

# On the Hanna Neumann Conjecture

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## Abstract

The **Hanna Neumann conjecture** states that if  $F$  is a free group, then for all nontrivial finitely generated subgroups  $H, K \leq F$ ,

$$\text{rank}(H \cap K) - 1 \leq [\text{rank}(H) - 1][\text{rank}(K) - 1]$$

Where most papers to date have considered a direct graph theoretic interpretation of the conjecture, here we consider the use of monomorphisms. We illustrate the effectiveness of this approach with two results. First, we show that for any finitely generated groups  $H, K \leq F$  either the pair  $H, K$  or the pair  $H^-, K$  satisfy the Hanna Neumann conjecture; here  $-$  denotes the automorphism which sends each generator of  $F$  to its inverse. Next, using particular monomorphisms from  $F$  to  $F_2$ , we obtain that if the Hanna Neumann conjecture is false then there is a counterexample  $H, K \leq F_2$  having the additional property that all the branch vertices in the foldings of  $H$  and  $K$  are of degree 3, and all degree 3 vertices have the same local structure or “type”.

## 1 Introduction

H. Neumann proved in [12] that any nontrivial subgroups  $H, K \leq_{\text{f.g.}} F$  (finitely generated) must satisfy

$$\text{rank}(H \cap K) - 1 \leq 2[\text{rank}(H) - 1][\text{rank}(K) - 1], \quad (1)$$

and so improved Howson’s earlier result [5] that  $H \cap K$  is finitely generated. The stronger assertion obtained by omitting the factor of 2 in (1) has come to be known as the Hanna Neumann conjecture. In [1], R. Burns improved H. Neumann’s bound by showing that

$$\begin{aligned} \text{rank}(H \cap K) - 1 &\leq 2[\text{rank}(H) - 1][\text{rank}(K) - 1] \\ &\quad - \min(\text{rank}(H) - 1, \text{rank}(K) - 1). \end{aligned}$$

In 1983, J. Stallings introduced the notion of a *folding* and showed how to apply these objects to the study of subgroups of free groups [16]. Stallings’s approach was applied by S. Gersten in [4] to solve certain special cases of the conjecture, and similar techniques were developed over a sequence of papers by W. Imrich [7, 6], P. Nickolas [14], and B. Servatius [15] who gave alternate

proofs of Burns' bound and resolved special cases of the conjecture. In 1989, W. Neumann showed that the conjecture is true "with probability 1" for randomly chosen subgroups of free groups [13], and proposed a strengthened form of the conjecture which bounds the quantity

$$\tilde{r}(H, K) \stackrel{\text{def}}{=} \sum_{g \in X} \tilde{r}(H^g \cap K)$$

where  $X \subset F$  is a set of double coset representatives for the double cosets  $HgK$ , and  $\tilde{r}$  is the reduced rank, defined as  $\tilde{r}(G) = \max(\text{rank}(G) - 1, 0)$  for subgroups  $G \leq F$ . Specifically, the strengthened Hanna Neumann conjecture asserts:

$$\tilde{r}(H, K) \leq \tilde{r}(H)\tilde{r}(K)$$

In 1992, G. Tardos proved in [17] that the strengthened conjecture is true if one of the two subgroups has rank 2. Then, in 1994, W. Dicks showed that the strengthened Hanna Neumann conjecture is equivalent to a conjecture on bipartite graphs, which he termed the Amalgamated Graph conjecture [2]. In 1996, G. Tardos used Dicks' method to give the first new bound for the general case in [18], where he proved that for any  $H, K \leq F$  with  $\text{rank}(H), \text{rank}(K)$  at least 2,

$$\tilde{r}(H, K) \leq 2\tilde{r}(H)\tilde{r}(K) - \tilde{r}(H) - \tilde{r}(K) + 1$$

Since then, W. Dicks and E. Formanek [3] resolved the conjecture for the case when one of the subgroups has rank at most 3, by proving that

$$\tilde{r}(H, K) \leq \tilde{r}(H)\tilde{r}(K) + \tilde{r}_{-3}(H)\tilde{r}_{-3}(K),$$

where  $\tilde{r}_{-k}(G) \stackrel{\text{def}}{=} \max(\text{rank}(G) - k, 0)$  for subgroups  $G \leq F$  and  $k \in \mathbb{N}$ .

