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$

$$G_F(r_1, \ldots, r_n) = \left(\sum_i r_i\right) F\left(\frac{r_1}{\sum_i r_i}, \ldots, \frac{r_n}{\sum_i r_i}\right)$$

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Theorem

Shannon entropy, min-entropy, guessing entropy and Arimoto-Rényi entropies are leakage-supermodular

Leakage-Supermodular Entropies

- Relation to leakage:

$$H(\pi, C) = \eta \left(\sum_y G_F \left(p(x_1, y), \ldots, p(x_n, y) \right) \right)$$

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π
$1/3$
$1/4$
$1/4$
$1/6$

C	y_1	y_2
x_1	1	0
x_2	0	1
x_3	0	1
x_3	1	0

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p	y_1	y_2
x_1	$1/3$	0
x_2	0	$1/4$
x_3	0	$1/4$
x_4	$1/6$	0

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$$H(\pi, C) = \eta\left(\sum_y G_F\left(p(x_1, y), \ldots, p(x_n, y)\right)\right)$$

p	y_1	y_2
x_1	$1/3$	0
x_2	0	$1/4$
x_3	0	$1/4$
x_4	$1/6$	0

$$H(\pi, C) = \eta\Big(G_F(1/3, 0, 0, 1/6) + G_F(0, 1/4, 1/4, 0)\Big)$$

A greedy solution

The k-hypergraph Problem

Let $\mathcal{Y} = \{\mathcal{A} \subset \mathcal{X} \mid |\mathcal{A}| \leq k\}$. Find $C : \mathcal{X} \to \mathcal{Y}$ that maximizes $H(\pi, C)$, subject to $\Omega = \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid x \in y\}$

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Greedy solution for 8 secret values, and $k = 3$

π
0.25
0.20
0.15
0.13
0.10
0.08
0.07
0.02

C	y_1	y_2	y_3
x_1	1	0	0
x_2	1	0	0
x_3	1	0	0
x_4	0	1	0
x_5	0	1	0
x_6	0	1	0
x_7	0	0	1
x_8	0	0	1

A greedy solution

Theorem

For leakage-supermodular entropies, the greedy solution is optimal

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For leakage-supermodular entropies, the greedy solution is optimal

Proof idea.

π
0.3
0.3
0.2
0.1
0.1

C	y_1	y_2	y_3
x_1	1	0	0
x_2	0	1	0
x_3	1	0	0
x_3	0	1	0
x_3	0	0	1

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p	y_1	y_2	y_3
x_1	0.3	0	0
x_2	0	0.3	0
x_3	0.2	0	0
x_3	0	0.1	0
x_3	0	0	0.1

A greedy solution

Theorem

For leakage-supermodular entropies, the greedy solution is optimal

Proof idea. $H(\pi, C) = \eta\big(G_F(0.3, 0.2) + G_F(0.3, 0.1) + G_F(0.1)\big)$

p	y_1	y_2	y_3
x_1	0.3	0	0
x_2	0	0.3	0
x_3	0.2	0	0
x_3	0	0.1	0
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p'	y_1	y_2	y_3
x_1	0.3	0	0
x_2	0.3	0	0
x_3	0	0.2	0
x_3	0	0.1	0
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$$(0.3, 0.3) = (0.3, 0.2) \vee (0.1, 0.3)$$

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For leakage-supermodular entropies, the greedy solution is optimal

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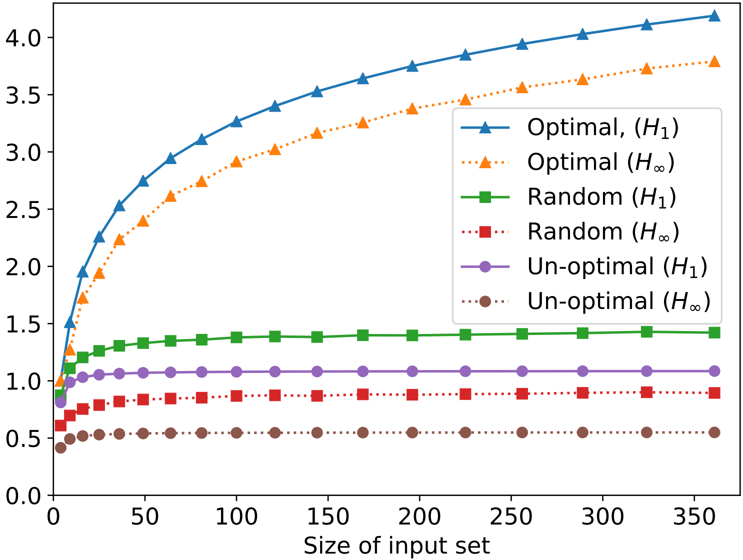
$$H(\pi, C') \geq H(\pi, C)$$

Experimental comparison



Deterministic Channels as a Solution for Multiple Executions

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- Often, a system is executed multiple times for a fixed secret value

Deterministic Channels as a Solution for Multiple Executions

- Often, a system is executed multiple times for a fixed secret value
- How do we design an optimal system in this scenario?

Deterministic Channels as a Solution for Multiple Executions

C	y_1	y_2	y_3
x_1	?	?	0
x_2	?	0	?
x_3	0	?	?
x_4	?	0	?
x_5	?	?	?

Deterministic Channels as a Solution for Multiple Executions

C	y_1	y_2	y_3
x_1	$1/2$	$1/2$	0
x_2	$1/3$	0	$2/3$
x_3	0	$3/4$	$1/4$
x_4	$1/3$	0	$2/3$
x_5	$3/5$	$1/5$	$1/5$

Deterministic Channels as a Solution for Multiple Executions

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x_2	$1/3$	0	$2/3$
x_3	0	$3/4$	$1/4$
x_4	$1/3$	0	$2/3$
x_5	$3/5$	$1/5$	$1/5$

$\longrightarrow$

C	$[x_1]$	$[x_2]$	$[x_3]$	$[x_5]$
x_1	1	0	0	0
x_2	0	1	0	0
x_3	0	0	1	0
x_4	0	1	0	0
x_5	0	0	0	1

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Proposition

Let C be a probabilistic channel respecting the operational constraints. Then, there is a deterministic channel D that respects the same constraints and asymptotically leaks at most as much information as C

Deterministic Channels as a Solution for Multiple Executions

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Let C be a probabilistic channel respecting the operational constraints. Then, there is a deterministic channel D that respects the same constraints and asymptotically leaks at most as much information as C

Thus, the deterministic solution to the design problem is asymptotically optimal

Conclusions and Contributions

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  - *Channel Ordering and Supermodularity* – to appear at IEEE ITW 2019
- Proved that, if a system is to be executed multiple times, the deterministic solution is optimal when the number of executions is very large