Subgroups of fully residually free groups: algorithmic problems

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Abstract. In [**16**] we introduced graph-theoretic techniques for finitely generated subgroups of $F^{\mathbb{Z}[t]}$ and solved effectively the membership problem in finitely generated fully residually free groups. In the present paper we prove that finitely generated fully residually free groups satisfy Howson property and show how one can effectively find the intersection of two finitely generated subgroups, we solve the conjugacy problem, the malnormality problem, and provide an algorithm to compute ranks of centralizers.

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1. Introduction

Finitely generated fully residually free groups play a crucial role in the theory of equations and first-order formulas over a free group. It is remarkable that these groups, which have been widely studied before, turn out to be the basic objects in newly developing areas of algebraic geometry and model theory of free groups. Recall that a group G is called *fully residually free* (or *freely discriminated* [**1**], or ω*-residually free* [**18**]) if for any finitely many non-trivial elements $g_1, \ldots, g_n \in G$

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there exists a homomorphism ϕ of G into a free group F, such that $\phi(g_i) \neq 1$ for $i = 1, \ldots, n$.

Studying equations in free groups Lyndon introduced in [**12**] the notion of a group with parametric exponents in an associative unitary ring. In particular, he described free exponential groups $F^{\mathbb{Z}[t]}$ over the ring of integer polynomials $\mathbb{Z}[t]$ and showed that these groups are fully residually free. In [**18**] Remeslennikov established a connection between residual properties of groups and their universal theories, namely, he proved that a finitely generated group H is fully residually free if and only if H has exactly the same universal theory as F. It follows then immediately from the Lyndon's result, that all finitely generated subgroups of $F^{\mathbb{Z}[t]}$ have the same universal theory as F. This once more emphasized the role of Lyndon's group $F^{\mathbb{Z}[t]}$ in the investigation of the elementary theory of F. A modern treatment of exponential groups is contained in [**13**], where Myasnikov and Remeslennikov proved that the group $F^{\mathbb{Z}[t]}$ can be obtained from F by an infinite chain of HNN-extensions of a very specific type, so-called *extensions of centralizers*. This implies that finitely generated subgroups of $F^{\mathbb{Z}[t]}$ are, in fact, subgroups of fundamental groups of graphs of groups of a very particular type, hence one can apply Bass-Serre theory to describe the structure of these subgroups. For instance, it is routine now to show that all such subgroups are finitely presented [**11**] (see, also [**17**] for another proof of this result). Exploiting relations between HNN-extensions and length functions it has been shown in [**14**] that the group $F^{\mathbb{Z}[t]}$ has a free Lyndon's length function with values in $\mathbb{Z}[t]$, thus finitely generated subgroups of $F^{\mathbb{Z}[t]}$ act freely on $\mathbb{Z}^n$-trees, hence on $\mathbb{R}^n$-trees. Recently, Giurardel proved this result independently using different techniques [**7**].

In [**11**] Kharlampovich and Myasnikov proved the converse of the Lyndon's result mentioned above, namely, they showed that every finitely generated fully residually free group is embeddable into $F^{\mathbb{Z}[t]}$. This provides a complete description of finitely generated fully residually free groups and gives a lot of information about their algebraic structure. In particular, all these groups, except for abelian and surface groups, have a non-trivial cyclic JSJ-decomposition.

A new technique to deal with $F^{\mathbb{Z}[t]}$ became available recently when Myasnikov, Remeslennikov, and Serbin showed that elements of this group can be viewed as reduced *infinite* words in the generators of F [**15**]. It turned out that many algorithmic problems for finitely generated fully residually free groups can be solved by the same methods as in the standard free groups. Indeed, in [**16**] an analog of the Stallings' folding was introduced for an arbitrary finitely generated subgroup of $F^{\mathbb{Z}[t]}$, which allows one to solve effectively the membership problem in $F^{\mathbb{Z}[t]}$, as well as in an arbitrary finitely generated subgroup of it. Following [**8**] and [**16**] we further develop this method here, focusing mostly on its algorithmic aspects.

In this paper we solve some principal algorithmic problems for subgroups of a fully residually free group G. In Section 3 we show that G satisfies *Howson property*: the intersection of two finitely generated subgroups H and K of G is finitely generated. Moreover, we show that the *Intersection Problem* is algorithmically decidable in G, i.e., for any finitely generated subgroups H and K of G (given by finite generating sets) one can effectively find a finite generating set of $H \cap K$. Furthermore, similar technique shows that one can find effectively the intersection of cosets of finitely generated subgroups of G. In Section 6 we prove that the *Conjugacy Problem* is decidable in G. Notice that this result also follows from [**6**] and [**5**].

