ISOMORPHISM PROBLEM FOR FINITELY GENERATED FULLY RESIDUALLY FREE GROUPS

INNA BUMAGIN, OLGA KHARLAMPOVICH, AND ALEXEI MIASNIKOV

Abstract. We prove that the isomorphism problem for finitely generated fully residually free groups (or $\mathcal{F}$-groups for short) is decidable. We also show that each $\mathcal{F}$-group G has a decomposition that is invariant under automorphisms of G, and obtain a structure theorem for the group of outer automorphisms $Out(G)$.

Contents

1. Introduction

The isomorphism problem - find an algorithm that for any two finite presentations determines, whether or not the groups defined by these presentations are isomorphic - is the hardest of the three algorithmic problems in

1991 *Mathematics Subject Classification.* 20E36,20F65,20F67,20E08,20E06,20F05,20F34.
Key words and phrases. isomorphism problem, fully residually free group, equations over groups, decomposition of a group, splitting, graph of groups.
The first and the second author were supported by NSERC Grant.
The third author was supported by NSERC Grant and by NSF GrantDMS-9970618.

group theory formulated by Max Dehn at the beginning of the 20th century. It is easy to see that solvability of the isomorphism problem in the class of finitely presented groups implies solvability of the word problem (find an algorithm to determine, whether or not a given product of generators of a group represents the trivial element of the group). The isomorphism problem is unsolvable in the entire class of finitely presented groups, because there exist finitely presented groups with unsolvable word problem; this latter assertion is the fundamental result of Novikov and Boone. One can still try to solve the isomorphism problem restricted to a certain class $\mathcal{C}$ of finitely presented groups: *find an algorithm that for any two finite presentations of groups from the class $\mathcal{C}$ determines, whether or not the groups defined by these presentations are isomorphic.* There are only few classes of groups for which the isomorphism problem is known to be solvable. This is a classical result that the isomorphism problem is solvable for finitely generated Abelian groups. Solvability of the isomorphism problem for finitely generated free groups has been known since 1950ties due to the work of Nielsen. Among the most significant results in this area is Segal's solution to the isomorphism problem for polycyclic-by-finite groups [30]. One should also mention the positive solution to the isomorphism problem for finitely generated nilpotent groups, which is an earlier result obtained by Segal and Grunewald [31]. Another profound result was obtained by Sela [32] who proved that the isomorphism problem is solvable for torsion-free word hyperbolic groups which do not split over a cyclic subgroup. One of the most important ingredients of Sela's solution to the isomorphism problem is the decidability of equations over free groups proved by Makanin [22] and Razborov [27], and extended by Rips and Sela [28] to torsion-free word hyperbolic groups.

We consider the class of *finitely generated fully residually free groups* ($\mathcal{F}$-groups for short) defined as follows.

Definition 1.1. [2] A group G is called *fully residually free* if for any finite number of non-trivial elements $g_1, \dots, g_n$ in G there exists a homomorphism $G \to F$ from G to a free group F that maps $g_1, \dots, g_n$ to non-trivial elements of F.

The first examples of non-free fully residually free groups are due to Lyndon [19], where he introduced free Lyndon's $\mathbb{Z}[t]$-groups and proved that they are fully residually free. In the same year 1960, in a very influential paper [20] he used these groups to describe completely the solution sets of one-variable equations over free groups.

A finitely generated fully residually free group G is word hyperbolic, if any maximal Abelian subgroup of G is cyclic [13]. However, in this latter case G has one of the following decompositions: a non-trivial free decomposition, or a non-trivial JSJ decomposition, or G is the fundamental group of a closed surface and has a non-trivial cyclic splitting. Therefore, the case of a word hyperbolic fully residually free group is not covered by Sela's solution to the isomorphism problem. Our main result is the following theorem.

Theorem 4.13. *Let $G \cong \langle \mathcal{S}_G \mid \mathcal{R}_G \rangle$ and $H \cong \langle \mathcal{S}_H \mid \mathcal{R}_H \rangle$ be finite presentations of fully residually free groups. There exists an algorithm that determines whether or not G and H are isomorphic. If the groups are isomorphic, then the algorithm finds an isomorphism $G \to H$.*

The most important ingredients of our proof are computability of a JSJ decomposition of an $\mathcal{F}$-group, and solvability and the structure of the solution sets of equations over $\mathcal{F}$-groups, obtained by the second and the third authors [14], [15] (see also Theorem 3.12 and Section 4 in the present paper). To deduce solvability of the isomorphism problem, we prove that a one-ended $\mathcal{F}$-group G has a canonical Abelian JSJ decomposition that is invariant under automorphisms of G. Moreover, using results obtained in [15], we deduce that the canonical decomposition can be constructed effectively. More precisely, in Theorem 3.13 we define an Abelian JSJ decomposition $\Gamma(V, E)$ of G that has the following property.

Theorem 1.2. *Let G be a one-ended $\mathcal{F}$-group, and let $\Gamma(V, E)$ be an Abelian JSJ decomposition of G that satisfies the conditions of Theorem 3.13. If a graph of groups $\Delta(U, P)$ is another Abelian JSJ decomposition of G that satisfies the conditions of Theorem 3.13 also, then Δ can be obtained from Γ by conjugation and modifying boundary monomorphisms.*

Theorem 1.2 follows from Theorem 3.17. Hyperbolic groups have canonical JSJ decompositions over virtually cyclic subgroups as was shown by Bowditch [4], this result was first proved by Sela [32] for torsion-free hyperbolic groups. Another class of groups that possess canonical JSJ decompositions was introduced by Forester [9] (Guirardel [12] gave an alternate proof of this latter result). Not all finitely presented groups have canonical JSJ decompositions, as shown by Forester [10]. Using Theorem 1.2, we obtain the following structure theorem for $Out(G)$ (cf. Theorem 5.3).

Theorem 1.3. *Let G be a one-ended $\mathcal{F}$-group. $Out(G)$ is virtually a direct product of a finitely generated free Abelian group, subgroups of $GL_n(\mathbb{Z})$, and the quotient of a direct product of mapping class groups of surfaces with boundary by a central subgroup isomorphic to a finitely generated free Abelian group.*

Recall that similar results for torsion-free hyperbolic groups were obtained by Sela [33] and for a more general class of groups by Levitt [18, Theorem 1.2].

The first author wishes to thank Ilya Rips, Zlil Sela and Daniel Wise for numerous useful conversations preceding the work on the present paper.

2. Graphs of groups and splittings

Definition 2.1. A *directed graph* $(V, E, \mathcal{O})$ consists of a set of vertices V, a set of edges E and an orientation $\mathcal{O}$ determined by two functions $i\colon E \to V$

and $\tau\colon E \to V$. For an edge $e \in E$ the vertex $i(e)$ is the *initial vertex* of e, and $\tau(e)$ is the *terminal vertex* of e. We call $i(e)$ and $\tau(e)$ the *endpoints of* e.

Definition 2.2. A graph of groups $\Gamma(V, E, \mathcal{O})$ is a directed graph $(V, E, \mathcal{O})$ where to each vertex $v \in V$ (or to each edge $e \in E$) we assign a group called G_v (or G_e) so that for each edge $e \in E$ there are monomorphisms

$$\alpha\colon G_e \to G_{i(e)} \quad \text{and} \quad \omega\colon G_e \to G_{\tau(e)}$$

called the *boundary monomorphisms from the edge group* G_e to the *vertex groups* $G_{i(e)}$ and $G_{\tau(e)}$. We refer to G_v and G_e the *stabilizer* of v and e, respectively.

Definition 2.3. By a *splitting of* G we mean a *triple* $\Sigma = (\Gamma(V, E, \mathcal{O}), T, \varphi)$ where $\Gamma(V, E, \mathcal{O})$ is a graph of groups, T is a *maximal subtree of the graph* (V, E) and $\varphi\colon \pi_1(\Gamma(V, E, \mathcal{O}); T) \to G$ is an *isomorphism.*

We recall that the fundamental group of a graph of groups $\pi_1(\Gamma(V, E, \mathcal{O}); T)$ with respect to a maximal subtree T is given by

$$\langle G_v(v \in V), t_e(e \in E_0) \mid \forall e \in E_0(t_e t_{\bar{e}} = 1, \alpha(g)t_e = t_e\omega(g), \forall g \in G_e), \\ \forall e \in T(\alpha(g) = \omega(g), \forall g \in G_e)\rangle$$

where $E_0 = \{e \in E \mid e \notin T\}$ denotes the set of edges that do not belong to the maximal tree.

Let G be a group and let $\mathcal{G}$ be a set of splittings of G into a graph of groups. One introduces an equivalence relation on $\mathcal{G}$ generated by the following operations (we refer the reader to [29] and to [15, Section 2.4] for more details):

(1) *Conjugation* is a usual conjugation;
(2) *Modifying boundary monomorphisms by conjugation* is defined as follows. Let $G = \langle A, t \mid t\alpha(c)t^{-1} = \omega(c)\forall c \in C\rangle$. For an arbitrary element $a \in A$ one defines $\alpha'\colon C \to A$ by $\alpha'(c) = a^{-1}\alpha(c)a$, and replaces α by α'. One replaces also the isomorphism φ by the isomorphism φ_a defined by $\varphi_a(t) = \varphi(ta)$ and $\varphi_a(g) = \varphi(g)$ for all $g \neq t$. If $G = A *_C B$, then one replaces the monomorphism $\alpha\colon C \to A$ by $\alpha'\colon C \to A$ defined as above and φ by the isomorphism φ_a defined by $\varphi_a(g) = \varphi(g)$ for $g \in A$ and $\varphi_a(g) = \varphi(a^{-1}ga)$ for all $g \in B$. For a general graph of groups, let e be the edge stabilized by C; one collapses all edges but e and defines α' and φ_a as above, with the only restriction that $a \in G_{i(e)}$.
(3) *Sliding* corresponds to the relation
$$(A *_{C_1} B) *_{C_2} D \cong (A *_{C_1} D) *_{C_2} B$$
in the case when $C_1 \subseteq C_2$.
(4) By a *refinement of* $\Delta \in \mathcal{G}$ *at a vertex* $v \in \Delta$ we mean replacing v by a non-degenerate graph of groups $\gamma(V_\gamma, E_\gamma)$ which is compatible with Δ and has the fundamental group G_v (where G_v is the stabilizer of

v in Δ). A vertex v is *flexible* if there exists a refinement of Δ at v; otherwise, v is *rigid*.