The strengthened conjecture was also recently solved in the special case when one of the two groups, say  $H$ , has a generating set consisting of positive words (i.e. a set of words in which no generator of  $F$  has negative exponent). Specifically, it was shown by J. Meakin and P. Weil [11], and independently by B. Khan [9] that if there is some automorphism of  $F$  which carries a generating set of  $H$  to a set of positive words, then the conjecture holds for  $H$  and any nontrivial  $K \leq_{\text{f.g.}} F$ .

Recall that an automorphism  $\sigma$  of  $F(X)$  is called *length-preserving* if  $\forall u \in F$ ,  $|u^\sigma| = |u|$ , i.e.  $(X^\pm)^\sigma = X^\pm$  where  $X^\pm = X \cup X^{-1}$ . In section 3, we shall prove the following two theorems:

**Theorem 1.** *Let  $F_2 = F(a, b)$ . Take  $\sigma \in \text{Aut}(F_2)$  to be any length-preserving automorphism having no non-trivial fixed points, and let  $\tau$  be any monomorphism*

$$\tau : \begin{array}{ll} a & \mapsto aw_aa \\ b & \mapsto bw_bb, \end{array}$$

where  $w_a, w_b \in F_2$  are arbitrary elements for which the words  $aw_aa$  and  $bw_bb$  are reduced as written. Then for all nontrivial  $H, K \leq_{f.g.} F_2$ , either the pair  $H, K$  or the pair  $H^{\tau^\sigma}, K$  satisfy the strengthened Hanna Neumann conjecture.

**Theorem 2.** Let  $F = F(X)$  and  $-$  be the automorphism given by  $x \mapsto x^{-1}$  (for each  $x \in X$ ). Then for all nontrivial  $H, K \leq_{f.g.} F$ , either the pair  $H, K$  or the pair  $H^-, K$  satisfy the strengthened Hanna Neumann conjecture.

Note that in the special case when  $|X| = 2$ , Theorem 2 follows immediately from Theorem 1 by considering  $\tau$  to be the identity automorphism, and  $\sigma$  to be the automorphism sending  $x \mapsto x^{-1}$  for every  $x$  in  $X$ .

Recall that given  $H = \langle w_1, \dots, w_n \rangle \leq F$ , one may determine the associated Stallings' folding  $\Gamma_H = (V_H, E_H)$ , by the following constructive procedure (see [16]): Construct  $n$  directed cycles  $c_1 = (V_1, E_1), \dots, c_n = (V_n, E_n)$ , where  $|V_i| = |w_i|$ . Then pick one vertex from each of the cycles, and identify this subset of vertices, denoting the resulting vertex as the identity vertex  $1_H$ . Label the edges of cycle  $c_i$  by successive letters of  $w_i$ , starting at vertex  $1_H$ . Finally, repeatedly identify pairs of edges  $e, e'$  for which

$$\text{label}(e) = \text{label}(e') \wedge [\text{head}(e) = \text{head}(e') \vee \text{tail}(e) = \text{tail}(e')].$$

Each such identification is called an *edge-folding* and we say that the edge  $e$  (as well as  $e'$ ) was *folded*. Figure 1 illustrates the process, which terminates in finitely many steps yielding the folding  $\Gamma_H$ . It is easy to verify that the folding so obtained is well-defined, and moreover, is independent of the choice of generating set for  $H$ . It is not hard to see that the rank of  $H$  is precisely  $|E_H| - |V_H| + 1$ .

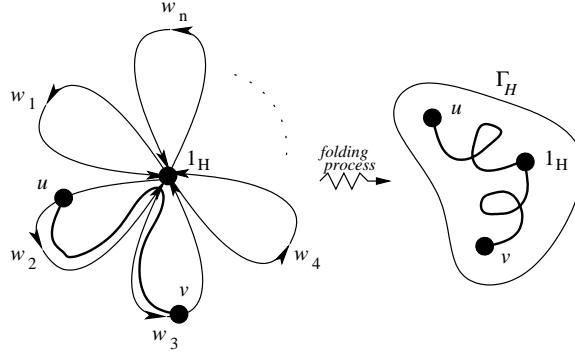


Figure 1: Constructing a folding from a rose.