Indeed, Dahmani showed in [**6**] that G is relatively hyperbolic and Bumagin proved in [**5**] that the Conjugacy Problem is decidable in relatively hyperbolic groups. In Section 4 we prove that for finitely generated subgroups H, K of G there are only finitely many conjugacy classes of intersections $H^g \cap K$ in G. Moreover, one can find a finite set of representatives of these classes effectively. This implies that one can effectively decide whether two finitely generated subgroups of G are conjugate or not, and check if a given finitely generated subgroup is malnormal in G. Observe, that the malnormality problem is decidable in free groups [**2**], but is undecidable in torsion-free hyperbolic groups - Bridson and Wise constructed corresponding examples in [**4**]. In Section 5 we provide an algorithm to find the centralizers of finite sets of elements in finitely generated fully residually free groups and compute their ranks. In particular, we prove that for a given finitely generated fully residually free group G the *centralizer spectrum* $Spec(G) = \{rank(C) \mid C = C_G(g), g \in G\}$, where $rank(C)$ is the rank of a free abelian group C, is finite and one can find it effectively.

2. Preliminaries

Here we introduce basic definitions and notations which are to be used throughout the whole paper. For more details see [**15, 16**].

2.1. Lyndon's free $\mathbb{Z}[t]$-group and infinite words. Let $F = F(X)$ be a free non-abelian group with basis X and $\mathbb{Z}[t]$ be a ring of polynomials with integer coefficients in a variable t. In [**12**] Lyndon introduced a $\mathbb{Z}[t]$-*completion* $F^{\mathbb{Z}[t]}$ of F, which is called now the Lyndon's free $\mathbb{Z}[t]$-group.

It turns out that $F^{\mathbb{Z}[t]}$ can be described as a union of a sequence of extensions of centralizers [**13**]

$$F = G_0 < G_1 < \cdots < G_n < \cdots , \tag{1}$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i by a free abelian group of countable rank.

In [**15**] it was shown that elements of $F^{\mathbb{Z}[t]}$ can be viewed as *infinite words* defined in the following way. Let A be a discretely ordered abelian group. By 1_A we denote the minimal positive element of A. Recall that if $a, b \in A$ then the closed segment $[a, b]$ is defined as

$$[a, b] = \{x \in A \mid a \le x \le b\}.$$

Let $X = \{x_i \mid i \in I\}$ be a set. An A-*word* is a function of the type

$$w : [1_A, \alpha_w] \to X^{\pm},$$

where $\alpha_w \in A,\ \alpha_w \geqslant 0$. The element α_w is called the *length* $|w|$ of w. By ε we denote the empty word. We say that w is *reduced* if $w(\alpha) \neq w(\alpha+1)^{-1}$ for any $1 \le \alpha < \alpha_w$. Then, as in a free group, one can introduce a partial multiplication $*$, an inversion, a word reduction etc., on the set of all A-words (infinite words) $W(A, X)$. We write $u \circ v$ instead of uv if $|uv| = |u| + |v|$. All these definitions make it possible to develop infinite words techniques, which provide a very convenient combinatorial tool (for all the details we refer to [**15**]).

It was proved in [**15**] that $F^{\mathbb{Z}[t]}$ can be canonically embedded into the set of reduced infinite words $R(\mathbb{Z}[t], X)$, where $\mathbb{Z}[t]$, an additive group of polynomials with integer coefficients, is viewed as an ordered abelian group with respect to the standard lexicographic order $\leqslant$ (that is, the order which compares the degrees of polynomials first, and if the degrees are equal, compares the coefficients of corresponding terms starting with the terms of highest degree). More precisely, the embedding of $F^{\mathbb{Z}[t]}$ into $R(\mathbb{Z}[t], X)$ was constructed by induction, that is, all G_i from the series (1) were embedded step by step in the following way. Suppose, the embedding of G_i into $R(\mathbb{Z}[t], X)$ is already constructed. Then, one chooses a *Lyndon's set* $U_i \subset G_i$ (see [**15**]) and the extension of cenralizers of all elements from U_i produces G_{i+1}, which is now also naturally embedded into $R(\mathbb{Z}[t], X)$.