In what follows, by a splitting of G we mean a graph of groups $\Gamma(V, E)$; when there is no ambiguity, we identify the groups assigned to edges G_e with their images $\alpha(G_e) \subseteq G_{i(e)}$ and the groups assigned to vertices with their images in G under the isomorphism φ. Usually, we do not specify a maximal tree and an orientation in the graph (V, E). Observe that conjugation corresponds to an inner automorphism of G, whereas operations (2)- (4) change the presentation of G as a graph of groups and usually do not lead to an automorphism of G. However, there is an exception. If we have an operation of type (2) so that a is in the centralizer $C_A(\alpha(C))$ of $\alpha(C)$ in A, then $\alpha'(C) = \alpha(C)$ which means that we actually do not modify the graph of groups. Then, in the above notation, the composition of the isomorphisms $\varphi_a \circ \varphi^{-1}$ is well-defined and results in an automorphism of G called a *generalized Dehn twist*. More precisely, we have the following definition.

Definition 2.4. Let $\Gamma(V, E)$ be an Abelian splitting G of a group G, and let $e \in E$ be an edge with the endpoints $i(e) = v$ and $\tau(e) = u$ and the stabilizer G_e. By a *generalized Dehn twist along the edge* e we mean an automorphism $\beta_a \colon G \to G$ with $a \in C_{G_v}(G_e)$, defined as follows.

If e is a separating edge, let Δ_v (or Δ_u) denote the connected component of $(V, E) \setminus \{e\}$ that contains v (or u). Then $\beta_a(g) = g$ for $g \in G_w$ with $w \in \Delta_v$ and $\beta_a(g) = aga^{-1}$ for $g \in G_w$ with $w \in \Delta_u$.

If e is a non-separating edge, then one can choose a maximal tree T in (V, E) so that $e \notin T$. Let t be the stable letter that corresponds to e. We set $\beta_a(t) = at$ and $\beta_a(g) = g$ for all $g \neq t$.

In particular, if the edge group C is cyclic and $\alpha(C) = C_A(\alpha(C))$, then our definition coincides with the definition of a *Dehn twist* (see [29]).

Definition 2.5. A splitting is *elementary* if the graph $\Gamma(V, E)$ is either an edge of groups or a loop of groups so that either $G \cong A *_C B$, or $G \cong A*_C$. A splitting is called *Abelian* if the edge groups are all Abelian.

2.1. **G-tree.** By a *tree* we mean a simplicial tree i.e., a graph with no circuits. One assigns unit length to each edge of a tree, to make a tree into a geodesic metric space.

Definition 2.6. A tree equipped with an action of a group G is called a *G-tree*. An action $G \times X \to X$ is *Abelian*, if edge stabilizers in X are all Abelian subgroups of G. A G-tree X is *minimal* if it contains no G-invariant proper subtrees. Two vertices (or edges) x_1 and x_2 in X are *G-equivalent*, if they belong to a G-orbit.

By the *fixed set of* $g \in G$ we mean $Fix(g) = \{x \in X \mid g.x = x\}$. A G-tree is *k-acylindrical*, if $diam(Fix(g)) \leq k$ for all $g \in G$.

Convention 2.7. In what follows, we consider Abelian splittings and Abelian actions, only.

The central result of the Bass-Serre theory [35],[1] tells that to each splitting $\Sigma = (\Gamma(V,E), T, \varphi)$ of a group G one can associate a minimal G-tree, which is the covering space of the graph of groups $\Gamma(V,E)$, and vice versa, G inherits a splitting from its action on a minimal G-tree with no inversions.

2.2. **Extended fundamental domain and natural lift.**

Definition 2.8. An *extended fundamental domain* D is a finite subtree of X so that the G-orbit of D is the whole tree X, and different edges of D belong to different G-orbits.

Lemma 2.9. *Vertices v and $u \neq v$ of an extended domain D are G-equivalent if and only if either $v = t.u$ or $u = t.v$, where t is a stable letter in the presentation of G determined by Δ.*

Proof of this lemma is straightforward and we omit it.

Definition 2.10. A graph of groups Δ is *reduced*, if for each vertex v of valency one or two, G_v properly contains the groups of adjacent edges. We say that Δ is *semi-reduced*, if for each edge $e \in E$ with the endpoints v and u, the equality $G_e = G_v$ implies that $v \neq u$, $val(v) \geq 2$ and $G_e \subsetneq G_u$. We say that a G-tree X is *(semi-)reduced*, if the corresponding graph of groups $\Delta = G\backslash X$ is (semi-)reduced.

Definition 2.11. Let X be 2-acylindrical and semi-reduced. A *natural lift λ of Δ to X* is defined as follows. The image of a vertex $v \in \Delta$ with the stabilizer G_v is the vertex $\lambda(v) \in X$ with $Stab(\lambda(v)) = G_v$. Let e be an edge with the endpoints $i(e) = v$ and $\tau(e) = u$. If $e \in T$, then $\lambda(e)$ is the edge of X joining $\lambda(v)$ and $\lambda(u)$, and if $e \notin T$, then $\lambda(e)$ is the edge of X joining $\lambda(v)$ and $t_e.\lambda(u)$ where t_e is the stable letter corresponding to e.

Lemma 2.12. (1) *The natural lift of Δ to X is well-defined.*
(2) *The natural lift of Δ to X is an extended domain.*

Proof. Assume that there are two vertices x_1 and x_2 in X with $Stab(x_1) = Stab(x_2) = G_v$. The path p joining x_1 and x_2 in X is stabilized by G_v. Since X is 2-acylindrical, the length of p is either 1 or 2. If p is an edge, we get a contradiction as X is semi-reduced. Let the length of p equal 2, and let $v = \pi(x_1)$ and $u = \pi(x_2)$ be the natural projections of x_1 and x_2 to Δ. Assume that $val(v) > 1$. The stabilizer of an edge $f \notin \pi(p)$ incident on v is a non-trivial subgroup B of G_v. The edge $f \in \Delta$ lifts to an edge $q_f \in X$ so that $q_f \notin p$ with $Stab(q_f) = B$, so that the subgroup B fixes 3 edges in X, a contradiction. Therefore, $val(v) = val(u) = 1$ while X is semi-reduced, a contradiction. Thus, the image of each vertex in Δ under a natural lift is defined uniquely. Since the images of edges are determined uniquely by the images of their endpoints, the assertion (1) follows. Furthermore, the definition of the Bass-Serre tree X as a covering space of Δ implies the assertion (2). Indeed, the G-orbit of the natural lift of Δ is the whole X. Moreover, the edges of Δ are representatives of different G-orbits of edges in X, hence their lifts to X are not G-equivalent. □

2.3. Morphisms of graphs.

Definition 2.13. Let (V,E) and (U,B) be two graphs. A map $\chi\colon (V,E)\to (U,B)$ is *simplicial*, if χ maps each vertex $v\in V$ to a vertex $u\in U$ and each edge $e\in E$ to a (possibly, empty) path in (U,B) so that the incidence relations are preserved. A simplicial map $\chi\colon (V,E)\to (U,B)$ is an *isomorphism of graphs* if χ maps each edge $e\in E$ to an edge $b\in B$ and is bijective on both the set of vertices and the set of edges.

Remark 2.14. It follows immediately from the definition that for finite graphs (V,E) and (U,B) one can find effectively the (possibly, empty) set of all isomorphisms $\chi\colon (V,E)\to (U,B)$.

Definition 2.15. Let $\psi\colon G\to H$ be an isomorphism of groups, and let $\mathcal{G}$ (or $\mathcal{H}$) be the set of all splittings of G (or H) into a graph of groups. With the isomorphism ψ we associate a map $\psi_*\colon \mathcal{G}\to\mathcal{H}$, where the *image* $\psi_*(\Gamma)$ *of* $\Gamma(V,E)\in\mathcal{G}$ is the graph of groups $\Delta(U,B)\in\mathcal{H}$ defined as follows:

(1) The underlying graphs (V,E) and (U,B) are isomorphic, and we identify each vertex and each edge of (V,E) with its image in (U,B) under an isomorphism.
(2) The group assigned to a vertex or to an edge in Δ is the ψ-image of the group assigned to that vertex or edge in Γ.
(3) Let e be an edge with the endpoints $v=i(e)$ and $u=\tau(e)$. The boundary monomorphisms $\alpha_\psi\colon G_e\to G_v$ and $\omega_\psi\colon G_e\to G_u$ in Δ are defined by $\alpha_\psi(\psi(b))=\psi(\alpha(b))$ and $\omega_\psi(\psi(b))=\psi(\omega(b))$ for all $b\in G_e$.

2.4. Universal decomposition of a group.

Definition 2.16. [6] By a *universal decomposition* of G we mean a decomposition of G into a graph of groups $\Gamma=\Gamma(V,E)$ that has the following property. Given a minimal G-tree T, one can find refinements at flexible vertices of Γ and obtain a decomposition Γ_r of G so that there exists a G-equivariant simplicial map from the Bass-Serre tree $\tilde{\Gamma}_r$ onto T.

Example 2.17. Obviously, every group G has a *trivial universal decomposition* that consists of a unique flexible vertex stabilized by G. It can be readily seen that if G is a free (Abelian or non-Abelian) group or a closed surface group, then in fact, the only universal decomposition of G is the trivial decomposition. More precisely, G is indecomposable in the sense of Definition 2.19 below.

In what follows, we will be interested in an Abelian universal decomposition of a group G with maximal number of vertices. For instance, the Grushko free decomposition is a maximal universal decomposition in the class of all free decompositions of G. For a freely indecomposable group, a JSJ decomposition has the universal property (see Section 3 for more details).

Definition 2.18. We say that a graph of groups Δ is *non-degenerate*, if Δ is semi-reduced and the set of edges of Δ is not empty.

Definition 2.19. A group G is *decomposable* if G has a non-degenerate universal decomposition. Otherwise, G is *indecomposable*. In particular, if G is an indecomposable group which is not a free non-Abelian group, then G is freely indecomposable.

3. Properties of fully residually free groups

As before, we denote by $\mathcal{F}$ the *class of finitely generated fully residually free groups* (also called *limit groups* by Sela [34]), and say that G is an $\mathcal{F}$-group if G belongs to the class $\mathcal{F}$. In Theorem 3.1 below we mention only those properties of $\mathcal{F}$-groups which we use in our proof.