Now we consider subgroups of  $F_2$ : If  $H \leq_{f.g.} F_2$  then  $\Gamma_H$  has vertices of undirected degree  $\leq 4$ , where by “undirected degree”  $d = d_H(v)$  of a vertex  $v$  we mean the sum of the number of outgoing and incoming edges at  $v$ . Put  $d_i(\Gamma_H) = |\{v \in V_H \mid d_H(v) = i\}|$ , for  $i = 1, 2, 3, 4$ . Vertices of degree 3 may be

classified into 4 types, denoted  $C_a, C_b, C_{a^{-1}}, C_{b^{-1}}$ , based on the labels of the incident edges (see figure 2). For each  $x \in \{a^\pm, b^\pm\}$ , we define  $C_x(\Gamma_H)$  to be the number of degree 3 vertices of type  $C_x$  in  $\Gamma_H$ . The rank of  $H$  can be computed by the formula

$$\text{rank}(H) = d_4(\Gamma_H) + \frac{d_3(\Gamma_H)}{2} - \frac{d_1(\Gamma_H)}{2} + 1$$

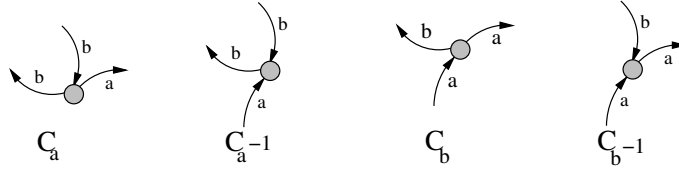


Figure 2: Local structure of vertex labels in a folding.

The graph-theoretic approach to the Hanna Neumann conjecture is based on the following key observations [16, 8]. Consider the product automaton  $\Gamma_H \times \Gamma_K$ , whose vertex set is  $V_H \times V_K$  and two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are connected by an edge labelled  $x$  iff both  $(u_1, v_1) \in E_H$  and  $(u_2, v_2) \in E_K$  have label  $x$ . Given a folding  $\Gamma$ , its *core* is defined to be the graph obtained by repeated deletion of non-identity degree 1 vertices. We denote this graph as  $\Gamma^\circ$ . Let  $(1_H, 1_K)$  play the role of identity vertex in  $\Gamma_H \times \Gamma_K$ . Then it is not hard to see that the components of  $(\Gamma_H \times \Gamma_K)^\circ$  are in one to one correspondence with  $\{\Gamma_{H^g \cap K}\}$  where  $g$  varies over a suitably chosen set of double coset representatives for the double cosets  $HgK$ . More specifically, the connected component of  $(\Gamma_H \times \Gamma_K)^\circ$  which contains  $(1_H, 1_K)$ , is isomorphic to  $\Gamma_{H \cap K}$ .

Many proofs of partial results towards the Hanna Neumann conjecture require a case-by-case analysis based on the numbers and types of degree 3 vertices present in the foldings of  $H, K$ . The next theorem has implications on the number of cases which need to be considered in such arguments; it is proved in section 3.

**Theorem 3.** *Let  $F = F(X)$  be a non-abelian free group. There is a monomorphism  $\phi_0 : F \rightarrow F_2$  into the free group of rank 2 such that: for any groups  $H, K \leq F$ , the foldings of  $H^{\phi_0}, K^{\phi_0} \leq_{f.g.} F_2$  have the property that all their branch vertices are of degree 3, and all degree 3 vertices have the same type.*

The previous theorem has the following immediate corollary:

**Corollary 1.** *If the Hanna Neumann conjecture is false then there is a counterexample  $H, K \leq_{f.g.} F_2$  having the additional property that all the branch vertices in the foldings of  $H$  and  $K$  are of degree 3, and all degree 3 vertices have the same type.*

## 2 Preliminaries

The numbers  $C_x(\Gamma_H)$  and  $C_x(\Gamma_K)$  allow one to compute upper bounds on the numbers of vertices of degree 3 in  $\Gamma_H \times \Gamma_K$  and hence in  $\Gamma_{H \cap K}$ . By considering a suitable conjugate of  $H$  and  $K$  we can assume  $d_1(\Gamma_H) = d_1(\Gamma_K) = 0$ .