The existence of an embedding of $F^{\mathbb{Z}[t]}$ into the set of infinite words implies automatically the fact that all subgroups of $F^{\mathbb{Z}[t]}$ are also subsets of $R(\mathbb{Z}[t], X)$, that is, their elements can be viewed as infinite words. From now on we assume the embedding $\rho : F^{\mathbb{Z}[t]} \longrightarrow R(\mathbb{Z}[t], X)$ to be fixed. Moreover, for simplicity we identify $F^{\mathbb{Z}[t]}$ with its image $\rho(F^{\mathbb{Z}[t]})$.

2.2. Reduced forms for elements of $F^{\mathbb{Z}[t]}$. Following [**15**] and [**16**] we introduce various normal forms for elements in $F^{\mathbb{Z}[t]}$ in the following way.

We may assume that the set

$$U = \bigcup_i U_i$$

is well-ordered. Let

$$U_i = \{u_{i_1}, u_{i_2}, \ldots\} \subset G_i,$$

be enumeration of elements of U_i in increasing order. Denote by I_i the set of indices $i_1, i_2, \ldots$ of elements from U_i. Now $g \in G_{n+1} - G_n$ has the following representation as a reduced infinite word:

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}, \tag{2}$$

where $n_1, n_2, \ldots, n_l \in I_n$, $g_k \in G_n$, $k \in [1, l+1]$, $[g_k, u_{n_k}] \neq \varepsilon$, $[g_{k+1}, u_{n_k}] \neq \varepsilon$, $k \in [1, l]$, $|\alpha_k| >> 0$, $k \in [1, l]$ (recall that $\alpha >> 0$ if $\alpha \in \mathbb{Z}[t] - \mathbb{Z}$). Representation (2) is called U_n*-reduced* if the ordered l-tuple $\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_l|\}$ is maximal with respect to the right lexicographic order among all possible such representations of g.

From (2) one can obtain another representation of g. Fix any u from the list $u_{n_1}, u_{n_2}, \ldots, u_{n_l}$. Then

$$g = h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}, \tag{3}$$

where $\beta_j = \alpha_{m_j}, m_j \in [1, l], j \in [1, p]$, $h_1 = g_1 \circ u_{n_1}^{\alpha_1} \circ \cdots \circ g_{m_1}$, $h_{p+1} = g_{m_p+1} \circ \cdots \circ g_{l+1}$, $h_k = g_{m_k+1} \circ \cdots \circ g_{m_{k+1}}, k \in [2, p]$. Representation (3) is called a u*-representation* or a u*-form* of g. In other words, to obtain a u-form one has to "mark" in (2) only nonstandard exponents of u. Representation (3) is called u*-reduced* if the ordered p-tuple $\{|\beta_1|, |\beta_2|, \ldots, |\beta_p|\}$ is maximal with respect to the right lexicographic order among all possible u-forms of g.

Observe that if (3) is a u-form for g and g is cyclically reduced then obviously

$$(h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}) \circ (h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}) \tag{4}$$

is a u-form for g^2. So, we call (3) *cyclically* u*-reduced* if (4) is u-reduced.

LEMMA 1. [**16**] *For any given u-reduced form of $g \in G_{n+1} - G_n, u \in U_n$, there exists a cyclic permutation of g such that its u-reduced form is cyclically u-reduced.*

Let $g \in G_{n+1} - G_n$ have a U_n-reduced form

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1},$$

where $u_{n_1}, u_{n_2}, \ldots, u_{n_l} \in U_n$, $g_k \in G_n$, $k \in [1, l+1]$, $[g_k, u_{n_k}] \neq \varepsilon$, $[g_{k+1}, u_{n_k}] \neq \varepsilon$, $|\alpha_k| >> 0$, $k \in [1, l]$. Now, recursively one has a U_{n-1}-reduced form for g_i

$$g_i = g(i)_1 \circ u_{m_1}^{\beta_{m_1}} \circ g(i)_2 \circ \cdots \circ u_{m_s}^{\beta_{m_s}} \circ g(i)_s,$$

where $u_{m_1}, \ldots, u_{m_s} \in U_{n-1}$, $|\beta_{m_k}| >> 0, k \in [1, s]$, $g(i)_k \in G_{n-1}, k \in [1, s+1]$ and one can get down to the free group F with such a decomposition of g, where step by step subwords between nonstandard powers of elements from U_i are presented as U_{i-1}-forms, $i \in [1, n]$. Thus, from this decomposition one can form the following series for g:

$$F < H_{0,1} < H_{0,2} < \cdots < H_{0,k(0)} < H_{1,1} < \cdots < H_{1,k(1)} < \ldots \tag{5}$$
$$\ldots < H_{n-1,k(n-1)} < H_{n,1} < \ldots < H_{n,k(n)},$$

where $H_{j,1}, \ldots, H_{j,k(j)}$ are subgroups of G_{j+1}, which do not belong to G_j and $H_{j,i}$ is obtained from $H_{j,i-1}$ by a centralizer extension of a single element $u_{j,i-1} \in H_{j,i-1} < G_j$. Element g belongs to $H_{n,k(n)}$ and does not belong to the previous terms. Series (5) is called an *extension series* for g.

Using the extension series above we can decompose g in the following way: $g \in H_{n,k(n)}$ has a $u_{n,k(n)}$-reduced form

$$g = h_1 \circ u_{n,k(n)}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n,k(n)}^{\beta_l} \circ h_{l+1},$$

where all $h_j, j \in [1, l+1]$ in their turn are $u_{n,k(n)-1}$-reduced forms representing elements from $H_{n,k(n)-1}$. This gives one a decomposition of g related to its extension series. We call this decomposition a *standard decomposition* or a *standard representation* of g.

Observe that for any $g \in F^{\mathbb{Z}[t]}$, its standard decomposition can be viewed as a finite product $b_1 b_2 \cdots b_m$, where

$$b_i \in B = \{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t] - \mathbb{Z}\}.$$

We denote this product by $\pi(g)$ so we have

$$\pi(g) = \pi(h_1)\ u_{n,k(n)}^{\beta_1}\ \pi(h_2)\ \cdots\ u_{n,k(n)}^{\beta_l}\ \pi(h_{l+1}),$$

where $\pi(h_i)$ is a finite product in the alphabet B corresponding to h_i, and from now on, by a standard decomposition of an element g we understand not the representation of g as a reduced infinite word but the finite product $\pi(g)$.

By $U(g)$ we denote a finite subset of U such that if $\pi(g)$ contains a letter $b_i \in B$ such that $b_i = u^\alpha$ then $u \in U(g)$. Observe that $U(g)$ is ordered with an order induced from U, so we have

$$U(g) = \{u_1, \ldots, u_m\},$$

where $u_i < u_j$ if $i < j$ and $u_m = u_{n,k(n)}$. By $\max\{U(g)\}$ we denote the maximal element of $U(g)$.

If $u \in U(g)$ then by $deg_u(g)$ we denote the maximal degree of infinite exponents of u, which appear in $\pi(g)$.

It is easy to see that in general $\pi(g_1 \circ g_2) \neq \pi(g_1)\pi(g_2)$ and $\pi(g \circ g) = \pi(g)\pi(g)$ if and only if the u-reduced form of g is cyclically u-reduced, where $u = \max\{U(g)\}$.

From the definition of a Lyndon's set and the results of [**15**] it follows that if $R \subset G_n$ is a Lyndon's set then a set R' obtained from R by cyclic decompositions of its elements is also a Lyndon's set. Thus, by Lemma 1 we can assume a w-reduced form of any $u \in U_n$ to be cyclically w-reduced, where $w = \max\{U(u)\}$. Hence, we can assume

$$\pi(u \circ u) = \pi(u)\pi(u)$$

for any $u \in U$.

2.3. Embedding theorems. There are three results which play an important role in this paper. The first embedding theorem is due to Kharlampovich and Myasnikov.

THEOREM 1 (The first embedding theorem ([**11**])). *Given a finite presentation of a finitely generated fully residually free group G one can effectively construct an embedding $\phi : G \to F^{\mathbb{Z}[t]}$ (by specifying the images of the generators of G).*

Combining Theorem 1 with the result on the representation of $F^{\mathbb{Z}[t]}$ as a union of a sequence of extensions of centralizers one can get the following theorem.

THEOREM 2 (The second embedding theorem). *Given a finite presentation of a finitely generated fully residually free group G one can effectively construct a finite sequence of extension of centralizers*

$$F < G_1 < \ldots < G_n,$$

where G_{i+1} is an extension of the centralizer of some element $u_i \in G_i$ by an infinite cyclic group $\mathbb{Z}$, and an embedding $\psi^* : G \to G_n$ (by specifying the images of the generators of G).

