Solutions to 235 Assignment 6

1.a) Yes, G is a group. Multiplication is associative as it is in $\mathbb{Q}$. Closure is easy to show. Identity is 1. And the inverse of 2^x is simply 2^{-x}. In fact, G is a subgroup of the multiplicative group of the ring $\mathbb{Q}$.

b) G is a group. Show first that $*$ is a binary operation on G, i.e., show that if $a, b \in G$, then $a * b \in G$. Associativity is easy to show by direct computation. The multiplicative identity is 0, $a^{-1} = -\frac{a}{a+1}$.

c) G is a subgroup of $GL_2(\mathbb{R})$, the group of 2×2 invertible real matrices. Indeed, the determinant of a matrix from G is never zero (since $a^2 + b^2 \neq 0$ when either a or $b \neq 0$), now it suffices to show that G is closed under multiplication and inversion (straightforward computations).

2.a) The group of units U_n of the ring $\mathbb{Z}_n$ consists of elements of the type $[k]$ where $\gcd(k, n) = 1$. So $U_{10} = \{1, 3, 7, 9\}$, and their orders are $1, 4, 4, 2$ respectively.

b) and c) Straightforward computations.

3.a) The elements of $S_3 \times \mathbb{Z}_4$ of order 6 are of the form $(\sigma, 2)$, where σ is an element of order 3 in S_3, i.e. a 3-cycle, i.e., $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ or $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

b) Non-abelian groups of given orders:
12: D_6
16: D_8
30: $S_3 \times \mathbb{Z}_{10}$
48: $S_4 \times \mathbb{Z}_2$

4) Let $a, b \in G$. Then $(ab)^2 = abab$ and also $(ab)^2 = a^2b^2$. Thus

$$abab = aabb \Rightarrow aba = aab \Rightarrow ba = ab$$

(multiplication on the right by b^{-1} and on the left by a^{-1}).

5) Let $a, b \in G$ and $ab = ba$. Denote $m = |a|$, $n = |b|$, $k = |ab|$. We can assume that $m, n \geq 2$. Then

$$(ab)^{mn} = a^{mn}b^{mn} = (a^m)^n (b^n)^m = 1^n 1^m = 1$$

So k divides mn. Since $gcd(m,n) = 1$ there are integers u, v such that $mu+nv = 1$. It follows that $k = kmu + knv$ and

$$1 = (ab)^k = (ab)^{kmu+knv} = a^{knv}b^{kmu}$$

Taking m-s power of the equality above

$$1 = a^{knvm}b^{kmum} = (a^m)^{knv}b^{kmum} = b^{kmum}$$

Therefore n divides $kmum$, hence n divides k (observe that $gcd(m,n) = 1$ and $gcd(u,n) = 1$). Similarly, m divides k, so $k = mn$, as required.

6.a) The center of S_3 is $\{1\}$ (check that for any element $a \in S_3$ if $a \neq 1$ then there exists $b \in S_3$ such that $ab \neq ba$).

b) H is normal, because for all $h \in H$, $g \in S_3$, we have $g^{-1}hg \in H$ (straightforward verification. In fact, it suffices to check this only for the generator $h = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$).

c) Since $|S_3|/|H| = 2$, and cosets are disjoint and span the entire group, there are only two cosets: namely H and Hg, where g is any element not in H.

7) No, it is not. Indeed, suppose $g = x + y\sqrt{2}$ generates G for some $x, y \in \mathbb{Z}$. Then (using additive notations) there exists an integer m such that $mg = \sqrt{2}$, i.e.,

$$mx + my\sqrt{2} = \sqrt{2}.$$

So (since $\sqrt{2}$ is irrational) $mx = 0$ and $my = 1$. This implies that $x = 0$ and $y = \pm 1$. But then for any integer n $ng \neq 1$ -contradiction.

8.a) No, they are not isomorphic, since, for example, $\mathbb{Z}_4 \times \mathbb{Z}_2$ is abelian and D_4 is not.

b) Denote vertices of the triangle by $1, 2, 3$. Then each element d in D_3 gives a bijection, say σ_d, between the vertices, i.e., a permutation on symbols $1, 2, 3$. Check that the map $d \to \sigma_d$ is an isomorphism between D_3 and S_3.