Theorem 3.1. *Let G be an $\mathcal{F}$-group. Then G possesses the following properties.*

1. *G is torsion-free;*
2. *Each subgroup of G is an $\mathcal{F}$-group;*
3. *G has the CSA property. Namely, each maximal Abelian subgroup of G is malnormal, so that if M is a maximal Abelian subgroup of G then $M \cap gMg^{-1} \neq \{1\}$ for $g \in G$ implies that $g \in M$;*
4. *Each Abelian subgroup of G is contained in a unique maximal finitely generated Abelian subgroup, in particular, each Abelian subgroup of G is finitely generated;*
5. *G is finitely presented, and has only finitely many conjugacy classes of its maximal Abelian subgroups.*
6. *G has solvable word problem, conjugacy problem and uniform membership problem.*
7. *G has the Howson property. Namely, if K_1 and K_2 are finitely generated subgroups of G, then the intersection $K_1 \cap K_2$ is finitely generated. Moreover, for given finitely generated subgroups K_1 and K_2 of G, there is an algorithm to find the intersection $K_1 \cap K_2$.*
8. *There is an algorithm to find the centralizer of a given element $g \in G$.*

Proof. Properties (1) and (2) follow immediately from the definition of an $\mathcal{F}$-group. A proof of property (3) can be found in [3]; property (4) is proven in [13]. Properties (4) and (5) are proved in [13]. Alternative proofs of properties (3), (4) and (5) can be found in [34]. Solvability of the word problem is shown in [23], an algorithm to solve conjugacy problem can be found in [25]. Recall that by a theorem proved by Dahmani [7], $\mathcal{F}$-groups are relatively hyperbolic which allows one to use alternative algorithms to solve word problem [8] and conjugacy problem [5]. Observe that results proved in [8] imply finite presentability of $\mathcal{F}$-groups, and a theorem proved in [26] implies solvability of the conjugacy problem. Solvability of the uniform membership problem and properties (7) and (8) are proved in [16]. $\square$

The following Lemma 3.2 asserts that we can consider only those Abelian splittings of an $\mathcal{F}$-group G where each maximal Abelian non-cyclic subgroup of G is elliptic. We denote by $\mathcal{D}(G)$ the set of all Abelian splittings of G that have this latter property.

Lemma 3.2. *Let G be an $\mathcal{F}$-group, let M be a maximal Abelian non-cyclic subgroup of G, and let A be an Abelian subgroup of G. If $G = G_1 *_A G_2$, then M can be conjugated into either G_1 or G_2. If $G = G_1 *_A$, and the intersection $M \cap A^g$ is a proper subgroup of M for some $g \in G$, then M can be conjugated so that $G = G_1 *_A M$. If $G = G_1 *_A$ and for any $g \in G$, the intersection $M \cap A^g$ is either trivial or coincides with M, then M can be conjugated into G_1.*

Proof. The first statement follows from the description of commuting elements in a free product with amalgamation. Now, let G have the presentation as follows: $G = \langle G_v, t \mid tat^{-1} = \omega(a) \forall a \in A\rangle$.

If $M \cap gAg^{-1}$ is not trivial, then by Theorem 3.1(4), $g^{-1}Mg$ is the maximal Abelian subgroup containing A. Denote by M_t the maximal Abelian subgroup containing tAt^{-1}. Since the intersection $g^{-1}Mg \cap t^{-1}M_t t = A$ is not trivial, by Theorem 3.1(3), we conclude that $M_t = tg^{-1}Mgt^{-1}$. If $t \notin g^{-1}Mg$, then $A = g^{-1}Mg$, so that $g^{-1}Mg$ is elliptic, as claimed. In this case, $G = \langle G_v, t \mid tat^{-1} = \omega(a) \forall a \in M_1\rangle$ and $\omega(M_1) = M_2$ where both M_1 and M_2 are maximal Abelian subgroups of G_v.

If $t \in g^{-1}Mg$, then $g^{-1}Mg \subseteq C_G(t)$, where $C_G(t)$ is the centralizer of t in G. According to the presentation of G as an HNN-extension, $C_G(t) = \langle A, t\rangle \subseteq g^{-1}Mg$, hence $\langle A, t\rangle = g^{-1}Mg$, in particular A is a proper subgroup of $g^{-1}Mg$ and $G = G_1 *_A g^{-1}Mg$ (cf. also [11, Theorem 5]).

If M intersects no conjugate of A and M is hyperbolic when acting on the Bass-Serre tree corresponding to the splitting of G as the HNN-extension, then M inherits a non-trivial splitting as a free product, a contradiction. □

Definition 3.3. We say that an Abelian splitting $\mathcal{S} = (\mathcal{G}(V,E); T, \theta)$ of a group G is an *Abelian cycle of groups* if the following conditions hold:

(1) G can be obtained as a series of amalgamated products
$$\tilde{G} = (((G_1 *_{A_1} G_2) *_{A_2} G_3) * \dots) *_{A_{n-1}} G_n$$
and an HNN-extension $G = \langle \tilde{G}, t \mid A = t^{-1}\alpha(A_n)t\rangle$ with $A \subset G_1$ and $\alpha(A_n) \subset G_n$. In particular, the graph (V,E) is a cycle.

(2) The edge groups $A_1, \dots, A_n$ $(n \geq 1)$ are all subgroups of a maximal Abelian subgroup $M \subset G$.

We also call such a splitting $\mathcal{S}$ an *M-cycle of groups* to stress that all edge groups in Γ are subgroups of the group M. Thus, if G is an M-cycle, then G has the following presentation:
$$G = \langle G_1, \dots, G_n, t \mid \alpha(A_i) = \omega(A_i), i = 1, \dots, n-1, A = t^{-1}\alpha(A_n)t\rangle,$$
where $\alpha(A_i) \subseteq G_i \cap M$ (for $i = 1, \dots, n$), $\omega(A_i) \subseteq G_{i+1}$ (for $i = 1, \dots, n-1$) and $A \subset G_1$.

Definition 3.4. A graph of groups $\Upsilon(V,E)$ is a *star of groups*, if (V,E) is a tree T which has diameter 2. If Υ is a star of groups, then the fundamental group of Υ is as follows:

$$\pi(\Upsilon) = \langle M, K_1, \ldots, K_n \mid \alpha(A_i) = \omega(A_i), i = 1, \ldots, n\rangle,$$

meaning that $\alpha(A_i) \subseteq M$ and $\omega(A_i) \subseteq K_i$. The vertex $v_0 \in V$ with the stabilizer M is called the *center* and vertices v_i with stabilizers K_i are called *leaves of* $\Upsilon(V,E)$. If $\Upsilon(V,E)$ is an Abelian star of groups, then M is a maximal Abelian subgroup of G.

Definition 3.5. A graph of groups $\Psi(V, E\cup E_s)$ is a *constellation of groups*, if $\Psi(V, E\cup E_s)$ can be obtained by taking finitely many amalgamated products of stars of groups over leaves and HNN-extensions where both associated subgroups are stabilizers of the centers of those stars. In other words, $\Psi(V, E\cup E_s)$ can be obtained by iterations of the following construction:

$$\pi(\Psi) = \langle \pi(\Upsilon_1), \pi(\Upsilon_2), t \mid K_i^{(1)} = K_j^{(2)}, M^{(1)} = tM^{(2)}t^{-1}\rangle,$$

where $\pi(\Upsilon_l) = \langle M^{(l)}, K_1^{(l)}, \ldots, K_{n_l}^{(l)} \mid \alpha(A_i^{(l)}) = \omega(A_i^{(l)}), 1 \le i \le n_l\rangle$ for $l = 1, 2$ is a star of groups as in Definition 3.4. We call an edge e a *silver edge* if e corresponds to an HNN-extension where associated subgroups are maximal Abelian. E_s denotes the set of all silver edges in Ψ.

Remark 3.6. In what follows, we focus on Abelian stars of groups and constellations of groups, meaning that edge groups are all Abelian.

(1) Since maximal Abelian subgroups of G are malnormal by Theorem 3.1(3), two Abelian stars of groups are never amalgamated over two different pairs of leaves, and the silver subgraph of $(V, E\cup E_s)$ is a tree.

(2) We *do not* consider an amalgamated product of two stars of groups with no HNN-extension a constellation of groups. However, it is convenient to regard a star of groups as a particular case of a (trivial) constellation of groups. We also regard an edge of groups $M *_A G_v$ with $A \subseteq M$ and M a maximal Abelian subgroup of G as an Abelian star of groups.

Lemma 3.7. *If G is an $\mathcal{F}$-group and $\Delta(V,E)$ is a splitting of G which is an Abelian cycle of groups, then one can effectively modify Δ so as to obtain a splitting Ψ of G which is an Abelian constellation of groups.*

Proof. Contract all edges of Δ but one to a point. The new splitting of G that we obtain is an HNN-extension $G = G_v *_A$, hence G has the presentation as follows: $G = \langle G_v, t \mid tat^{-1} = \omega(a) \forall a \in A\rangle$. Let M be the maximal Abelian subgroup containing A.

First, assume that $A \subsetneqq M$. By Lemma 3.2, $G = G_v *_A M$. Furthermore, G_v is an $\mathcal{F}$-group that splits into a series of amalgamated products over Abelian subgroups. Observe that all these Abelian subgroups and also A are contained in a maximal Abelian subgroup $M_v \subset G_v$. Lemma 3.2 implies that

M_v can be conjugated to a vertex group in the splitting of G_v, in particular A is elliptic in this splitting. Therefore, the splitting of G_v extends to a splitting of the whole group G into a graph of groups that has a tree as the underlying graph, with a vertex stabilized by M. Since all edge groups in the graph are subgroups of M, by a sequence of slidings one obtains a star of groups in the sense of Definition 3.3, as follows. If there is a vertex $v \in V$ such that $M = G_v$, then define $u = v$, otherwise add a vertex u with $G_u = M$ and an edge f with $G_f = M$ so that $i(f) = u$ and $\tau(f) = v$ (this is a refinement of Δ at the vertex v). Having introduced the vertex u with the stabilizer $G_u = M$, we make the following finite sequence of slidings in Δ. Let $v_i \in V$ be a vertex adjacent to u (we set $v_1 = v$), denote by f_i the edge connecting them (clearly, $f_1 = f$), and assume that $val(v_i) > 1$ (for otherwise, we are done). Choose an edge $e \neq f_i$ in $Star(v_i)$ and slide this edge to u. W.l.o.g., we can assume that we had $i(e) = v_i$, so that having made the sliding we have $i(e) = u$. If $G_{\tau(e)} \subseteq M$, then collapse e, so that u and $\tau(e)$ get identified. None of these operations changes the fundamental group of Δ. We end up with a star of groups centered at u.