Combining Theorem 1 with the result on the effective embedding of $F^{\mathbb{Z}[t]}$ into $R(\mathbb{Z}[t], X)$ obtained in [**15**] one can get the following theorem.

THEOREM 3 (The third embedding theorem). *Given a finite presentation of a finitely generated fully residually free group G one can effectively construct an embedding $\psi : G \to R(\mathbb{Z}[t], X)$ (by specifying the images of the generators of G).*

2.4. Graphs labeled by infinite $\mathbb{Z}[t]$-words. By an *$(\mathbb{Z}[t], X)$-labeled directed graph* (*$(\mathbb{Z}[t], X)$-graph*) Γ we understand a combinatorial graph Γ where every edge has a direction and is labeled either by a letter from X or by an infinite word $u^\alpha \in F^{\mathbb{Z}[t]}, u \in U, \alpha \in \mathbb{Z}[t], \alpha > 0$, denoted $\mu(e)$.

For each edge e of Γ we denote the origin of e by $o(e)$ and the terminus of e by $t(e)$.

For each edge e of $(\mathbb{Z}[t], X)$-graph we can introduce a formal inverse e^{-1} of e with the label $\mu(e)^{-1}$ and the endpoints defined as $o(e^{-1}) = t(e), t(e^{-1}) = o(e)$, that is, the direction of e^{-1} is reversed with respect to the direction of e. For the new edges e^{-1} we set $(e^{-1})^{-1} = e$. The new graph, endowed with this additional structure we denote by $\widehat{\Gamma}$. Usually we will abuse the notation by disregarding the difference between Γ and $\widehat{\Gamma}$.

A *path* p in Γ is a sequence of edges $p = e_1 \cdots e_k$, where each e_i is an edge of Γ and the origin of each e_i is the terminus of e_{i-1}. Observe that $\mu(p) = \mu(e_1) \ldots \mu(e_k)$ is a word in the alphabet $\{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t]\}$ and we denote by $\overline{\mu(p)}$ a reduced infinite word $\mu(e_1) * \cdots * \mu(e_k)$ (this product is always defined).

A path $p = e_1 \cdots e_k$ in Γ is called *reduced* if $e_i \neq e_{i+1}^{-1}$ for all $i \in [1, k-1]$.

A path $p = e_1 \cdots e_k$ in Γ is called *label reduced* if

1) p is reduced;
2) if $e_{k_1} \cdots e_{k_2},\ k_1 \leq k_2$ is a subpath of p such that $\mu(e_i) = u^{\alpha_i}, u \in U, \alpha_i \in \mathbb{Z}[t],\ i \in [k_1, k_2]$ and $\mu(e_{k_1-1}) \neq u^\beta, \mu(e_{k_2+1}) \neq u^\beta$ for any $\beta \in \mathbb{Z}[t]$, provided $k_1 - 1, k_2 + 1 \in [1, k]$, then $\alpha = \alpha_{k_1} + \cdots + \alpha_{k_2} \neq 0$ and $\mu(e_{k_1-1}) * u^\alpha = \mu(e_{k_1-1}) \circ u^\alpha, u^\alpha * \mu(e_{k_2+1}) = u^\alpha \circ \mu(e_{k_2+1})$.

A $(\mathbb{Z}[t], X)$-graph Γ is called *partially folded* if there are no edges e_1 and e_2 in Γ with $\mu(e_1) = \mu(e_2)$ such that $o(e_1) = o(e_2)$ or $t(e_1) = t(e_2)$.

In [**16**] *partial foldings* on $(\mathbb{Z}[t], X)$-graphs were introduced and the following result was proved.