**Remark 1.** It follows from the definition of the product folding that

$$\begin{aligned} d_4(\Gamma_{H \cap K}) &\leq d_4(\Gamma_H \times \Gamma_K) = d_4(\Gamma_H)d_4(\Gamma_K), \text{ and} \\ d_3(\Gamma_{H \cap K}) &\leq d_3(\Gamma_H \times \Gamma_K) \\ &= d_4(\Gamma_H)d_3(\Gamma_K) + d_3(\Gamma_H)d_4(\Gamma_K) + \sum_{x \in \{a,b\}^\pm} C_x(\Gamma_H)C_x(\Gamma_K) \end{aligned}$$

**Definition 1.** Given two subgroups  $H, K \leq_{f.g.} F_2$  and  $x \in \{a, b\}^\pm$ , we define

$$\begin{aligned} \delta_x(H, K) &= \min \left\{ \frac{C_x(\Gamma_H)}{d_3(\Gamma_H)}, \frac{C_x(\Gamma_K)}{d_3(\Gamma_K)} \right\} \\ \delta(H, K) &= \max_{x \in \{a^\pm, b^\pm\}} \delta_x(H, K) \end{aligned}$$

and put  $\mu(H, K)$  to be any  $x \in \{a^\pm, b^\pm\}$  for which  $\delta_x(H, K) = \delta(H, K)$ .

**Remark 2.** Walter Neumann [13] showed that if  $H, K \leq_{f.g.} F_2$  are a counterexample to the strengthened conjecture, then  $\delta(H, K) > \frac{1}{2}$ . We outline his argument here, in graph-theoretic notation. Using a simple and beautiful argument from convexity theory, he showed that if  $\delta(H, K) \leq \frac{1}{2}$  then

$$\sum_{x \in \{a,b\}^\pm} C_x(\Gamma_H)C_x(\Gamma_K) \leq \frac{1}{2}d_3(\Gamma_H)d_3(\Gamma_K).$$

Since the components of  $(\Gamma_H \times \Gamma_K)^\circ$  are in one to one correspondence with  $\{\Gamma_{H^g \cap K}\}$  (where  $g$  varies over a suitably chosen set of double coset representatives for the double cosets  $HgK$ ), we see that

$$\tilde{r}(H, K) = d_4((\Gamma_H \times \Gamma_K)^\circ) + d_3((\Gamma_H \times \Gamma_K)^\circ)/2 - d_1((\Gamma_H \times \Gamma_K)^\circ)/2$$

which is at most  $d_4(\Gamma_H \times \Gamma_K) + d_3(\Gamma_H \times \Gamma_K)/2$ . By Remark 1 then,

$$\begin{aligned} \tilde{r}(H, K) &\leq d_4(\Gamma_H \times \Gamma_K) + d_3(\Gamma_H \times \Gamma_K)/2 \\ &\leq d_4(\Gamma_H)d_4(\Gamma_K) + \frac{1}{2} \left[ d_4(\Gamma_H)d_3(\Gamma_K) + d_3(\Gamma_H)d_4(\Gamma_K) \right. \\ &\quad \left. + \sum_{x \in \{a,b\}^\pm} C_x(\Gamma_H)C_x(\Gamma_K) \right] \\ &\leq d_4(\Gamma_H)d_4(\Gamma_K) + d_4(\Gamma_H)d_3(\Gamma_K)/2 + d_3(\Gamma_H)d_4(\Gamma_K)/2 \\ &\quad + d_3(\Gamma_H)d_3(\Gamma_K)/4 \\ &\leq \tilde{r}(H)\tilde{r}(K). \end{aligned}$$

and thus the strengthened conjecture holds.

### 3 Results

**Remark 3.** Given an endomorphism  $\phi : F_2 \rightarrow F_2$ , the folding  $\Gamma_{H^\phi}$  can be obtained from  $\Gamma_H$  as follows. First construct a labelled directed graph  $\phi(\Gamma_H)$  by replacing each edge with label  $x$  in  $\Gamma_H$  by a sequence of edges labelled by the successive letters of  $x^\phi$  (for  $x = a, b$ ). Then, apply the previously described folding procedure to transform the graph  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ . One may verify that this yields a folding which is isomorphic to the one obtained by constructing  $\Gamma_{H^\phi}$  directly from the set  $\{w_1^\phi, \dots, w_n^\phi\}$ .

For example, if  $\phi$  is a length-preserving automorphism, then  $\Gamma_{H^\phi}$  can be obtained from  $\Gamma_H$  by replacing every label  $x$  by  $x^\phi$  and changing the orientation of the edges if necessary.