Now, let $A = M$. Since M is malnormal in G, $M^t \neq M$ for each $t \in G \backslash M$. Therefore, by the property (1) of an Abelian cycle (see Definition 3.3 for the notation), there is a unique edge $e \in E$ with $i(e) = v_n$ and $\tau(e) = v_1$, so that the boundary monomorphisms are as follows: $\alpha(G_e) = A_n = M$ and $\omega(G_e) = A = M^t$. To modify Δ, we add a vertex u stabilized by M and a vertex u_t stabilized by M^t, join u to v_n by an edge f_n with the edge group $G_n = M$ and join u_t to v_1 by an edge f_t with the edge group $G_t = M^t$. Next, we slide the edge e along the edges f_n and f_t so that $i(e) = u$ and $\tau(e) = u_t$; so e becomes a *silver* edge in the meaning of Definition 3.5. Clearly, none of the above operations changes the fundamental group of Δ. The graph spanned by the vertices $v_1, \dots, v_n, u$ is now a linear tree (with no branch points) with all edge groups being subgroups of M, hence one can transform this subgraph by a series of slidings to an M-star of groups. Observe that $M \subset G_1$, since G_1 contains M^t and intersects with M non-trivially. Therefore, each edge group in this star of groups equals M. The graph spanned by v_1 and u_t is an edge of groups which is a particular case of a star of groups with the center u_t and a unique leaf v_1. Thus, we have obtained a splitting Ψ of G which is an Abelian constellation of groups.

It remains to notice that an Abelian M-cycle Δ can be transformed to a constellation of groups (and not to a star of groups) if and only if each edge group in Δ equals M.

To show that Ψ can be found effectively, observe that we need to use the following algorithms. First, for a given Abelian subgroup A of G which is an edge group in a splitting of G, one should find effectively the maximal Abelian subgroup M containing A. Existence of this algorithm follows from Theorem 3.1 (8), as by Theorem 3.1 (4), M is the centralizer of any non-trivial element of A. The other problem which is to be solved effectively is to find the intersection of two given finitely generated subgroups of G. This algorithm is provided according to Theorem 3.1 (7). □

Corollary 3.8. *Let G be an $\mathcal{F}$-group, and let M be a maximal Abelian subgroup of G. If G does not split as an HNN-extension where M is one of the two associated subgroups, then each splitting of G contains at most one Abelian M-cycle.*

Proof. By the proof of Lemma 3.7, $G = G_v *_M$ if and only if G has a splitting with an Abelian M-cycle where $A_1 = \cdots = A_n = M$. Assume there are two Abelian cycles in a splitting of G. One can find in each cycle an edge (denoted by e_1 and e_2) that does not belong to the other cycle, so that the edge group of both e_1 and e_2 are proper subgroups of M. Choose a maximal tree T in the underlying graph so that $e_1, e_2 \notin T$. Let t_1 and t_2 be stable letters corresponding to e_1 and e_2. Since each edge group in both cycles is a subgroup of M, according to the proof of Lemma 3.7, both t_1 and t_2 belong to M, a contradiction. □

3.1. **Universal decomposition.** The following theorem 3.12 which is the main result of [15] is crucial for our proof. Before we state the theorem, we need to introduce some more definitions.

Definition 3.9. (QH-vertex) Let P be a planar subgroup of G which admits one of the following presentations:

(1) $\langle p_1, \ldots, p_m, a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{k=1}^{m} p_k \prod_{j=1}^{g} [a_j, b_j] \rangle$;
(2) $\langle p_1, \ldots, p_m, v_1, \ldots, v_g \mid \prod_{k=1}^{m} p_k \prod_{j=1}^{g} v_j^2 \rangle$.

Let $\Gamma(V, E)$ be a graph of groups. Let $v \in V$ and let $e_1, \ldots, e_m$ be all edges with $i(e_i) = v$. We suppose that $G_v = P$ and that $\alpha(e_i) = p_i$. Such a vertex v is called a *QH-vertex*.

Definition 3.10. (QH-subgroup) A subgroup P of G is a *QH-subgroup*, if there is a splitting $\mathcal{G}(V, E)$ of G and a QH-vertex $v \in \mathcal{G}$ (see Definition 3.9) such that P can be conjugated into the stabilizer of v. A subgroup P of G is a *maximal QH-subgroup* (denoted by MQH-subgroup for short), if for each elementary cyclic splitting $G = G_1 *_C G_2$ either P can be conjugated into G_1 or G_2, or C can be conjugated into P in such a way that there is an elementary splitting of P over a cyclic subgroup C_1 so that this splitting extends to an elementary splitting of the whole group G, and C is hyperbolic with respect to the splitting of G over C_1.

Definition 3.11. We say that Δ is *almost reduced*, if the equality $G_e = G_v$ implies that u is a QH-vertex (in particular, G_e is cyclic), $val(v) = 2$ and for the other edge f incident on v we have that $G_f \subsetneq G_v$ and the other endpoint of f is a QH-vertex as well.

Recall that if G is an $\mathcal{F}$-group, then $\mathcal{D}(G)$ denotes the set of all Abelian splittings of G where each maximal Abelian subgroup of G is elliptic.

Theorem 3.12. [15, Theorem 0.1 and Proposition 2.15]. *Let H be a freely indecomposable $\mathcal{F}$-group. There exists an almost reduced unfolded Abelian splitting $D \in \mathcal{D}(H)$ of H with the following properties:*

(1) *Every MQH-subgroup of H can be conjugated to a vertex group in D; every QH-subgroup of H can be conjugated into one of the MQH-subgroups of H; non-MQH subgroups in D are of two types: maximal abelian and non-abelian, every non-MQH vertex group in D is elliptic in every Abelian splitting in $\mathcal{D}(H)$.*
(2) *If an elementary cyclic splitting $H = A *_C B$ or $H = A*_C$ is hyperbolic in another elementary cyclic splitting, then C can be conjugated into some MQH subgroup.*
(3) *Every elementary Abelian splitting $H = A *_C B$ or $H = A*_C$ from $\mathcal{D}(H)$ which is elliptic with respect to any other elementary Abelian splitting from $\mathcal{D}(H)$ can be obtained from D by a sequence of collapses, foldings, conjugations and modifying boundary monomorphisms by conjugation.*
(4) *If $D_1 \in \mathcal{D}(H)$ is another splitting that has properties* (1) *and* (2)*, then it can be obtained from D by slidings, conjugations, and modifying boundary monomorphisms by conjugation.*

Moreover, given a presentation of H, there is an algorithm to find the splitting D.

In our proof, we use the slightly modified version of Theorem 3.12, stated in Theorem 3.13 below. It follows from [13, Theorem 6] (cf. also [34, Theorem 4.1]) that an indecomposable $\mathcal{F}$-group G is one of the following: the fundamental group of a closed surface, a free Abelian or a free non-Abelian group (cf. Example 2.17).

Theorem 3.13. *Let G be a one-ended decomposable $\mathcal{F}$-group. G has a semi-reduced Abelian splitting $\Gamma = (\Gamma(V, E), T, \varphi) \in \mathcal{D}(G)$ called a JSJ decomposition of G that satisfies the following properties:*

(1) *The decomposition Γ is universal, in the meaning of Definition 2.16.*
(2) *The Bass-Serre tree $\tilde{\Gamma}$ corresponding to Γ is 2-acylindrical (see Definition 2.6).*
(3) *Each rigid vertex group in Γ is of one of the following two types: a maximal Abelian subgroup (we call such a vertex elementary), or a non-Abelian subgroup.*
(4) *(V, E) is a bipartite graph: two elementary vertices and two non-elementary vertices are never joined by an edge $e \in E$.*
(5) *Each flexible vertex of Γ is a maximal QH-vertex. Let $G = G_1 *_C G_2$ or $G = G_1 *_C$ be a cyclic splitting of G. C can be conjugated into the stabilizer of a flexible vertex of Γ if and only if the splitting in question is hyperbolic with respect to another splitting of G.*

Moreover, there is an algorithm to obtain Γ.

Proof. Properties (1) and (5) follow immediately from Theorem 3.12 and the definitions.

Let $\Delta \in \mathcal{D}(G)$ be an Abelian splitting which is the output of the algorithm mentioned in Theorem 3.12. We modify the graph of groups Δ so as to obtain a new splitting Γ satisfying properties (3) and (4).

Let M be a maximal Abelian subgroup of G that contains either $\alpha(G_e)$ or $\omega(G_e)$ for some $e \in E$. Consider the set E_M of all edges e_i of Δ with $\alpha(A_i) \subseteq M$. Since M is elliptic in Δ, the union Δ_M of all edges $e \in E_M$ is a connected subgraph of Δ. It is easy to see that Δ_M can be found effectively. Indeed, it follows from Theorem 3.1 (4) that an edge with the stabilizer A belongs to Δ_M if and only if a non-trivial element of A commutes with a non-trivial element of $\alpha(A_1)$. By Theorem 3.1 (6), the word problem in G is decidable so that this latter problem is decidable also. If Δ_M is a tree, then by a series of slidings it can be transformed to an M-star of groups (cf. the proof of Lemma 3.7). Otherwise, Δ_M contains Abelian cycles. It follows immediately from Definition 3.3 that the union of all edges e_i of Δ_M with $\alpha(A_i) = \omega(A_i)$ form a maximal tree of Δ_M. Since $M^t \cap M^s = 1$ for two different stable letters $t \neq s$, the proof of Lemma 3.7 shows that Δ_M can be transformed effectively into an Abelian constellation of groups Ψ_M.

More generally, we have the following procedure. Since M is elliptic in Δ, there exists a vertex $v \in V$ with $M \subseteq G_v$. If $M \neq G_v$ for each G_v that contains it, then we add to V an elementary vertex z stabilized by M and connect z by an edge f with $G_f = M$ to v. When we have a vertex for each maximal Abelian subgroup M, then we produce a sequence of slidings as follows. If $\alpha(G_e)$ and $\omega(G_e)$ are both subgroups of a maximal Abelian subgroup M, then we slide e so that $i(e) = z$ with $G_z = M$ and don't change $\tau(e)$. If $\alpha(G_e) \subseteq M$ and $\omega(G_e) \subseteq N$ for $N \neq M$, then we slide e so that $i(e) = z$ with $G_z = M$ and $i(e) = y$ with $G_y = N$ and declare e a *silver edge*. The reason to introduce the more general procedure is that in Γ one can have cycles formed by an M-tree and an N-tree. In this latter case we have silver edges that do not belong to Abelian cycles in the sense of Definition 3.3. But the argument mentioned in Remark 3.6 remains valid in this case also, and we conclude that the silver subgraph of the modified graph Δ' is a forest. Therefore, we can collapse each silver M-subtree to a point stabilized by M. Obviously, the fundamental group of the new graph Γ is isomorphic to G, $\Gamma \in \mathcal{D}(G)$, and also properties (3) and (4) hold. Furthermore, in Γ each non-trivial Abelian subgroup fixes a subgraph of diameter at most 2 which, together with the CSA property (see Theorem 3.1) implies the assertion (2). □

Corollary 3.14. *Each edge group of* Γ *is elliptic in any splitting of* G.

