MATH 571: Higher Algebra I, Winter 2005

Solutions to Assignment 4

Chapter IV, Section 2, #4: Direct computations.

Chapter IV, Section 2, #7: If G is non-abelian then one can use conjugation by $g \in G$ which is a non-trivial inner automorpphism of G. If G is abelian and there exists at least one element $g \in G$ such that $g^{-1} \neq g$ then the map $f \to f^{-1}$ for all $f \in G$ defines an automorphism of G which is not trivial since g is not mapped to itself. Finally, we can assume that all elements of G have order 2. Hence, G coincides with its 2-torsion subgroup $G[2]$ which is a $\mathbb{Z}_2$-module. Thus,

$$G \simeq \sum_{i \in I} \mathbb{Z}_2$$

and $|I| \neq 1$ by the asumption that $|G| > 2$. As a $\mathbb{Z}_2$-module G has a basis $X = \{x_i \mid i \in I\}$ which contains at least two elements x_1 and x_2. Eventually, the map

$$\begin{cases} x_1 \to x_2, \\ x_2 \to x_1, \\ x \to x, \qquad x \in X,\ x \neq x_1, x_2 \end{cases}$$

is a linear transformation of G as a $\mathbb{Z}_2$-module and then is an automorphism of G as a group which is not trivial by construction.

Chapter IV, Section 2, #12: Let F be a free R-module, where R is a ring with identity. One can use the induction on k to prove that F has a basis of cardinality $n+k$ for any $k \geq 0$. The base of this induction when $k = 0$ is given as a precondition of the problem. Then, we assume that the statement is true for k and prove for $k+1$. That is, there exists a basis of cardinality $n+k$ so that

$$F \simeq \oplus_{i=1}^{n+k} R$$

since F is free, and we have to show that there exists another basis of cardinality $n+k+1$, or, equivalently

$$F \simeq \oplus_{i=1}^{n+k+1} R.$$

Indeed, we have

$$F \simeq \oplus_{i=1}^{n+k} R = (\oplus_{i=1}^{n} R) \oplus (\oplus_{i=1}^{k} R)$$

and

$$\oplus_{i=1}^{n} R \simeq F \simeq \oplus_{i=1}^{n+1} R.$$

Thus,

$$F \simeq \oplus_{i=1}^{n+k} R = (\oplus_{i=1}^{n} R) \oplus (\oplus_{i=1}^{k} R) \simeq (\oplus_{i=1}^{n+1} R) \oplus (\oplus_{i=1}^{k} R) = \oplus_{i=1}^{n+k+1} R.$$

Chapter IV, Section 3, #4: (a) Direct computations.

(b) If $n = |G|$ and $0 \neq g \in G$ then the equation $nx = g$ gas no solution in G.

(c) If G is a free abelian group then $G \simeq \oplus_{i \in I} \mathbb{Z}$ and G has a basis $X = \{x_i \mid i \in I\}$ as an free $\mathbb{Z}$-module. Take an element $y \in X$. If G is divisible then the equation $2x = y$ has asolution in G, that is, there exists a solution $x = \sum_{j=1}^{k} \alpha_j x_j$, where $x_i \in X,\ \alpha_j \in \mathbb{Z}$. Hence,

$$2(\sum_{j=1}^{k} \alpha_j x_j) = y$$

is a non-trivial linear combination of some elements from X. It follows that there exists $j_0 \in [1, k]$ such that $x_{j_0} = y$ and $2\alpha_{j_0} = 1$, while for the rest $x_j,\ j \neq j_0$ we have $\alpha_j = 0$. But $2\alpha_{j_0} = 1$ does not have a solution over $\mathbb{Z}$ - contradiction.

(d) Obvious.

