MATH 235 01: Basic Algebra   Fall 2002

Assignment 5

1) Prove that $F = \mathbb{R}[x]/(x^2+2)$ is a field and find $(x+1)^{-1}$ in F (here and below $\mathbb{R}$ is the field of real numbers).

2) Let $F = \mathbb{Z}_2[x]/(x^3+x+1)$ find $(x^2+x+1)^{-1}$ in F.

3) Is $\mathbb{Z}_3[x]/(x^3+2x^2+x+1)$ a field?

4) For a positive number $a \in \mathbb{Q}$ define

$$\mathbb{Q}(\sqrt{a}) = \{r + s\sqrt{a} \mid r, s \in \mathbb{Q}\}$$

(here and below $\mathbb{Q}$ is the field of rational numbers).

   a) Verify that $\mathbb{Q}(\sqrt{3})$ is a subring of $\mathbb{R}$;
   b) Show that $\mathbb{Q}(\sqrt{3})$ is isomorphic to $\mathbb{Q}[x]/(x^2-3)$.

5) Is $\mathbb{Q}(\sqrt{2})$ isomorphic to $\mathbb{Q}(\sqrt{3})$?

6) a) Show that the set $I = \{(k, 0) \mid k \in \mathbb{Z}\}$ is an ideal in the ring $\mathbb{Z} \times \mathbb{Z}$ (see Theorem 3.1 from the textbook for definition of $\mathbb{Z} \times \mathbb{Z}$);
   b) Show that the set $J = \{(k, k) \mid k \in \mathbb{Z}\}$ is not an ideal in the ring $\mathbb{Z} \times \mathbb{Z}$;
   c) Is the set

$$J = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} \mid r \in \mathbb{R} \right\}$$

   an ideal in the ring $M_2(\mathbb{R})$ of 2×2 matrices over $\mathbb{R}$?

7) List the **distinct** principal ideals in $\mathbb{Z}_9$.

8) Let J be an ideal in a ring R. Prove that

$$I = \{r \in R \mid rt = 0 \textit{ for every } t \in J\}$$

is also an ideal in R.

9) Let $J = \{0, 5\}$ in $\mathbb{Z}_{10}$. Verify that J is an ideal and show that $\mathbb{Z}_{10}/J$ is isomorphic to $\mathbb{Z}_5$.