MATH 235 01: Basic Algebra   Fall 2002

Assignment 6

1) Determine whether the set G is a group under operation $*$:

   a) $G = \{2^x \mid x \in \mathbb{Q}\}, a * b = ab$;

   b) $G = \{x \in \mathbb{R} \mid x \neq -1\}, a * b = ab + a + b$.

   c) $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R}, \textit{not both } 0 \right\}$ with matrix multiplication $*$.

2) Let U_n be the group of units of the ring $\mathbb{Z}_n$.

   a) List all elements of the group U_{10} and find their orders;

   b) List all elements of order 4 in D_4.

   c) Write out the operation tables for U_6, D_3 and the group $G = \{1, -1, i, -i\}$ of complex numbers with respect to multiplication.

3) a) List all elements of order 6 in $S_3 \times \mathbb{Z}_4$,

   b) Give examples of non-abelian groups of orders 12, 16, 30, and 48.

4) If $(ab)^2 = a^2b^2$ for all a, b in a group G, prove that G is abelian.

5) Let $a, b \in G$ and $ab = ba$. If the orders $|a|$ and $|b|$ are relatively prime, prove that ab has order $|a||b|$.

6) a) Find the center of S_3.

   b) Is the subgroup H generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ normal?

   c) Write down all distinct cosets of the subgroup H from b).

7) Is the additive group $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ cyclic?

8) a) Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_2$ and D_4 isomorphic?

   b) Show that D_3 is isomorphic to S_3.