Proof. Let Λ be a splitting of G, and let G_e be an edge stabilizer in Γ. We identify the edge e with its lifting to the Bass-Serre tree $\tilde{\Gamma}$. By Theorem 3.13 (1), there is a G-equivariant simplicial map κ from $\tilde{\Gamma}$ onto $\tilde{\Lambda}$. The image $\kappa(e) \in \Lambda$ of the edge $e \in \tilde{\Gamma}$ is a path λ in $\tilde{\Lambda}$; the path λ may be degenerate. As κ is G-equivariant, G_e is a subgroup of the stabilizer G_λ of λ, in particular, G_e fixes a point when acting on $\tilde{\Lambda}$, hence is elliptic in Λ, as claimed. □

Corollary 3.15. *Let T be a simplicial G-tree so that G acts on T with Abelian edge stabilizers.*

(1) *Let $t \in T$ be an edge with the stabilizer S_t. If S_t is elliptic in any splitting of G, then S_t can be conjugated into an elementary vertex group of Γ.*
(2) *If for each edge $t \in T$, the stabilizer S_t is a subgroup of G which is elliptic in any splitting of G, then each flexible vertex stabilizer of Γ fixes a point in T.*

Proof. By Theorem 3.13 (1), there is a G-equivariant simplicial map κ from $\tilde{\Gamma}$ onto T. If e is an edge of $\tilde{\Gamma}$ such that $\kappa(e)$ contains t, then G_e can be conjugated into S_t; in particular, G_e and a conjugate of S_t belong to a maximal Abelian subgroup of G. By Theorem 3.13 (4), one of the two endpoints of e in Γ is an elementary vertex with the stabilizer M which is an Abelian subgroup of G. By Lemma 3.7, M is a maximal Abelian subgroup of G, hence S_t can be conjugated into M, and the first assertion follows.

To prove the second assertion, assume that a flexible vertex stabilizer G_u of Γ does not fix a point in T. In this case, G_u inherits a non-trivial splitting Λ from its action on T. The edge groups in Λ are subgroups of the edge stabilizers of T. Collapse all the edges of Λ but one and denote by Λ_1 the obtained elementary splitting of G_u. By Corollary 3.14, the edge groups of G are elliptic when acting on T, so that Λ_1 extends to a splitting of G. Observe that the edge group of Λ_1 is elliptic in any splitting of G, which contradicts Theorem 3.13(5). □

3.2. **Uniqueness of a universal decomposition.**

Lemma 3.16. *Let G and H be two one-ended $\mathcal{F}$-groups, and let $\varphi\colon G \to H$ be an isomorphism. Let Γ (or Ξ) be an Abelian JSJ decomposition of G (or H). Then there exists a simplicial map $\mu\colon X \to Y$ between the Bass-Serre trees $X = \tilde{\Gamma}$ and $Y = \tilde{\Xi}$ so that the following diagram is commutative:*

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ {\scriptstyle (\varphi,\mu)}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ H \times Y & \longrightarrow & Y \end{array}$$

Proof. Observe that there are faithful actions $G \times Y \to Y$ defined by $\rho(g, y) = \varphi(g).y$ for all $g \in G$ and $y \in Y$, and $H \times X \to X$ defined by $\sigma(h, x) = \varphi^{-1}(h).x$ for all $h \in H$ and $x \in X$. Furthermore, by Corollary 3.14, each edge group H_e of H fixes a point in X. Therefore, by Corollary 3.15(1), each flexible vertex group of X fixes a point (i.e., is elliptic) when acting on Y. Observe that each rigid vertex group of X is elliptic also, by the definition. Each elementary vertex group M of X is a maximal Abelian subgroup of G, hence its image $\varphi(M)$ is a maximal Abelian subgroup of H. Since $\Gamma \in \mathcal{D}(G)$ and $\Xi \in \mathcal{D}(H)$, $\varphi(M)$ fixes a vertex in Y. Moreover, since G splits over a subgroup $A \subseteq M$, we have that $H = \varphi(G)$ splits over $\varphi(A)$, so that according to the proof of Lemma 3.7 and Theorem 3.13, $\varphi(M)$ fixes a unique elementary vertex in Y.

Our argument above allows one to define a simplicial map $\mu\colon X \to Y$ as follows. If $v \in X$ is a vertex with the stabilizer G_v, then $\mu(v) = y$ is the vertex with $\varphi(G_v) \subseteq H_y$. If $e \in X$ is an edge with the endpoints v and u, then $\mu(e) = f$ is the path joining $\mu(v)$ and $\mu(u)$. Furthermore, we claim that the diagram in the assertion of the theorem is commutative. Let $g \in G$ be a non-trivial element, and let $v \in X$ be a vertex with the stabilizer G_v. The image $u = g.v \in X$ is the vertex with the stabilizer $G_u = g^{-1}G_vg$, hence $\mu(g.v) = y_u \in Y$ so that $\varphi(g^{-1}G_vg) \subseteq H_u$, where H_u denotes the stabilizer of y_u. On the other hand, $\mu(v) = y_v$ with $\varphi(G_v) \subseteq H_v$, and g maps $y_v \in Y$ to the vertex $\bar{y}_v = \varphi(g).y_v$ with the stabilizer $H_{\bar{v}} = \varphi(g)^{-1}H_v\varphi(g)$. Observe that both H_u and $H_{\bar{v}}$ contain $\varphi(g^{-1}G_vg)$ as a subgroup. If G_v (hence, $\varphi(G_v)$) is non-elementary, then it cannot fix an edge in either X or Y. If G_v is elementary, then $g.v$, $y_u = \mu(g.v)$, $y_v = \mu(v)$ and $\varphi(g).\mu(v)$ are elementary vertices. In either case, we conclude that $H_u = H_{\bar{v}}$, and since the vertex of Y stabilized by H_u is unique, we have that $\mu(g.v) = \varphi(g).\mu(v)$, as claimed. □

Theorem 3.17. *Let $\varphi\colon G \to H$ be an isomorphism of two one-ended $\mathcal{F}$-groups, and let $\Gamma = \Gamma(V,E)$ and $\Xi = \Xi(U,B)$ be Abelian JSJ decompositions of G and H, respectively. Then the equivariant map $\mu\colon \tilde{\Gamma} \to \tilde{\Xi}$ between the Bass-Serre trees, defined in Lemma 3.16, is a one-to-one isometry.*

Proof. Denote $X = \tilde{\Gamma}$ and $Y = \tilde{\Xi}$. First, observe that the length of the image $\mu(e) \in Y$ of an edge $e \in X$ does not exceed 2 since Y is 2-acylindrical. Moreover, according to Theorem 3.13 (4), we can assume that one of the endpoints u and v of e is an elementary vertex, so that the image of this endpoint in Y is an elementary vertex as well. As Ξ is a bipartite graph (hence, Y is a bipartite tree) and different elementary vertex stabilizers have only trivial intersections, it follows that $\mu(e)$ has length 1 or 0.

Now, we claim that the non-degenerate images of two edges of X cannot get folded in Y. More precisely, let e and f be two edges of X, both incident on a vertex v so that $i(e) = i(f) = v$, hence $\alpha(G_e), \alpha(G_f) \subseteq G_v$, with different terminal points: $\tau(e) = u$ and $\tau(f) = w$. Assume that the images of e and f under μ get folded, so that $\mu(e) = \mu(f) = c$ and $\mu(u) = \mu(w) = y$. Let $\mu(v) = y_v$. Since $\varphi(g).\mu(x) = \mu(g.x)$ for all $x \in X$ and $g \in G$, both $\varphi(G_e)$ and $\varphi(G_f)$ are subgroups of H_c. Since the edge stabilizers in Y are Abelian, H_c is an Abelian subgroup of H and therefore, is a subgroup of a unique maximal Abelian subgroup of H which we denote by M_H. It follows that both $\varphi(G_e)$ and $\varphi(G_f)$ are subgroups of M_H, so that both G_e and G_f are subgroups of a maximal Abelian subgroup $M = \varphi^{-1}(M_H)$. Hence, by our construction, v is an elementary vertex of X (and $M = G_v$). Therefore,

G_u and G_w are non-elementary, and neither is H_y as both $\varphi(G_u)$ and $\varphi(G_w)$ are subgroups of H_y. On the other hand, H_y inherits a non-trivial elementary splitting from its action on X, a contradiction.

Next, we show that the image of an edge $e \in X$ cannot have length 0 in Y. Assume that $\mu(v) = \mu(u) = y$, where u and v are the endpoints of e, and $i(e) = v$ is an elementary vertex. If $\alpha(G_e) \subsetneqq G_v$, then we get a contradiction, because H_y acts non-trivially on X, hence splits over an Abelian subgroup. Let $G_e = G_v$. In this case the valence of v is at least 2 since X is semi-reduced; let $f \neq e$ be another edge incident on v. As we have just shown, the images of edges incident on an elementary vertex in X cannot get folded in Y. If the image of f under μ collapses also, then we have three vertices of X mapped to a vertex $y \in Y$, so that H_y acts non-trivially on X, a contradiction. Thus, $\mu(f)$ is not degenerate, so that in Y there is an edge stabilized by $\varphi(G_f) \subset \varphi(G_v)$. By our construction of the graph Ξ in Theorem 3.13, there is an elementary vertex z in Y with the stabilizer $\varphi(G_v)$. By the definition of μ, $z = \mu(v) \neq \mu(u)$, a contradiction.

So far, we have shown that μ is a local immersion. Finally, assume that there are two edges (or vertices) of X which are mapped to the same edge (or vertex) in Y. Consider the path p connecting them in X and its image $\mu(p)$ in Y. Since $\mu(p)$ is a closed path in Y, p has either an edge e incident on a vertex v so that $\mu(e) = \mu(v)$, or two edges e and f incident on v so that $\mu(e) = \mu(f)$. In either case, μ restricted to $Star(v)$ is not a local immersion, a contradiction. □

3.3. **Isomorphism of groups and splittings of the groups.**

Theorem 3.18. *Let G and H be two $\mathcal{F}$-groups, and let $\varphi\colon G \to H$ be an isomorphism. Let $\Gamma = \Gamma(V, E)$ (or $\Xi = \Xi(U, B)$) be the Abelian JSJ decomposition of G (or H). The image $\varphi_*(\Gamma)$ of Γ under φ can be obtained from Ξ by conjugation and modifying boundary monomorphisms.*

Proof. Fix the natural lift D of Γ into $X = \tilde{\Gamma}$ (see Definition 2.11), and let $\mu(D)$ be the image of D in $Y = \tilde{\Xi}$, where μ is the G-equivariant isometry defined in Lemma 3.16 (see also Theorem 3.17); recall that G acts on X by left multiplications and on Y via the isomorphism φ and left multiplications. Observe that $\mu(D)$ is a fundamental domain of Y. Indeed, since $X = G.D$, and the map μ is G-equivariant and onto, we conclude that $Y = G.\mu(D)$. Moreover, as μ is G-equivariant, $x_1 = g.x_2$ iff $\mu(x_1) = \varphi(g).\mu(x_2)$ for all $x_1, x_2 \in X$ and $g \in G$, so that two vertices (or two edges) of X are G-equivalent if and only if their images in Y are $\varphi(G) = H$-equivalent. Therefore, two different edges of $\mu(D)$ are never H-equivalent, and two vertices $\mu(v)$ and $\mu(u)$ of $\mu(D)$ are H-equivalent if and only if v and u are G-equivalent. This latter argument shows that the underlying graph of Γ is the underlying graph of Ξ. Therefore, we can assume that the maximal trees of Γ and of Ξ coincide and the orientation of edges is the same. It can be readily seen that $\varphi_*(\Gamma)$ can be obtained from $\mu(D)$ by identifying the H-equivalent vertices.

