MATH 571: Higher Algebra I, Winter 2005

Solutions to Assignment 5

Chapter IV, Section 5, #2: **(a)** Consider the map $\phi : A \times \mathbb{Z}_m \to A/mA$ defined by $(a, k + m\mathbb{Z}) \to ka + m\mathbb{Z}$. ϕ is a well-defined bilinear map, so by the universal property of tensor product it follows that there exists a unique homomorphism $\widetilde{\phi} : A \otimes \mathbb{Z}_m \to A/mA$. The map $\psi : A/mA \to A \otimes \mathbb{Z}_m$ defined by $a + m\mathbb{Z} \to a \otimes (1 + m\mathbb{Z})$ is a well-defined homomorphism and it can be seen easily that ψ is inverse to $\widetilde{\phi}$. Thus, $\widetilde{\phi}$ is bijective.

(b) Consider the map $\phi : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_c$ defined by $(k + m\mathbb{Z}, l + n\mathbb{Z}) \to kl + c\mathbb{Z}$. ϕ is a well-defined bilinear map, so by the universal property of tensor product it follows that there exists a unique homomorphism $\widetilde{\phi} : \mathbb{Z}_m \otimes \mathbb{Z}_n \to \mathbb{Z}_c$. The map $\psi : \mathbb{Z}_c \to \mathbb{Z}_m \otimes \mathbb{Z}_n$ defined by $k + c\mathbb{Z} \to (k + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})$ is a well-defined homomorphism and it can be seen easily that ψ is inverse to $\widetilde{\phi}$. Thus, $\widetilde{\phi}$ is bijective.

(c) If A and B are f.g. abelian groups then

$$A \simeq \mathbb{Z}^k \oplus (\bigoplus_{i=1}^{l} \mathbb{Z}_{p_i}^{k_i}),\ B \simeq \mathbb{Z}^m \oplus (\bigoplus_{j=1}^{n} \mathbb{Z}_{q_j}^{m_j}).$$

Now, since

$$\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{\gcd(m,n)},\ \mathbb{Z} \times \mathbb{Z}_n \simeq \mathbb{Z}_n,\ \mathbb{Z} \times \mathbb{Z} \simeq \mathbb{Z}$$

and

$$(K \oplus M) \otimes N \simeq (K \otimes N) \oplus (M \otimes N)$$

then it can be concluded that $A \otimes B$ is finitely generated, in which the free abelian part has rank km.

Chapter IV, Section 5, #3(b): Consider the map $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ defined by $(p, q) \to pq$. ϕ is a well-defined bilinear map, so by the universal property of tensor product it follows that there exists a unique homomorphism $\widetilde{\phi} : \mathbb{Q} \otimes \mathbb{Q} \to \mathbb{Q}$. The map $\psi : \mathbb{Q} \to \mathbb{Q} \otimes \mathbb{Q}$ defined by $p \to p \otimes 1$ is a well-defined homomorphism and it can be seen easily that ψ is inverse to $\widetilde{\phi}$. Thus, $\widetilde{\phi}$ is bijective.

Chapter IV, Section 5, #8: In all cases it is enough to show that $1_D \otimes f$ is injective.

(a) Since $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits then there exists an isomorphism $\widetilde{f} : B \to A \oplus C$ such that $\widetilde{f} \circ f = 1_A$. Hence, $(1_D \otimes \widetilde{f}) \circ (1_D \otimes f) = 1_D \otimes 1_A = 1_{D \otimes A}$, that is, $1_D \otimes f$ is injective.

(b) D is free, hence, $D \simeq \oplus_{i\in I} R$. Thus,

$$D \otimes A = (\oplus_{i\in I} R) \otimes A = \oplus_{i\in I}(R \otimes A) \simeq \oplus_{i\in I} A,$$

$$D \otimes B = (\oplus_{i\in I} R) \otimes B = \oplus_{i\in I}(R \otimes B) \simeq \oplus_{i\in I} B.$$

Now, since $f : A \to B$ is injective, it follows that $1_D \otimes f : \oplus_{i\in I} A \to \oplus_{i\in I} B$ is injective too.

(c) Since D is projective it follows that there exists a free module F such that $F = D \oplus K$ for some K. From **(b)** it follows that $1_F \otimes f : F \otimes A \to F \otimes B$ is injective. Finally, observe that $F \otimes A \simeq (D \otimes A) \oplus (K \otimes A)$, $F \otimes B \simeq (D \otimes B) \oplus (K \otimes B)$, and $1_D \otimes f$ is the restriction of $1_F \otimes f$, so is injective.

Chapter IV, Section 5, #11(b)(i): Since every bilinear map $\phi : A \times B \to C$ induces a unique map $\widetilde{\phi} : A \otimes B \to C$ then the map $\theta : \mathcal{L}(A, B; C) \to Hom_R(A \otimes B, C)$ is well-defined by $\phi \to \widetilde{\phi}$. θ is obviously a homomorphism of modules. On the other hand, for each $\psi \in Hom_R(A \otimes B, C)$ one can consider $\widehat{\psi} = \psi \circ i \in \mathcal{L}(A, B; C)$, where i is a canonical projection of $A \times B$ onto $A \otimes B$. That is, the homomorphism $\eta : Hom_R(A \otimes B, C) \to \mathcal{L}(A, B; C)$ is well-defined by $\psi \to \widehat{\psi}$. Finally, it is easy to verify that η and θ are inverse to each other.

Chapter IV, Section 6, #3: Direct computations.

Chapter IV, Section 6, #5: One can take $\mathbb{Z}_4$ as a counterexample. Indeed, the minimal annihilator of $\mathbb{Z}_4$ is 4 which is divisible by 2, but $\mathbb{Z}_2$ is not a direct summand of $\mathbb{Z}_4$.