Solutions to assignment 2

In the following, we write (a, b) for $gcd(a, b)$.

1) Let $d = (n, n+1)$. Then $n = dk, n+1 = dl$ for some integers k and l. We have $d(l-k) = 1$, so $d = 1$ since d is positive and $l - k$ is an integer.

2) We apply Th1.3/page 9 for the integers 6 and 15. The answer is 3.

3) If $a = 2k + 1$, then $4|(4k^2 + 4k) = a^2 - 1$.

4) In general, $(a, b)|(a + b, a - b)$. Let $d = (a + b, a - b)$. Since a is odd and b is even, d must be odd. Write $a + b = dk, a - b = dl$ for some integers k and l. Then $d(k + l) = 2a$, so $d|(2a)$. Since $(d, 2) = 1$ it follows (Theorem 1.5) that $d|a$. Similarly, $d(k - l) = 2b$ and $d|b$. Therefore $d|(a, b)$, so $d = (a, b)$.

5) Let $a = 3p + r$, $0 \leq r < 3$, and $b = 3q + s$, $0 \leq s < 3$. Then $a^2 + b^2 = 3(something) + r^2 + s^2$. Since $3|(a^2 + b^2)$ then $3|(r^2 + s^2)$, but this implies $r = s = 0$ (check all the cases for r, s).

6) Suppose that $\sqrt{30}$ is rational, say $\sqrt{30} = \frac{m}{n}$, with m, n integers. We may assume $(m, n) = 1$. Decompose m and n into primes: $m = p_1 ... p_r$, $n = q_1 ... q_s$. Since $(m, n) = 1$, we have $p_i \neq q_j$, for all $1 \leq i \leq r$, $1 \leq j \leq s$. Then $2 \times 3 \times 5 \times q_1^2 ... q_s^2 = p_1^2 ... p_r^2$, so $\{2, 3, 5\}$ is contained in $\{p_1, ..., p_r\}$. Simplifying we obtain that $\{2, 3, 5\}$ is contained in $\{q_1, ..., q_s\}$ too, which is a contradiction.

7) $2^{11} - 1 = 2047 = 23 \times 89$.

8) Since n is prime $n \neq 1$. $n = 2$ does not satisfy conditions of the problem ($n + 2 = 4$ is not prime). We see that $n = 3$ satisfies the problem. Let $n > 3$. Then n is of the form $3k + 1$ or $3k + 2$, for some natural number $k \neq 0$. If $n = 3k + 1$, then $n + 2 = 3k + 3$ is not prime. If $n = 3k + 2$, then $n + 4 = 3k + 6$ is not prime. Therefore $n = 3$ is the only solution.