Now, let K be the natural lift of Ξ into Y. Fix a vertex $d \in D$. There is a vertex $k \in K$ so that d and k are G-equivalent. Observe (cf. Lemma 2.9) that there may be more than one vertex G-equivalent to d. To specify our choice, we also require that there is an isomorphism of graphs λ with $\lambda(k) = d$ that maps each vertex (or edge) of K to a G-equivalent vertex (or edge) in D.

The stabilizers H_d and H_k are conjugate in H, let $H_d = H_k^h$ for some $h \in H$. Let e_k be an edge in K incident on k, and let k_1 be the other endpoint of e_k. Recall that by our construction, precisely one of the vertices k and k_1 is elementary, so that k and k_1 are never G-equivalent. Denote $e_d = \lambda(e_k)$ and $d_1 = \lambda(k_1)$. We have that $H_{d_1} = H_{k_1}^{h_1}$ for some $h_1 \in H$, so that $H_{e_d} = H_d \cap H_{d_1} = H_k^h \cap H_{k_1}^{h_1} = (H_k \cap H_{k_1}^{h_1 h^{-1}})^h = (H_k^{h h_1^{-1}} \cap H_{k_1})^{h_1}$. Since the tree Y is 2-acylindrical and λ is an isometry, this latter intersection is non-empty if and only if either $h_1 h^{-1} \in H_k$ so that $H_{e_d} = (H_{e_k}^{h_1 h^{-1}})^h = H_{e_k}^{h_1}$, or $h_1 h^{-1} \in H_{k_1}$ so that $H_{e_d} = H_{e_k}^h$. In either case, we need to modify a boundary monomorphism.

Consider a particular case when the natural projections of d and d_1 into Γ are joined by two edges. We use the above notation. Let $f \neq e_d$ be the other edge joining d and d_1 in (V, E), and let t be the stable letter that corresponds to f in Γ. W.l.o.g., we can assume that $i(f) = d$. Since the graphs (V, E) and (U, P) are isomorphic, there is a unique edge $p \in P$ so that $\lambda(f) = p$: this is the edge joining (the natural projections of) k and k_1 in (U, P). We denote by s the stable letter that corresponds to p in Ξ. Let $A = \alpha(G_f) \subseteq G_d$ and $B = \omega(G_f) \subseteq G_{d_1}$, so that $A^t = B$. As we have just shown, $l = h_1 h^{-1}$ is either in H_k or in H_{k_1}. If $l \in H_k$, then $s = \varphi(t) l h_f$ with $h_f \in H_k$ being non-trivial if we need to modify the boundary monomorphism as follows: $\alpha(H_{\mu(f)}) = h_f \alpha(H_p) h_f^{-1}$. If $l \in H_{k_1}$, then $s = l^{-1}\varphi(t) h_f$ with $h_f \in H_k^l$ and $\alpha(H_{\mu(f)}) = h_f \alpha(H_p)^l h_f^{-1}$.

We proceed with the other edges incident on k and check that the assertion holds for $Star(k)$. The assertion follows by induction on the number of vertices. □

4. Algorithm to solve the isomorphism problem

Our algorithm is based on the following result.

Theorem 4.1. [15, Theorem 0.1 and Theorem 13.1] *Let $\langle \mathcal{S} \mid \mathcal{R} \rangle$ be a finite presentation of an $\mathcal{F}$-group G; we regard this presentation as the input of Elimination process. The Elimination process determines whether or not G is freely indecomposable, and the output of the process is a finite presentation $\langle S \mid R \rangle$ of G that can be described as follows:*

(1) *If G is a free non-Abelian group, then $R = \emptyset$.*

(2) *If G is freely decomposable but not free, then there are partitions $S = S_1 \sqcup ... \sqcup S_k \sqcup S_{k+1}$ and $R = R_1 \sqcup ... \sqcup R_k$, so that $\langle S \mid R\rangle = \langle S_1 \mid R_1\rangle * \cdots * \langle S_k \mid R_k\rangle * \langle S_{k+1} \mid -\rangle$, where $\langle S_i \mid R_i\rangle$ is a presentation of a freely indecomposable non-cyclic group for $1 \leq i \leq k$, and $\#S_{k+1} \geq 0$. In other words, the presentation $\langle S \mid R\rangle$ corresponds to the Grushko decomposition of G.*

(3) *If G is freely indecomposable, then the output of the Elimination process is a presentation of G as a JSJ-graph of groups. If G is also indecomposable in the meaning of Definition 2.19, then the presentation $\langle S \mid R\rangle$ of G has the following properties.*
   (a) *If G is the fundamental group of a closed surface, then R is a set of quadratic words, in the standard form.*
   (b) *If G is a free Abelian group, then the cardinality of S is minimum possible; in other words, $\#S = rank(G)$.*

In what follows, we assume that we are given a presentation of $G = \langle \mathcal{S}_G \mid \mathcal{R}_G\rangle$ and a presentation of $H = \langle \mathcal{S}_H \mid \mathcal{R}_H\rangle$, both presentations are output of the Elimination process.

Lemma 4.2. *Let G and H be indecomposable $\mathcal{F}$-groups. There exists an effective procedure to decide whether or not G and H are isomorphic.*

Proof. We apply the Elimination process to both presentations of G and of H to determine whether or not the corresponding group is a free group. If both G and H are free, then they are isomorphic if and only if the cardinalities of their generating sets coincide. Now, assume that neither of G and H is a free group. Since the equalities $[g_i, g_j] = 1$ for all pairs of generators of G hold in G if and only if G is a free Abelian group, and the word problem for $\mathcal{F}$-groups is solvable by Theorem 3.1(6), one can effectively decide whether or not G and H are free Abelian groups. Moreover, if G is a free Abelian group, then by Theorem 4.1(3b), one can effectively determine the rank of G. If both groups G and H are free Abelian, then they are isomorphic if and only if their ranks are equal. If neither of G and H is free Abelian, then both G and H are fundamental groups of closed surfaces. By Theorem 4.1(3a), one can effectively find standard quadratic presentations for both G and H. The groups are isomorphic if and only if their standard presentations coincide, up to permutation of generators. □

In what follows, we assume that both G and H are decomposable groups. Lemma 4.3 below allows us to reduce the problem to the case when both G and H are freely indecomposable groups.

Lemma 4.3. [17] *Let $G = G_1 * G_2 * ... * G_k * F_r$ and $H = H_1 * H_2 * ... * H_l * F_s$ be the Grushko decompositions. The groups G and H are isomorphic if and only if $k = l$, $r = s$ and there exists a permutation σ of the set $\{1, \ldots, k\}$ so that G_i is isomorphic to $H_{\sigma(i)}$ for each $i = 1, \ldots, k$.*

4.1. **Freely indecomposable groups.** Our solution to the isomorphism problem relies upon Theorem 3.18. According to Theorem 4.1(3), the above presentations define G and H as fundamental groups of graphs of groups: $G \backsimeq \pi_1(\Gamma)$ and $H \backsimeq \pi_1(\Xi)$, which are Abelian JSJ decompositions of G and H, respectively. Our algorithm is built so as to compare the two graphs of groups and conclude whether or not their fundamental groups are isomorphic; the algorithm is described in Theorem 4.13 below. It consists of a sequence of smaller procedures, some of these we describe now. First, we classify and compare the vertex groups.

Lemma 4.4. *There is an algorithm to determine the type of a given vertex G_v in an Abelian JSJ decomposition Γ of an $\mathcal{F}$-group.*

Proof. If each pair of generators commute, then G_v is free Abelian. If G_v is flexible, then the given presentation of G_v is a presentation of a QH-subgroup of one of the two possible kinds 3.10, up to permutation of the generators. If G_v is neither Abelian nor flexible, then according to Theorem 3.13, G_v is rigid non-elementary. □

Definition 4.5. Let G and H be two isomorphic groups, let $A_1, \dots, A_n$ be subgroups of G, and let $B_1, \dots, B_n$ be subgroups of H. An isomorphism $\phi\colon G \to H$ is an *extendable isomorphism* (or *e-isomorphism* for short), if there is one-to-one correspondence $A_i \to B_{j_i}$ between the sets of the subgroups so that ϕ maps A_i onto a conjugate of B_{j_i}. Pairs $(G, \{A_1, \dots, A_n\})$ and $(H, \{B_1, \dots, B_n\})$ are called *e-isomorphic*, if there is an e-isomorphism $\phi\colon G \to H$.

To find e-isomorphisms of QH-subgroups, we use the Elimination process that gives their standard presentations, and the following classical result.

Lemma 4.6. *Let $G_v \subset G$ and $H_u \subset H$ be two QH-subgroups in the Abelian JSJ decompositions of one-ended $\mathcal{F}$-groups G and H, and let $A_1, \dots, A_n \subset G_v$ and $B_1, \dots, B_n \subset G_u$ be their sets of peripheral subgroups. Then G_v and H_u are e-isomorphic if and only if their standard presentations (see Definition 3.10) are the same, up to permutation of generators. In particular, if φ_v is an e-isomorphism, then $\varphi_v(A_i) = B_i$ for all $i = 1, 2, \dots, n$.*

4.2. **Rigid vertices.** To find out whether or not two rigid vertex groups are e-isomorphic, we use Theorem 4.9 below. To state the theorem, we need some more definitions.

Definition 4.7. Two monomorphisms $\psi\colon G \to H$ and $\phi\colon G \to H$ are *equivalent* if ψ is a composition of ϕ and conjugation by an element from H.