**Lemma 1.** *Let  $\Gamma_H$  be the folding of a subgroup  $H \leq_{f.g.} F_2$ , and  $\phi : F_2 \rightarrow F_2$  an endomorphism. If two edges  $e, f$  from  $\phi(\Gamma_H)$  get folded during the folding process  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ , then there must exist a path  $p$  in  $\phi(\Gamma_H)$  beginning at  $e$  and ending at  $f$  with the property that every edge in  $p$  was folded during the folding process.*

*Proof.* The lemma is proved by induction on the number  $n$  of edge-foldings which take place during the folding process—note that this number does not depend on the folding process since it is equal to  $|E_{\phi(\Gamma_H)}| - |E_{\Gamma_{H^\phi}}|$  and the resultant folded graph  $\Gamma_{H^\phi}$  is unique). For  $n = 1$ , the path  $p$  consists of just edges  $e, f$ . Now suppose the first edge-folding occurs when edges  $d_1$  and  $d_2$  are merged into an edge  $d'$ , and denote the folding obtained after this identification as  $\Gamma'$ . By induction, there exists a path  $p'$  in  $\Gamma'$  connecting  $e$  and  $f$ . There are two cases to consider: either  $d'$  appears in  $p'$ , or it does not. In the first case, let  $p_1$  (resp.  $p_2$ ) be the path obtained by replacing  $d'$  with  $d_1$  (resp  $d_2$ ) in  $p'$ . It is clear that either  $p_1$  or  $p_2$  must fulfill the requirements of the lemma. In the second case, we simply take  $p = p'$ .  $\square$

**Lemma 2.** *Let  $H, K \leq_{f.g.} F_2$  be subgroups which satisfy  $\delta(H, K) > \frac{1}{2}$  (and hence are a potential counterexample to the strengthened Hanna Neumann conjecture) and take  $*$ ,  $\circ$  to be two length-preserving automorphisms of  $F_2$  whose values differ on  $\mu(H, K)$ , i.e.  $\mu(H, K)^* \neq \mu(H, K)^\circ$ . Then the groups  $H^\circ, K^*$  must satisfy the strengthened Hanna Neumann conjecture.*

*Proof.* Set  $x_0 = \mu(H, K)$ . Since  $\delta(H, K) > \frac{1}{2}$ , it follows that

$$\begin{aligned} C_{x_0}(\Gamma_H) &> \frac{1}{2}d_3(\Gamma_H) \\ C_{x_0}(\Gamma_K) &> \frac{1}{2}d_3(\Gamma_K). \end{aligned}$$

In light of Remark 3,  $\Gamma_{K^*}$  is the same graph as  $\Gamma_K$ , except that all  $a$  edges have been relabelled as  $a^*$ , and  $b$  edges have been relabelled as  $b^*$ , and an

analogous statement is true about the relationship between  $\Gamma_{H^\circ}$  and  $\Gamma_H$ . So

$$\begin{aligned} C_x(\Gamma_K) &= C_{x^*}(\Gamma_{K^*}) \\ C_x(\Gamma_H) &= C_{x^\circ}(\Gamma_{H^\circ}) \end{aligned}$$

for all  $x \in \{a^\pm, b^\pm\}$ . It follows that

$$C_{x_0^*}(\Gamma_{K^*}) = C_{x_0}(\Gamma_K) > \frac{1}{2}d_3(\Gamma_K) = \frac{1}{2}d_3(\Gamma_{K^*}), \quad (2)$$

$$C_{x_0^\circ}(\Gamma_{H^\circ}) = C_{x_0}(\Gamma_H) > \frac{1}{2}d_3(\Gamma_H) = \frac{1}{2}d_3(\Gamma_{H^\circ}). \quad (3)$$

Since  $x_0^\circ \neq x_0^*$ , it follows that  $\delta_x(H^\circ, K^*) < \frac{1}{2}$  for every  $x \in \{a, b\}^\pm$ . Thus,  $\delta(H^\circ, K^*) < \frac{1}{2}$ , and hence by Remark 2 the groups  $H^\circ, K^*$  cannot be a counterexample to the strengthened Hanna Neumann conjecture.  $\square$

The proofs of **Theorems 1 and 2** (see page 2) now follow from Lemma 2.