PROPOSITION 1. [**16**] *Let Γ be a $(\mathbb{Z}[t], X)$-graph, which has only a finite number of edges. Then there exists a partially folded $(\mathbb{Z}[t], X)$-graph Δ, which can be obtained from Γ by a finite number of partial foldings.*

Let Γ be a $(\mathbb{Z}[t], X)$-graph and $u \in U$ be fixed. Vertices $v_1, v_2 \in V(\Gamma)$ are called *u-equivalent* (denoted $v_1 \sim_u v_2$) if there exists a path $p = e_1 \cdots e_k$ in Γ such that $o(e_1) = v_1, t(e_k) = v_2$ and $\mu(e_i) = u_i^\alpha, \alpha_i \in \mathbb{Z}[t], i \in [1, k]$. $\sim_u$ is an equivalence relation on vertices of Γ, so if Γ is finite then all its vertices can be divided into a finite number of pairwise disjoint equivalence classes. Suppose, $v \in V(\Gamma)$ is fixed. One can take the subgraph of Γ spanned by all the vertices which are u-equivalent to v and remove from it all edges with labels not equal to $u^\alpha, \alpha \in \mathbb{Z}[t]$. We denote the resulting subgraph of Γ by $Comp_u(v)$ and call a *u-component of v*. If $v \in V(\Gamma), v_0 \in V(Comp_u(v))$ then one can define a set

$$H_u(v_0) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } Comp_u(v) \text{ from } v_0 \text{ to } v_0\}.$$

LEMMA 2. [**16**] *Let Γ be a $(\mathbb{Z}[t], X)$-graph and $v \in V(\Gamma), v_0 \in V(Comp_u(v))$. Then*

(1) *$H_u(v_0)$ is a subgroup of $R(\mathbb{Z}[t], X)$;*
(2) *$H_u(v_0)$ is isomorphic to a subgroup of $\mathbb{Z}[t]$;*
(3) *if $Comp_u(v)$ is a finite graph, then $H_u(v_0)$ is finitely generated;*
(4) *if $v_1 \in V(Comp_u(v))$ then $H_u(v_0) \simeq H_u(v_1)$.*

Following [**16**] one can introduce operations on u-components which are called *u-foldings*. One of the most important properties of u-foldings is that they do not change subgroups associated with u-components.

LEMMA 3. [**16**] *Let Γ be a $(\mathbb{Z}[t], X)$-graph, $v \in V(\Gamma)$ and $C = Comp_u(v)$ be finite. Then there exist a $(\mathbb{Z}[t], X)$-graph Δ obtained from Γ by finitely many u-foldings such that $v' \in V(\Delta)$ corresponds to v and $C' = Comp_u(v')$ consists of a simple positively oriented path $P_{C'}$, and some edges that are not in $P_{C'}$ connecting some pairs of vertices in $P_{C'}$.*

C' in Lemma 3 is called a *reduced* u-component. Since $P_{C'}$ is a simple path there exists a vertex $z_{C'} \in V(P_{C'})$ which is an origin of only one positive edge in $P_{C'}$. $z_{C'}$ is called a *base-point* of C'.

It turns out that any finite reduced u-component C in a $(\mathbb{Z}[t], X)$-graph is characterized completely by the pair $(P_C, H_u(z_C))$ in the following sense. For any reduced path p in C there exists a unique reduced subpath q (denoted $q = [p]$) of P_C with the same endpoints as p, such that $\overline{\mu(p)} * \overline{\mu(q)}^{-1} \in H_u(z_C)$. Moreover, let $P_C = f_1 \cdots f_m$, where $o(f_1) = z_C, v_0 = z_C, v_i = t(f_i), i \in [1, m]$ and let $p_0, p_1, \ldots, p_m$ be reduced subpaths of P_C such that $o(p_i) = z_C, t(p_i) = v_i, i \in [0, m]$. The set of paths

$p_0, p_1, \ldots, p_m$ is called a *set of path representatives associated with* C (denoted by $Rep(C)$).

LEMMA 4. [**16**] *Let C be a finite reduced u-component in a $(\mathbb{Z}[t], X)$-graph Γ, $v \in V(C)$ and let $\alpha \in \mathbb{Z}[t]$. If $\overline{\mu(p_i)} * \overline{\mu(p_j)}^{-1} \notin H_u(z_C)$ for any $p_i, p_j \in Rep(C), i \neq j$ then either there exists a unique reduced path p in P_C such that $o(p) = v$ and $u^\alpha \in \overline{\mu(p)} * H_u(z_C)$ or there exists no path q in C with this property.*

If C is reduced and $Rep(C)$ satisfies the condition from Lemma 4 then we call C *a u-folded u-component.*