Definition 4.8. Let G be a group and $\mathcal{K} = \{K_1, \dots, K_n\}$ be a set of subgroups of G. An Abelian splitting Δ of G is called a *splitting modulo $\mathcal{K}$* if all subgroups from $\mathcal{K}$ are conjugated into vertex groups in Δ.

Observe that a rigid vertex group in an Abelian JSJ decomposition of a group has no non-degenerate Abelian splittings modulo its peripheral subgroups.

Theorem 4.9. [15, Theorem 15.1] *Let G (or H) be an $\mathcal{F}$-group, and let $S_A = \{A_1, \ldots, A_n\}$ (respectively, $S_B = \{B_1, \ldots, B_n\}$) be a finite set of non-conjugated maximal Abelian subgroups of G (respectively, H) such that the Abelian decomposition of G modulo S_A is trivial. The number of equivalence classes of monomorphisms from G to H that map subgroups from S_A onto conjugates of the corresponding subgroups from S_B is finite. A set of representatives of the equivalence classes can be effectively found.*

Corollary 4.10. *Let G be an $\mathcal{F}$-group, and let $S = \{A_1, \ldots, A_n\}$ be a finite set of maximal Abelian subgroups of G. Denote by $Out(G; S)$ the set of those outer automorphisms of G which map each $A_i \in S$ onto a conjugate of it. If $Out(G; S)$ is infinite, then G has a non-trivial Abelian splitting modulo S. There is an algorithm to decide if $Out(G; S)$ is infinite and if it is, to find the splitting.*

Lemma 4.11. *Let G (or H) be an $\mathcal{F}$-group, and let $S_A = \{A_1, \ldots, A_n\}$ (respectively, $S_B = \{B_1, \ldots, B_n\}$) be a finite set of non-conjugated maximal Abelian subgroups of G (respectively, H) such that the Abelian decomposition of G modulo S_A is trivial. Then there is an algorithm to decide whether or not G and H are e-isomorphic, and if they are, then the algorithm finds all the equivalence classes of extendable isomorphisms from G to H.*

Proof. We apply Theorem 4.9 and find all the representatives $\phi_1, \ldots, \phi_k$ (if exist) of the equivalence classes of monomorphisms from G_v to H_u that map subgroups from S_A onto the subgroups from S_B.

If a monomorphism $\phi\colon G_v \to H_u$ that maps the edge groups of G_v onto the conjugates of the corresponding edge groups of H_u exists, one can effectively check whether or not it is onto. First, we apply [15, Theorem 3.21] to obtain a presentation for the image $\phi(G) \subseteq H$ which is the subgroup of H generated by $x_1, \ldots, x_k$. Now, we apply [25] to see whether or not $h_j \in \phi(G)$ for each j. The monomorphism ϕ is onto if and only if ϕ is an isomorphism. □

4.3. **Algorithm.** Let $\hat{\Gamma}(V, E)$ and $\hat{\Xi}(U, P)$ be Abelian JSJ decompositions of two one-ended $\mathcal{F}$-groups G and H, respectively (see Theorem 3.13). Assume that there is an isomorphism of graphs $\lambda\colon (V, E) \to (U, P)$. We denote the image $\lambda(e)$ of an edge e by the same letter e. For each vertex $v \in V$, we order all the edges incident on v end fix the same order for the edge subgroups of G_v, so that $A_i = \alpha(G_{e_i})$. (Since our ordering is local and the graph is bipartite, we can always assume that $i(e_i) = v$.) Similarly, we order all the edge subgroups of H_u where $u = \lambda(v)$ when we assume that λ respects the ordering of edges incident on v and on u. Further, we assume that for each $v \in V$ and $u = \lambda(v)$, there is an e-isomorphism $\varphi_v\colon G_v \to H_u$ that preserves ordering of the edge subgroups of G_v and G_u, so that $\varphi_v(A_i)$ is conjugate to B_i in H_u.

We fix a maximal tree T in (V, E) (hence, in (U, P)) and introduce comparative labelling of edges $L_u^{(\varphi)}\colon P \cap T \to H_u$ defined as follows. Let $v \in V$

be a *rigid non-elementary* vertex, and let $A_1, \dots, A_n \subset G_v$ be the edge subgroups. For $u = \lambda(v)$, let $B_1, \dots, B_n \subset H_u$ be the edge subgroups. Fix an e-isomorphism $\varphi\colon G_v \to H_u$ and set $L_u^{(\varphi)}(p_i) = h_i \in H$ if $p_i \in P$ is an edge incident on u with the edge group B_i and $\varphi(A_i) = h_i B_i h_i^{-1}$. Notice that labelling depends on the e-isomorphism φ. We assign the trivial label $1 \in H$ to each edge $e \in T$ incident on a flexible vertex. By a *star of a vertex v* in the tree T we mean the subgraph $Star(v)$ of T where the set of edges consists of the edges of T incident on v and the set of vertices consists of the endpoints of those edges.

Lemma 4.12. *With the above notation and assumptions, e-isomorphisms between vertices of $\Gamma(V,E)$ and $\Xi(U,B)$ can be extended to an isomorphism between the fundamental groups $\pi_1(\Gamma)$ and $\pi_1(\Xi)$ if and only if there are e-isomorphisms of vertices so that in the star of each elementary vertex, at most one label h_i is not trivial.*

Proof. To show that the condition is necessary, suppose there is an elementary vertex u with two different edges $e_1, e_2 \in Star(u)$ stabilized by B_1, B_2, so that their labels h_1 and h_2 are not trivial. Observe that $B_1, B_2 \subset H_u$, so that $B_i^{h_i} \subset H_u^{h_i}$ for $i = 1, 2$. Therefore, $H_u^{h_1} = H_u^{h_2}$, hence $h_1 h_2^{-1} \in H_u$, a contradiction.

To show that the condition is also sufficient, we extend e-isomorphisms $\varphi_v\colon G_v \to H_{\lambda(v)}$ between vertices of the graphs of groups $\Gamma(V,E)$ and $\Xi(U,B)$ to an isomorphism between the fundamental groups of the trees of groups $\varphi_T\colon \hat{\Gamma}(T) \to \hat{\Xi}(T)$. These trees of groups are obtained from the graphs of groups Γ and Ξ by removing the edges that do not belong to T. The map φ_T defines the images of the vertex groups G_v of Γ under an isomorphism $\varphi\colon G \to H$ that we are constructing. Having defined images of G_v in H, we assign images to the stable letters in the presentation of G as the fundamental group of $\Gamma(V,E)$, and get the isomorphism $\varphi\colon G \to H$.

Fix elementary vertices $u \in U$ and $v \in V$ so that $u = \lambda(v)$. First, we extend e-isomorphisms between vertices of $Star(v)$ and $Star(u)$ to an e-isomorphism between the fundamental groups $\pi_1(Star(v))$ and $\pi_1(Star(u))$. Assume that in $Star(u)$, precisely one label h is not trivial. Let $u_o = \tau(e_0)$ where e_0 is the labelled edge, and let T_u denote the connected component of $T \setminus \{e_0\}$ that contains u. We replace the e-isomorphism $\varphi_x\colon G_x \to H_{\lambda(x)}$ by $\hat{h} \circ \varphi_x$ where $\hat{h}$ is conjugation by h, for each x with $\lambda(x) \in T_u$. Let $v_0 \in V$ be so that $u_0 = \lambda(v_0)$ and $\varphi_v^{(0)}\colon G_{v_0} \to H_{u_0}$ be the e-isomorphism that corresponds to the labelling in question. Observe that all vertices of $Star(u)$ but u_0 are in T_u, and e-isomorphisms $\hat{h} \circ \varphi_v$ and $\phi \in \{\varphi_v^{(0)}, \hat{h} \circ \varphi_x \mid x \in Star(v), x \neq v_0\}$ agree on edge subgroups. Therefore, the e-isomorphisms $\hat{h} \circ \varphi_v$ and $\varphi_v^{(0)}, \hat{h} \circ \varphi_x$ $(x \in Star(v), x \neq v_0)$ define an e-isomorphism ψ_v between the fundamental groups $\pi_1(Star(v))$ and $\pi_1(Star(u))$, since replacing e-isomorphisms at the vertices $x \in V$ with $\lambda(x) \in \Delta_u$, does not affect the labelling of P. If there is no non-trivial label in $Star(u)$, then the e-isomorphisms φ_x where $x \in Star(v)$ agree on edge subgroups, hence extend to an e-isomorphism $\psi_v \colon \pi_1(Star(v)) \to \pi_1(Star(u))$.

We proceed to other elementary vertices by induction on the distance from v in T and end up with the isomorphism φ_T. Now, let $e \in E$ do not belong to T, and let t (or s) be the stable letter that corresponds to e in G (or H). Let $v = i(e)$ and $x = \tau(e)$ be the endpoints of e, and $A = \alpha(G_e)$ and $C = \omega(G_e)$. Recall that $\alpha(G_e) = \omega(G_e)^t$ in G and $\alpha(H_e) = \omega(H_e)^s$ in H. Our assumptions and the above procedure imply that $\varphi_T(\alpha(G_e)) = \alpha(H_e)^h$ and $\varphi_T(\omega(G_e)) = \omega(H_e)^b$ for some h and b in H. Hence, we can set $\varphi_T(t) = b^{-1}sh$ to preserve the relations. Obviously, the map $\varphi \colon G \to H$ that we obtain is an isomorphism. □

Theorem 4.13. *Let $G \cong \langle \mathcal{S}_G \mid \mathcal{R}_G \rangle$ and $H \cong \langle \mathcal{S}_H \mid \mathcal{R}_H \rangle$ be finite presentations of fully residually free groups. There exists an algorithm that determines whether or not G and H are isomorphic. If the groups are isomorphic, then the algorithm finds an isomorphism $G \to H$.*