*Proof. (Theorem 1)* Suppose  $H, K$  do not satisfy the strengthened conjecture. By remark 2 we have  $\delta(H, K) > \frac{1}{2}$ . By definition of  $\tau$  we have  $\delta(H^\tau, K) = \delta(H, K)$ . We apply Lemma 2 to  $H^\tau, K$ , taking  $\circ$  to be the fixed-point-free length-preserving automorphism  $\sigma$ , and  $*$  to be the identity automorphism. The theorem follows.  $\square$

*Proof. (Theorem 2)* Suppose  $X = \{a_1, \dots, a_n\}$ . We consider the embedding  $\psi : F(X) \rightarrow F_2 = F(a_1, a_2)$  defined by  $\psi : a_i \mapsto a_1^i a_2 a_1^i$ . If  $H, K \leq_{\text{f.g.}} F(X)$  are a counterexample to the strengthened conjecture, then so are  $H^\psi, K^\psi \leq_{\text{f.g.}} F_2$ . Let  $-$  be the automorphism of  $F(X)$  given by  $a_i \mapsto a_i^{-1}$  (for each  $a_i \in X$ ). Restricting  $-$  to  $F_2$  and applying the previous lemma, we see that either  $H^\psi, K^\psi$  or  $(H^\psi)^-, K^\psi$  must satisfy the strengthened conjecture. But  $(H^-)^\psi = (H^\psi)^-$ ; here we think of  $H^\psi$  as a subgroup of  $F(X)$  under the canonical inclusion of  $F_2$  into  $F(X)$ . It follows that either  $H^\psi, K^\psi$  or  $(H^-)^\psi, K^\psi$  must satisfy the strengthened conjecture. Since  $\psi$  is a monomorphism, this implies that either  $H, K$  or  $H^-, K$  must satisfy the strengthened conjecture.  $\square$

Now towards the proof of Theorem 3, we introduce the following definition:

**Definition 2.** We say  $\phi : F_2(a, b) \rightarrow F_2(a, b)$  is a  $N$ -endomorphism if it has the property that  $U_\phi = \{a^\phi, b^\phi\}$  is  $N$ -reduced [10, pp.6], which is to say that every triple  $v_1, v_2, v_3$  in  $U_\phi^\pm$  satisfies

$$(N0) \ v_1 \neq 1,$$

$$(N1) \ v_1 v_2 \neq 1 \text{ implies } |v_1 v_2| \geq |v_1| + |v_2|,$$

$$(N2) \ v_1 v_2, v_2 v_3 \neq 1 \text{ implies } |v_1 v_2 v_3| > |v_1| + |v_2| + |v_3|.$$

**Remark 4.** It is well-known [10, pp.7] that if subset  $U$  of a free group  $F$  satisfies N0-N2, then one may associate with each  $u \in U$  words  $a(u), m(u) \in F$  with  $m(u) \neq 1$  such that  $u = a(u)m(u)a(u^{-1})^{-1}$  in  $F$  and having the property that for any  $w = u_1 \cdots u_t$ ,  $t \geq 0$ ,  $u_i \in U^\pm$  where  $u_i u_{i+1} \neq 1$ , the subwords  $m(u_1), \dots, m(u_t)$  remain uncanceled in the reduced form of  $w$ .

**Lemma 3.** *Every N-endomorphism of  $F_2$  is a monomorphism.*

*Proof.* Take  $w \in F_2$ , with  $w \neq 1$ . By (N2),  $|(w^\phi)^3| > 0$ , hence  $(w^\phi)^3 \neq 1$ . It follows that  $w^\phi \neq 1$ .  $\square$

**Lemma 4.** *Given  $H \leq_{f.g.} F_2$  and an N-endomorphism  $\phi$  of  $F_2$ , then for every edge  $e$  in  $\Gamma_H$ , at least one edge from the image of  $e$  under  $\phi$  does not get folded during the folding process  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ .*

*Proof.* Let  $e = (u, v)$  be any edge of  $\Gamma_H$ ; suppose  $e$  is labelled by  $x \in \{a, b\}^\pm$ . Consider the path  $\phi(e)$  in  $\phi(\Gamma_H)$ ; this path consists of a sequence of edges labelled by successive letters of  $x^\phi$ . Since  $\phi$  is an N-endomorphism,  $\{a^\phi, b^\phi\}$  is N-reduced and Remark 4 applies. Accordingly, let  $\bar{e}$  be the edge in  $\phi(e)$  which corresponds to the first letter of  $m(x^\phi)$  inside  $x^\phi$ . We claim that  $\bar{e}$  does not get folded during the folding process  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ .

Suppose towards contradiction, that  $\bar{e}$  gets folded with some edge  $f$  during the folding process  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ . Then by Lemma 1, there must exist a non-backtracking path  $p$  in  $\phi(\Gamma_H)$  beginning at  $\bar{e}$  and ending at  $f$ , with the property that every edge in  $p$  was folded during the folding process. Since  $p$  is a non-backtracking path in  $\phi(\Gamma_H)$ , it is a subpath of  $\phi(q)$  for some non-backtracking path  $q$  in  $\Gamma_H$ . It follows that the labels along  $\phi(q)$  are a word  $u_1 \cdots u_t$ ,  $t \geq 0$ ,  $u_i \in \{a^\phi, b^\phi\}^\pm$  and  $u_i u_{i+1} \neq 1$ . Since  $\bar{e}$  is labelled by the first letter of  $m(x^\phi)$ , by Remark 4 the edge  $\bar{e}$  was *not* folded during the folding process; this is a contradiction.  $\square$

We introduce the following notations: let  $\Gamma_H = (V_H, E_H)$  be the folding of  $H$ . Take any vertex  $v \in V_H$ , and let  $E_v$  be the edges incident to  $v$ . Define  $\Gamma_v$  to be the tree subgraph of  $\Gamma_H$  induced by edges  $E_v$ . Then  $\phi(\Gamma_v)$  is also a tree.

By Lemma 4, we may associate to each edge  $e \in E_v$ , an edge  $m(e) \in E_{\phi(\Gamma_H)}$  which does not get folded during the folding process  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ . We define  $\text{tr}\phi(\Gamma_v)$  to be the graph obtained by truncating the branches of  $\phi(\Gamma_v)$  so that they terminate with edges  $m(e)$ ,  $e \in E_v$ . Clearly, for all  $v \in V_H$ ,  $\text{tr}\phi(\Gamma_v)$  is a subgraph of  $\phi(\Gamma_H)$ .

The next lemma shows that N-endomorphisms do not cause large-scale disturbances in the neighborhood of branch vertices.

**Lemma 5.** *Given  $H \leq_{f.g.} F_2$  and an N-endomorphism  $\phi$  of  $F_2$ , then during the folding process  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ , no edge from  $\text{tr}\phi(\Gamma_v)$  gets folded with an edge from outside  $\text{tr}\phi(\Gamma_v)$ .*

*Proof.* Suppose, towards contradiction, that an edge  $e$  inside  $\text{tr}\phi(\Gamma_v)$  and an edge  $f$  outside  $\text{tr}\phi(\Gamma_v)$  get folded during the folding process  $\phi(\Gamma_H) \rightsquigarrow \Gamma_{H^\phi}$ .



Then by Lemma 1, there exists a path  $p$  beginning at  $e$  and ending at  $f$  with the property that every edge in  $p$  was folded during the folding process. Then  $p$  must pass through some edge  $m(e)$ ,  $e \in E_v$ . This contradicts the properties of  $m(e)$  as determined in Lemma 4. It follows that no edge inside  $\text{tr}\phi(\Gamma_v)$  gets folded with an edge outside  $\text{tr}\phi(\Gamma_v)$ .  $\square$

Informally stated, the previous lemma implies that for an  $N$ -endomorphism  $\phi$  and subgroup  $H \leq_{f.g.} F_2$ , the 5-tuple of values

$$C_a(\Gamma_{H\phi}), C_b(\Gamma_{H\phi}), C_{a^{-1}}(\Gamma_{H\phi}), C_{b^{-1}}(\Gamma_{H\phi}), d_4(\Gamma_{H\phi})$$

is completely determinable from the 5-tuple of values

$$C_a(\Gamma_H), C_b(\Gamma_H), C_{a^{-1}}(\Gamma_H), C_{b^{-1}}(\Gamma_H), d_4(\Gamma_H),$$

without knowledge of any further structure (e.g. the generating set) of  $H$ .

