MATH 570: Higher Algebra I, Winter 2005

Solutions to Assignment 3

Chapter III, Section 5, #2: a) Define

$$\phi : (M_n(R))[x] \to M_n(R[x])$$

in the following way: for every

$$A = \sum_{k=0}^{p} (a_{i,j})^{(k)} x^k \in (M_n(R))[x],$$

where $(a_{i,j})^{(k)} \in M_n(R),\ k \in [0,p]$ put

$$\phi(A) = (c_{i,j}) \in M_n(R[x]),$$

where $c_{i,j} = \sum_{k=0}^{p} a_{i,j}^{(k)} x^k$ and $a_{i,j}^{(k)}$ is an (i,j)-entry of $(a_{i,j})^{(k)}$.

ϕ is obviously a bijection. Finally, one has to show that ϕ is a homomorphism, that is, $\phi(A+B) = \phi(A) + \phi(B),\ \phi(AB) = \phi(A)\phi(B)$ but this is a usual technical procedure.

b) $\phi : (M_n(R))[[x]] \to M_n(R][x]])$ is defined as above, but in the sum instead of p one has to use ∞.

Chapter III, Section 5, #8.b): Factorization in $\mathbb{Z}[x]$ is obvious

$$x^2 + 3x + 2 = (x+1)(x+2),$$

where both $x+1$ and $x+2$ are irreducible and non-invertible. At the same time, by Proposition 5.9 (p.155), $x^2 + 3x + 2$ is irreducible in $\mathbb{Z}[[x]]$ since 2 is irreducible in $\mathbb{Z}$ (no contradiction with factorization in $\mathbb{Z}[x]$ because both $x+1$ and $x+2$ are units in $\mathbb{Z}[[x]]$: indeed, $\frac{1}{x+1}$ and $\frac{1}{x+1}$ can be represented as infinite (geometric) series).

Chapter III, Section 5, #9: It is easy to see that the map ϕ_0 defined as

$$\phi_0 : \begin{array}{l} x \to 0, \\ f \to f, \quad \forall\ f \in F \end{array}$$

induces the unique homomorphism $\phi : F[x] \to F$ which is surjective. Observe that for every $h(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$ we have $\phi(h(x)) = a_0$, that is, $ker(\phi)$ consists of all polynomials with the constant coefficient equal to zero or in other words all polynomials which have x as a factor. Hence, $ker(\phi) = (x)$ and $F[x]/(x) \simeq F$ is a field, so, (x) is maximal. Observe also that $x-1 \notin ker(\phi) = (x)$. On the other hand, considering the homomorphism $\psi : F[x] \to F$ induced by

$$\psi_0 : \begin{array}{l} x \to 1, \\ f \to f, \quad \forall\ f \in F \end{array}$$

one can show that $ker(\psi) = (x-1)$ is a maximal ideal of $F[x]$. But $x-1 \in (x-1)$, so $(x) \neq (x-1)$ and (x) is not the only maximal ideal of $F[x]$.

Chapter IV, Section 1, #2.a): "$\Rightarrow$" If for every homomorphisms $g, h : D \to A$ we have $(fg)(a) = (fh)(a)$ for every $d \in D$ then $f(g(d)) = f(h(d))$ and $f(g(d)) - f(h(d)) = f(g(d) - h(d)) = 0$, where $d \in D$ is any element. Since $f : A \to A$ is a monomorphism it follows that $g(d) - h(d) = 0$, that is, g and h agree on every element of D.

"$\Leftarrow$" Following the hint in the textbook define $D = ker(f)$, which is obviously a submodule of A, and two homomorphisms $g, h : D \to A$ such that $g(D) = D \subseteq A$ is an inclusion and $h(D) = 0$ is a zero map. Observe that $(fg)(d) = f(g(d)) = f(d) = 0$ and $(fh)(d) = f(h(d)) = f(0) = 0$ for every $d \in D$. Hence, by our assumprion $g = h$ and it follows that $D = 0$.