Proof. We apply the Elimination process to the given presentations. The output of the Elimination process are presentations $G \cong \langle S_G \mid R_G \rangle$ and $H \cong \langle S_H \mid R_H \rangle$ described in Theorem 4.1. If both G and H are indecomposable, then we apply Lemma 4.2. If both G and H have non-trivial Grushko decompositions with the same number of factors, then by Lemma 4.3, it is enough to compare the factors G_i and H_j of these decompositions. If G_i and H_j are free groups, then they are isomorphic if and only if their generating sets have the same cardinality. Otherwise, G_i and H_j are one-ended groups (in what follows, we still denote these groups by G and H), and we consider their Abelian JSJ decompositions $\Gamma(V, E)$ and $\Xi(U, P)$. Theorem 3.17 gives rise to the following algorithm. We find all possible isomorphisms between the graphs (V, E) and (U, P). If there are not any, then we are done as the groups are not isomorphic. Otherwise, fix an isomorphism $\lambda \colon (V, E) \to (U, P)$ and try to find an extendable isomorphism $\varphi_v \colon G_v \to H_{\lambda(v)}$ that preserves the ordering of the edge subgroups (see the beginning of this section), for each $v \in V$. This latter procedure depends on the type of the vertex group in question: Abelian (elementary), flexible or rigid non-elementary. Recall that by Lemma 4.4, we are able to determine the type of each vertex group effectively. If G_v and $H_{\lambda(v)}$ are either elementary or flexible groups, then it suffices to compare their canonical presentations that are output of the Elimination process. The groups are isomorphic if and only if a map sending the generators of G_v in the canonical presentation to the generators of $H_{\lambda(v)}$, sends the peripheral subgroups of G_v onto the peripheral subgroups of $H_{\lambda(v)}$, so that it remains to check that the ordering of the peripheral subgroups is preserved. An algorithm for rigid groups is the content of Lemma 4.11. Observe that each rigid non-elementary subgroup is an $\mathcal{F}$-group with the trivial Abelian decomposition modulo the set of peripheral subgroups, which makes Lemma 4.11

applicable in this case. If for each isomorphism of graphs λ there is a pair of vertices $(v, \lambda(v))$ with no e-isomorphism between G_v and $H_{\lambda(v)}$ preserving the ordering (which we can find out in a finite time), then G and H are not isomorphic. Otherwise, we fix λ and an e-isomorphism φ_v for each pair $(v, \lambda(v))$ and associate the comparative labelling as defined above, to each set of e-isomorphisms between the non-elementary vertices of (V, E) and (U, P). Since by Corollary 4.10, the set of e-isomorphisms between two rigid vertices is finite, we can apply Lemma 4.12 and obtain the claim. □

5. Structure of the automorphism group

Let G be a one-ended $\mathcal{F}$-group. By Theorem 3.18, an Abelian JSJ decomposition of G and its image under an automorphism of G differ by conjugation and modifying boundary monomorphisms. We apply this result to study the structure of $Out(G)$. To state our result, we introduce one more definition.

Definition 5.1. Let G be a freely indecomposable $\mathcal{F}$-group, and let $\Gamma(V, E \cup E_s; T)$ be the Abelian JSJ decomposition of G. We define the group $Out_\Gamma(G)$ to be the subgroup of $Out(G)$ generated by the following types of automorphisms of G:

1. Generalized Dehn twists along edges in Γ (see Definition 2.4).
2. Automorphisms of an elementary vertex group that preserve the peripheral subgroups of the group.
3. Automorphisms of a flexible vertex group G_u that preserve the peripheral subgroups of the group, up to conjugacy (geometrically, these are Dehn twists along simple closed curves on the punctured surface Σ with $\pi_1(\Sigma) \cong G_u$).

Lemma 5.2. *With the notation of Definition 5.1,* $[Out(G) : Out_\Gamma(G)] < \infty$.

Proof. According to Theorem 3.18, each automorphism $\psi \in Aut(G)$ preserves the maximal tree T of Γ. Therefore, ψ is the composition of e-automorphisms of vertices, automorphisms of type (1), and conjugation. Observe that the e-automorphisms of elementary and flexible vertices belong to $Out_\Gamma(G)$. Furthermore, according to Corollary 4.10, each rigid vertex has only finitely many e-automorphisms. Also observe that e-automorphisms of different vertices commute; the assertion follows. □

Let G be a one-ended $\mathcal{F}$-group, and let $\Gamma(V, E)$ be an Abelian JSJ decomposition of G. By an *e-automorphism of a vertex group* G_v we mean an automorphism $\psi \in Out(G_v)$ that maps each edge subgroup of G_v onto a conjugate of itself (cf. Definition 4.5). We denote by $V_M \subset V$ the subset of all elementary vertices and by $V_Q \subset V$ the subset of all flexible (or QH-)vertices of Γ (see [29, Definition] and Definition 3.10 in the present paper). With each vertex $v \in V_M \cup V_Q$ we associate the subgroup of e-automorphisms of G_v denoted by $\mathcal{M}_v$ if $v \in V_M$ and by $\mathcal{Q}_v$ if $v \in V_Q$. Since

G_v is a finitely generated free Abelian group, $\mathcal{M}_v$ is a subgroup of $GL_n(\mathbb{Z})$, where n is the maximal rank of an Abelian subgroup of G. Each flexible vertex group is the fundamental group of a punctured surface, so that $\mathcal{Q}_v$ is the mapping class group of a surface with boundary. Let $\mathcal{Q} = \prod_{v \in V_Q} \mathcal{Q}_v$ and $\mathcal{M} = \prod_{v \in V_M} \mathcal{M}_v$. Since the structure of $Out_\Gamma(G)$ is well understood, we have the following result.

Theorem 5.3. *Let G be a one-ended $\mathcal{F}$-group. The group $Out(G)$ is virtually a direct product $\mathbb{Z}^d \times \mathcal{M} \times \hat{\mathcal{Q}}$ where $\hat{\mathcal{Q}}$ is the quotient of $\mathcal{Q}$ by a central subgroup isomorphic to a f.g. free Abelian group $\mathbb{Z}^m$.*

References

[1] H. Bass, *Group actions on non-Archimedean trees*, Arboreal group theory (Berkeley, CA, 1988), 69–131, Math. Sci. Res. Inst. Publ., 19, Springer, New York, 1991.

[2] B. Baumslag, *Residually free groups*, Proc. London Math. Soc., **17(3)** (1967), 402–418.

[3] G. Baumslag, A. Miasnikov, V. Remeslennikov, *Algebraic geometry over groups I. Algebraic sets and ideal theory,* Journal of Algebra, 1999, v.219, 16–79.

[4] B. Bowditch, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180 (1998), no. 2, 145–186.

[5] I. Bumagin, *The conjugacy problem for relatively hyperbolic groups*, Algebraic and Geometric Topology, **4** (2004) 1013-1040.

[6] I. Bumagin and D. Wise, *Coherence of coherent-by-cyclic groups*, preprint.

[7] F. Dahmani, *Combination of convergence groups*, Geom. Topol. **7** (2003) 933-963.

[8] B. Farb, *Relatively hyperbolic groups*, GAFA, **8**(1998), 810–840.

[9] M. Forester, *Deformation and rigidity of simplicial group actions on trees*, Geom. Topol. **6** (2002) 219–267.

[10] M. Forester, *On uniqueness of JSJ decompositions of finitely generated groups*, Comment. Math. Helv. **78** (2003) 740–751.

[11] D. Gildenhuys, O. Kharlampovich and A. Myasnikov, *CSA-groups and separated free constructions*, Bull. Aistral. Math. Soc., **52**(1995), 63–84.

[12] V. Guirardel, *A very short proof of Forester's rigidity result*, Geom. Topol. **7** (2003) 321–328.

[13] O. Kharlampovich and A. Myasnikov, *Irreducible affine varieties over a free group II*, J. of Algebra, **200**(1998), 517–570.

[14] O. Kharlampovich and A. Myasnikov, *Implicit function theorem over free groups*, to appear in Journal of Algebra.

[15] O. Kharlampovich, A. Myasnikov, *Effective JSJ decompositions,* to appear in Group Theory: Algorithms, Languages, Logic (A. Borovik editor), Contemp. Math., Amer. Math. Soc., Vol.378, 2005, 87–212.

[16] O. Kharlampovich, A. Myasnikov, V. Remeslennikov, D. Serbin, *Subgroups of fully residually free groups: algorithmic problems,* Group theory, Statistics and Cryptography (A.G. Myasnikov and V. Shpilrain, editors), Contemp. Math., Amer. Math. Soc., Vol.360, 2004.

[17] A. G. Kurosh, *The theory of groups*, Translated from the Russian and edited by K. A. Hirsch. 2nd English ed., 1960, Chelsea Publishing Co., New York.

[18] G. Levitt, *Automorphisms of hyperbolic groups and graphs of groups*, preprint, available at http://xxx.arxiv.org/abs/math.GR/0212088.

[19] R.C. Lyndon, *Groups with parametric exponents*, Trans. Amer. Math. Soc., 96:518–533, 1960.

[20] R.C. Lyndon, *Equations in free groups*, Trans. Amer. Math. Soc., 96:445–457, 1960.

[21] R.C. Lyndon and P. Schupp, *Combinatorial Group Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. Springer-Verlag, Berlin-New York, 1977.

[22] G.S. Makanin, *Equations in a free group*, Math. USSR Izvestiya, **21** (1983).

[23] G.S. Makanin, *Decidability of the universal and positive theories of a free group (Russian),* Izv. Akad. Nauk SSSR, Ser. Mat., 48(1):735–749, 1985. transl. in Math. USSR Izv., V. 25, 1985; MR 86c:03009.

[24] B. Mal'cev, *On the faithful representation of infinite groups by matrices*, Math. Sb. (N.S.), **8(50)** (1940), 405–422; English translation in Amer. Math. Soc. Transl., **45(2)**(1965), 1–18.

[25] A. Myasnikov, V. Remeslennikov, D. Serbin, *Regular free length functions on Lyndon's free* $\mathbb{Z}[t]$*-group* $F^{\mathbb{Z}[t]}$, to appear in Contemp. Math. AMS

[26] D. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems* , preprint.

[27] A. Razborov, *On systems of equations in a free group*, Math. USSR-Izv. **25(1)**, (1985), 115–162.

[28] E. Rips and Z. Sela, *Canonical representatives and equations in hyperbolic groups*, Invent. Math. **120**(1995), no. 3, 489–512.

[29] E. Rips and Z. Sela, *Cyclic splittings of finitely presented groups and the canonical JSJ decomposition*, Annals of Mathematics, **146** (1997), 53–104.

[30] D. Segal, *Decidable properties of polycyclic groups*, Proc. London Math. Soc. (3) **61** (1990), no. 3, 497–528.

[31] D. Segal and F. J. Grunewald, *Some general algorithms. II. Nilpotent groups*, Ann. of Math. (2) **112** (1980), no. 3, 531–617.

[32] Z. Sela, *The isomorphism problem for hyperbolic groups. I.* Ann. of Math. **141(2)** (1995), no. 2, 217–283.

[33] Z. Sela, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups II*, GAFA **7** (1997), 561–593.

[34] Z. Sela, *Diophantine geometry I*, Publications Mathematiques de l'IHES, 93(2001), 31-105.

[35] J.P. Serre, *Trees*, Springer-Verlag, Berlin and New York, 1980.

School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, ON, Canada, K1S 5B6

E-mail address: bumagin@math.carleton.ca

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, QC, Canada, H3A2K6

E-mail address: olga@math.mcgill.ca

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, QC, Canada, H3A2K6

E-mail address: alexeim@att.net