**Lemma 6.** *Let  $\phi_0 : F_2 \rightarrow F_2$  be the endomorphism defined by  $\phi_0 a = a^2$  and  $\phi_0 b = [a, b]$ . Then for any finitely generated subgroup  $H \leq_{f.g.} F_2$ ,  $C_{b^{-1}}(\Gamma_{H\phi_0}) = d_3(\Gamma_{H\phi_0})$ , and  $\text{rank}(H^{\phi_0}) = \text{rank}(H)$ .*

*Proof.* It is straightforward to check that  $\{a^2, [a, b]\}$  is  $N$ -reduced, and hence  $\phi_0$  is an  $N$ -endomorphism. By Lemma 5 it suffices to consider the effect of  $\phi_0$  on the various types of branch vertices in  $\Gamma_H$ . Figure 3 depicts how  $\phi_0$  transforms

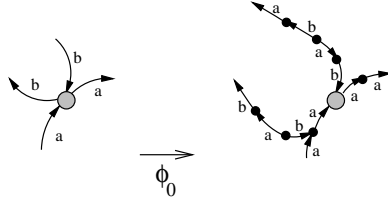


Figure 3: The effect of  $\phi_0$  on a vertex of degree 4.

the neighborhood of degree 4 vertex  $v$  to produce two vertices of type  $C_{b^{-1}}$ . The effect of  $\phi_0$  on each of the four types of degree 3 vertices may be determined simply by restricting our consideration to appropriate subgraphs of the depicted neighborhood. It follows that  $\phi_0$  transforms any degree 3 vertex in  $\Gamma_H$  into a vertex of type  $C_{b^{-1}}$  in  $\Gamma_{H\phi_0}$ . Thus,

$$C_{b^{-1}}(\Gamma_{H\phi_0}) = d_3(\Gamma_H) + 2d_4(\Gamma_H).$$

Since all branch vertices of  $\Gamma_H$  are seen to produce vertices of type  $C_{b^{-1}}$ , we get that  $C_{b^{-1}}(\Gamma_{H\phi_0}) = d_3(\Gamma_{H\phi_0})$ . Finally, by Lemma 3,  $\phi_0$  is a monomorphism, so  $\text{rank}(H^{\phi_0}) = \text{rank}(H)$ .  $\square$

The proof of **Theorem 3** (see page 4) now follows from Lemma 6.

*Proof. (Theorem 3)* Fix the embedding  $\psi : F(X) \rightarrow F_2 = F(a, b)$  defined by  $\psi : a_i \mapsto a^i b a^i$ . Put  $H' = (H^\psi)^{\phi_0}$  and  $K' = (K^\psi)^{\phi_0}$ , where  $\phi_0$  is the endomorphism of  $F(a, b)$  defined by  $a^{\phi_0} = a^2$  and  $b^{\phi_0} = [a, b]$ . Then by Lemma 6,

$$\begin{aligned} C_{b^{-1}}(\Gamma_{H'}) &= d_3(\Gamma_{H'}) \\ C_{b^{-1}}(\Gamma_{K'}) &= d_3(\Gamma_{K'}), \end{aligned}$$

and hence  $\delta(H', K') = 1$ . □

**Remark 5.** The following somewhat simpler proof for Theorem 3 was communicated to us by W. Neumann: Consider an  $N$ -endomorphism  $\phi_1 : F_2 \rightarrow F_2$  with the additional property that the graph for  $\phi_1(F_2)$  has only type  $b$  vertices (e.g.  $a \mapsto b^{-1}aba$  and  $b \mapsto b^{-1}aba^{-1}$ ). If  $H$  is any subgroup of  $F_2$  then the graph for  $\phi_1(H)$  is a subgraph of a covering graph of the graph for  $\phi_1(F_2)$ , so it also has only type  $b$  vertices, hence  $H' = (H^\psi)^{\phi_1}$  and  $K' = (K^\psi)^{\phi_1}$  satisfy  $\delta(H', K') = 1$  for all  $H, K \leq F(X)$ .

Corollary 1 follows immediately from Theorem 3, since  $\phi_0, \phi_1$  and  $\psi$  are monomorphisms and hence  $\text{rank}(H) = \text{rank}(H')$ ,  $\text{rank}(K) = \text{rank}(K')$ .

## 4 Acknowledgements

We are grateful to thank W. Neumann and A. Magidin for their insightful comments and suggestions. The first two authors are grateful to the Mathematics department at the City University of New York Graduate Center for funding this work as part of their ongoing doctoral research. The second author thanks both the Center for Computational Sciences at the Naval Research Laboratory in Washington DC, and the Advanced Engineering and Sciences division of ITT Industries for their support of these endeavors.

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