Chapter IV, Section 3, #6: Following the hint given in the textbook we can take

$$D = \langle \bigcup_{i \in I} D_i \rangle,$$

where $S_D = \{D_i \mid i \in I\}$ is a class of all divisible subgroups of G. D is a subgroup by definition and it is easy to show that D is divisible. Both G and D can be viewed as $\mathbb{Z}$-modules and since D is divisible then it is an injective $\mathbb{Z}$-module, so G splits into the direct sum

$$G \simeq D \oplus N,$$

where N is a subgroup of G. Finally, if N is not reduced, that is, if it contains a non-trivial divisible subgroup D' then D' is a divisible subgroup of G and $D' \in S_D$. It follows that $D' \leq D$ which contradicts to the fact that $D \cap N = 0$.

Chapter IV, Section 3, #8: Direct computations using the hint from the textbook.

Chapter IV, Section 3, #10: We use hint provided in the textbook. By **#4** from **Chapter IV, Section 2** it follows that $D[p]$ is a vector space over $\mathbb{Z}_p$ and, hence, has a basis $X = \{x_i \mid i \in I\}$. For every $x \in X$ one can define H_x to be

$$H_x = \langle\ x_n,\ n \in \mathbb{N}\ \rangle,$$

where $x_1 = x,\ px_{n+1} = x_n$. It is easy to see that any $g \in H_x$ has the form qx_{n+1} for some $n \geq 0$, where q and p are relativery prime. Hence, $p^n g = qx$ and one can view g as a fraction $\frac{q}{p^n}x$. This observation makes it possible to establish an isomorphism $H_x \simeq \mathbb{Z}(p^\infty)$.

Take $0 \neq g \in D$. Then there exists $n > 0$ such that $p^n g = 0$, or, equivalently $p(p^{n-1}g) = 0$. I t follows that $p^{n-1}g \in D[p]$ and can be wxpressed in terms of X, that is,

$$p^{n-1}g = \sum_{j=1}^{k} a_j y_j,$$

where $y_j \in X$. But observe that $\frac{y_j}{p^{n-1}} \in H_{y_j}$, so,

$$g = \sum_{j=1}^{k} \frac{a_j y_j}{p^{n-1}} \in \bigcup_{i \in I} H_{x_i}.$$

Finally, we have to show that

$$\bigcup_{i \in I} H_{x_i} = \oplus_{i \in I} H_{x_i},$$

that is,

$$H_x \bigcap \bigcup_{x_i \neq x} H_{x_i} = 0.$$

Let $g \in D$ be a non-trivial element of this intersection. Then we have

$$g = \frac{bx}{p^n} = \sum_{j=1}^{k} \frac{a_j x_j}{p^{n_j}},$$

where $x_j \neq x,\ j \in [1, k]$. Take $N = \max\{n, n_1, \dots, n_k\}$, so we have

$$p^N \left(\frac{bx}{p^n} \right) = p^N \left(\sum_{j=1}^{k} \frac{a_j x_j}{p^{n_j}} \right).$$

Observe that both sides of this equality cannot be equal to zero due to our choice of N and the fact that all a_j are relatively prime with p. Hence, one gets a non-trivial linear combination of elements from X - a contradiction.

Chapter IV, Section 3, #14: We take advantage of the hint given in the textbook and to solve the problem it is enough to show that D has no maximal left ideals. Since every D-module is free then every D-module is projective, so, by **#1** from **Chapter IV, Section 3** it follows that every D-module is injective.

If D has a maximal left ideal I, then I can be viewed as a D-module. Since I is free then I is injective and it follows that I is a direct summand of D. So,

$$D \simeq I \oplus I',$$

where I' is a D-submodule of D. Since D is a free module over itself then I' is a free D-module and $X \cup Y$ is the basis of D, where X is a basis for I and Y is a basis for I'. If $|X \cup Y|$ is infinite then we get a contradiction with Theorem 2.6 from **Chapter IV, Section 2**. If $|X \cup Y|$ is finite then it follows that D has a basis of any finite cardinality over itself and we can assume that $|Y| > 1$. But then I' contains a proper submodule I'', so, $I \oplus I''$ is a proper submodule of D containing I - contradiction.