Solution to assignment 1

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Question #1

Show that $5|2^{4n-2}+1$ for every positive integer n.
proof:
base case: let $n=1$. Then, $2^{4*1-2}+1=5$ which is divisible by 5. OK!

inductive hypothesis: assume true for n. i.e. $5|2^{4n-2}+1$ which is the same as $2^{4n-2}+1=5*k$ where $k \in \mathbf{Z}$
Now, we show that $5|2^{4(n+1)-2}+1$

$$\begin{aligned} 2^{4(n+1)-2}+1 &= 2^{4n-2}2^4+1 \\ &= 16*2^{4n-2}+1 \\ &= 15*2^{4n-2}+2^{4n-2}+1 \\ &= 15*2^{4n-2}+5*k \quad \text{by inductive hypothesis} \end{aligned}$$

$$\Longrightarrow \quad 5|2^{4(n+1)-2}+1$$

Hence, $5|2^{4n-2}+1$ by mathematical induction
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Question #2

Show that $2^n > n^2$ for $n \geq 5$
proof:
base case: for $n=5$, $2^5=32$ and $5^2=25$
So, $2^5 > 5^2$.
inductive hypothesis: assume that $2^n > n^2$ is true

Now, we must show that $2^{n+1} > (n+1)^2$. We will need the following claim.
claim: $n^2 > 2n+1$ for $n \geq 5$. Again, we use induction!
base case: $5^2 = 25$ and $2*5+1$. So, $25 > 11$
inductive hypothesis1: assume that $n^2 > 2n+1$

$$\begin{aligned} (n+1)^2 &= n^2+2n+1 \\ &> 2n+1+2n+1 \quad \text{by inductive hypothesis1} \\ &> 2n+3 = 2(n+1)+1 \quad \text{clearly, for} n \geq 5 \end{aligned}$$

Thus,

$$\begin{array}{rcll} 2^{n+1} & = & 2 * 2^n & \\ & > & 2 * n^2 & \text{by inductive hypothesis} \\ & = & n^2 + n^2 & \\ & > & n^2 + 2n + 1 & \text{using claim} \\ & = & (n+1)^2 & \end{array}$$

Therefore, $2^n > n^2$ by induction.
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Question #3

Suppose A is a set with $|A| = n$. Show that A has precisely 2^n subsets.
proof:
Again, we use induction!
base case: 2^n is true for n = 0 since A is the empty set and hence, A has only one subset, the empty set.
inductive hypothesis: assume that if A has n elements then it has 2^n subsets.
Now, suppose we add another element to A, a_{n+1}. Then, choose any subset B of $A \cup \{a_{n+1}\}$. Then, either a_{n+1} is in B or not in B.
case 1: $a_{n+1} \in B$ then $B = B' \cup \{a_{n+1}\}$. Since $a_{n+1} \notin B'$, the induction hypothesis tells us that there are 2^n possible such B'.
case 2: $a_{n+1} \notin B$ then B is a subset of A. Again, by the induction hypothesis, there are 2^n such subset.
Thus, combining case 1 and 2, we have a total of $2^n + 2^n = 2^{n+1}$ subsets of $A \cup \{a_{n+1}\}$.

Therefore, A has precisely 2^n subsets if it has n elements.
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Question #4

Show that $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = -2x + 3$ is a bijection.
proof:
injection:

$$\begin{array}{cl} if & f(a) = f(b) \\ then & -2a + 3 = -2b + 3 \\ \Longrightarrow & a = b \end{array}$$

surjection: $\forall c \in \mathbf{R}$ we must find $b \in \mathbf{R}$ such that $f(b) = c$.i.e we must solve the following equation for c:

$$c = -2b + 3 \Longrightarrow b = \frac{3 - c}{2}$$

Therefore, f is a bijection
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Question #5

Find a bijection $f : \mathbf{N} \to \mathbf{Z}$.
Consider,

$$f(x) = \begin{cases} \frac{x}{2} & \text{if x is even} \\ \frac{-(x+1)}{2} & \text{if x is odd} \\ 0 & \text{if x = 0} \end{cases}$$

claim: f is a bijection
proof:
injection: if $f(a) = f(b)$
case1: a and b are even $\Longrightarrow \frac{a}{2} = \frac{b}{2} \Longrightarrow a = b$
case2: a and b are odd $\Longrightarrow \frac{-a-1}{2} = \frac{-b-1}{2} \Longrightarrow a = b$
else the 0 case is trivial.

surjection: Must show that $\forall c \in \mathbf{Z} \quad \exists \quad b \in \mathbf{N}$ such that $f(b) = c$
case1: If c is ≥ 0 then $b = 2c$.
case2: if c < 0 then $b = -2c - 1$.

Therefore, f is a bijection.
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Question #6

Prove that $U - (A \cup B) = (U - A) \cap (U - B)$
proof:
$x \in U - (A \cup B)$
$\Longleftrightarrow x \in U, x \notin (A \cup B)$
$\Longleftrightarrow x \in U, x \notin A$ and $x \notin B$
$\Longleftrightarrow x \in U, x \notin A$ and $x \in U, x \notin B$
$\Longleftrightarrow x \in (U - A) \cap (U - B)$

Therefore, $U - (A \cup B) = (U - A) \cap (U - B)$
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