Chapter IV, Section 1, #5: a) We assume that $A \neq 0$. Thus, there exists $a \in A$ and Ra is a submodule of A. Hence, the projection $A \to Ra$ is onto and non-zero. Since A is simple it follows that $ker(A) = 0$ and $A \simeq Ra$.

b) Suppose $A \neq 0$ and $\phi : A \to A$ is an endomorphism. Since $ker(\phi)$ is a submodule of A it follows that either $ker(\phi) = 0$ or $ker(\phi) = A$. In the latter case ϕ is a zero map and we are done. In the former case ϕ is a monomorphism and taking into account the fact that $im(\phi)$ is a submodule of A it follows that $im(\phi) = A$ and ϕ is an isomorphism.

Chapter IV, Section 1, #7.b): Let $S = Hom_R(A, A)$. Define $0_S(a) = 0,\ 1_S(a) = a,\ \forall a \in A$ and for any $f \in S$ set $(-f)(a) = -f(a),\ \forall a \in A$. Direct verification of the ring axioms.

Chapter IV, Section 1, #9: At first observe that $ker(f) \cap im(f) = 0$. Indeed, if $0 \neq b \in ker(f) \cap im(f)$ then $f(b) = 0$ and there exists $c \in A$ such that $b = f(c)$. Hence, $0 = f(b) = f(f(c)) = f(c) = b$ - contradiction. Finally, any $a \in A$ can be represented as $a = f(a) + (a - f(a))$, where $f(a) \in im(f)$ and $a - f(a) \in ker(f)$ because $f(a - f(a)) = f(a) - f(f(a)) = f(a) - f(a) = 0$. Thus, $A \simeq ker(f) \oplus im(f)$.

Chapter IV, Section 1, #11: a),b) Direct computations.

c) As a counterexample take $A = R = \mathbb{Z}_6$. Then, for $2, 3 \in \mathbb{Z}_6$ we have $\mathcal{O}_2, \mathcal{O}_3 \neq 0$ $(3 \in \mathcal{O}_2, 2 \in \mathcal{O}_3)$, while $\mathcal{O}_{3-2} = \mathcal{O}_1 = 0$ because 1 is invertible in $\mathbb{Z}_6$. That is, $T(\mathbb{Z}_6)$ is not closed under addition, so, can not be a submodule of $\mathbb{Z}_6$.

d) If $a \in T(a)$ then there exists $r \in R$ such that $ra = 0$, so, $0 = f(ra) = rf(a)$ and $f(a) \in T(B)$.

e) Since $ker(f) = 0$ it follows that $ker(f_T) = 0$, that is, $T(A) \to T(B)$ is an embedding. We show that $im(f_T) = ker(g_T)$. If $b \in ker(g_T)$ then $b \in ker(g) \cap T(B) =$

$im(f) \cap T(B)$. Hence, there exists $a \in A$ such that $f(a) = b$ and it is enough to show that $a \in T(A)$. But since $b \in T(B)$ then there exists $r \in R$ such that $rb = rf(a) = f(ra) = 0$, hence, $ra = 0$ because $ker(f) = 0$, and $ker(g_T) \subseteq im(f_T)$. Now, $ker(g_T) = ker(g) \cap T(B) = im(f) \cap T(B)$. On the other hand, $im(f_T) \subseteq im(f)$ and also $im(f_T) \subseteq T(B)$ by **d)**. That is, $im(f_T) \subseteq im(f) \cap T(B) = ker(g_T)$.

f) Consider $\mathbb{Z}$ and $\mathbb{Z}_2$. Then both can be viewed as $\mathbb{Z}$-modules, moreover, $T(\mathbb{Z}) = 0$, $T(\mathbb{Z}_2) = \mathbb{Z}_2$. Hence, $\mathbb{Z} \to \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is an epimorphism but $T(\mathbb{Z}) \to T(\mathbb{Z}_2)$ is not.