Solutions to midterm

1) $gcd(2613, 2171) = 13$ (use, for example, the Euclidean algorithm).

2) $3^{2002} = 4(mod\ 5)$. Indeed, notice that $3^2 = 4(mod\ 5)$, $3^3 = 2(mod\ 5)$, $3^4 = 1(mod\ 5)$, so now the remainders repeat: $3^5 = 3(mod\ 5)$, $3^6 = 4(mod\ 5)$, $\ldots$. Since $2002 = 4 \times 500 + 2$ it follows that $3^{2002} = (3^4)^{500} \times 3^2 = 1^{500} \times 3^2(mod\ 5) = 4$.

3) Let $a = 5k + u, b = 5l + v, c = 5m + w$, where $k, l, m \in \mathbb{Z}, u, v, w \in \{0, 1, 2, 3, 4\}$. Now,

$$a^2 + b^2 + c^2 = u^2 + v^2 + w^2(mod\ 5)$$

One can check directly (only finitely many checks!) that

$$u^2 + v^2 + w^2 = 0(mod\ 5) \leftrightarrow u = 0\ or\ v =\ \ or\ w = 0$$

4) Observe, that $\sqrt[3]{7}$ is not rational. Indeed, suppose that $\sqrt[3]{7} = \frac{p}{q}$ and $gcd(p, q) = 1$. Then $7q^3 = p^3$, hence $7|p$ and $7^3|7q^3$. This implies that $7|q$ -contradiction. Now if $\sqrt[3]{7}$ is not rational then $3\sqrt[3]{7} - 1$ is not rational.

5) $(A \cup B) - (C \cup B) = A - (C \cap B)$ (straightforward verification)

6) We have $n^2 - n + 1 = (n + 1)(n - 2) + 3$, hence $(n + 1, n^2 - n + 1) = (n + 1, 3) = 1$ or 3.

7) Induction on n:
$n = 2$: $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is clear.
Suppose

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}\ >\ \sqrt{n}.$$

Then

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} =$$

$$= \frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}} > \sqrt{n+1}$$

since $\sqrt{n(n+1)} > n$.

8) We check the axioms on page 42 in Hungerford :
1) Clear.
2) $a \oplus (b \oplus c) = a \oplus (b + c - 1) = a + b + c - 2$.
$(a \oplus b) \oplus c = (a + b - 1) \oplus c = a + b + c - 2$.

3) $a \oplus b = b \oplus a$ is clear.
4) The zero element 0_R is 1, since $a \oplus 1 = 1 \oplus a = a$.
5) The equation $a \oplus x = 1$ has solution $2 - a$.
6) Clear.
7) $a \odot (b \odot c) = a(bc - (b+c) + 2) - (a + bc - (b+c) + 2) + 2 = abc - ab - ac - bc + a + b + c$.
$(a \odot b) \odot c = (ab - (a+b) + 2)c - (ab - (a+b) + 2 + c) + 2 = abc - ac - bc - ab + a + b + c$.
8) $a \odot (b \oplus c) = a \odot (b + c - 1) = ab + ac - 2a - b - c + 3$.
$a \odot b \oplus a \odot c = (ab - a - b + 2) \oplus (ac - a - c + 2) = ab + ac - 2a - b - c + 3$.
Similarly for the other.
9) Clear.
The identity element 1_R is 2. If $a \odot b = 0_R = 1$ then $ab - a - b + 1 = 0$, hence $a = 1$ or $b = 1$, that is, $a = 0_R$ or $b = 0_R$.

9) Since $gcd(24, 86) = 2$ and $2|16$ the equation $24x = 16(mod 86)$ has a solution (Theorem 2.11 from the textbook). Moreover, it has 2 distinct solutions $x = 58$ and $x = 15$.

10) a) $x^2 + 2x - 6$ is not surjective: there is no $x \geq 0$ such that $x^2 + 2x - 6 = -7$.
b) Consider $f(x) = 5x - 10$, for example.
c) Consider $f(x) = \tan(\frac{\pi}{2}x)$.