2.5. Languages associated with $(\mathbb{Z}[t], X)$-graphs. Let Γ be a $(\mathbb{Z}[t], X)$-graph and let v be a vertex of Γ. We define the language of Γ with respect to v as

$$L(\Gamma, v) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v\}.$$

LEMMA 5. [**16**] *Let Γ be a finite $(\mathbb{Z}[t], X)$-graph and let $v \in V(\Gamma)$. Then $L(\Gamma, v)$ is a subgroup of $F^{\mathbb{Z}[t]}$.*

LEMMA 6. [**16**] *Let Γ be a finite $(\mathbb{Z}[t], X)$-graph and let $v \in V(\Gamma)$. Let Δ_1 be a $(\mathbb{Z}[t], X)$-graph obtained from Γ by a single partial folding and let Δ_2 be a $(\mathbb{Z}[t], X)$-graph obtained from Γ by a single u-folding for some $u \in U$, so that $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$ correspond to v. Then*

$$L(\Gamma, v) = L(\Delta_1, v_1) = L(\Delta_2, v_2).$$

Let Γ be a $(\mathbb{Z}[t], X)$-graph and $p = e_1 \cdots e_k$ be a reduced path in Γ. Let $g \in G_{n+1} - G_n$ and let

$$\pi(g) = \pi(h_1) u^{\beta_1} \pi(h_2) \cdots u^{\beta_l} \pi(h_{l+1}),$$

be the standard decompostion of g, where $u = \max\{U(g)\}$. We write

$$\mu(p) = \pi(g)$$

if p can be subdivided into subpaths

$$p = p_1 d_1 p_2 \cdots d_l p_{l+1},$$

where d_i is a path in some u-component of Γ and p_i is a path in Γ which does not contain edges labeled by $u^\alpha, \alpha \in \mathbb{Z}[t]$, so that $\overline{\mu(d_i)} = u^{\beta_i}, i \in [1, l]$ and $\mu(p_i) = \pi(h_i), i \in [1, l+1]$ is defined recursively in the same way. Observe that if $g = x_1 \cdots x_r \in F$ then $\mu(p) = \pi(g)$ if $k = r$ and $\mu(e_i) = x_i$ for every $i \in [1, k]$.

Let Γ be a finite $(\mathbb{Z}[t], X)$-graph. Since Γ is finite, the set of elements $u \in U$ such that there exists an edge e in Γ labeled by $u^\alpha, \alpha \in \mathbb{Z}[t]$ is finite and ordered with the order induced from U. Thus one can associate with Γ an ordered set $U(\Gamma) = \{u_1, \ldots, u_N\}, N > 0, u_i \in U, u_i < u_j$ for $i < j$.

If $u \in U(\Gamma)$ then by $deg_u(\Gamma)$ we denote the maximal degree of infinite exponents of u, which are labels of edges in Γ, that is,

$$deg_u(\Gamma) = \max\{deg(\alpha) \mid \mu(e) = u^\alpha \text{ for some } e \in E(\Gamma)\}.$$

It is easy to see that $deg_u(\Gamma)$ is invariant under partial and U-foldings, and

$$deg_u(\Gamma) \geq \max_{g \in L(\Gamma, v)} \{deg_u(g)\}$$

for any $v \in V(\Gamma)$.

Let $u_i \in U(\Gamma)$ be fixed and $\Gamma(i)$ be a subgraph of Γ which consists only of edges $e \in E(\Gamma)$ such that either $\mu(e) = x \in X^{\pm}$ or $\mu(e) = u_j^{\alpha}, \alpha \in \mathbb{Z}[t], j \leq i$. $\Gamma(i)$ is called an *i-level graph of* Γ (by 0-level graph we understand a subgraph of Γ which consists only of edges with labels from X) and the *level* (denoted $l(\Gamma)$) of Γ is the minimal $n \in \mathbb{N}$ such that $\Gamma = \Gamma(n)$. Observe that $\Gamma(i)$ may not be connected for some $i < l(\Gamma)$, but still one can apply to $\Gamma(i)$ partial and u-foldings, $u \in U(\Gamma)$.

A finite connected $(\mathbb{Z}[t], X)$-graph Δ is called *U-folded* if for any $u_n \in U(\Delta)$ the following conditions are satisfied:

(i) Δ is partially folded;

(ii) all u_n-components of Δ are u_n-folded and isolated, that is, there exists no reduced path p with $\overline{\mu(p)} = u_n^k, k \in \mathbb{Z}$ in $\Delta(n-1)$ such that p connects two different u_n-components of Δ;

(iii) if C is a u_n-component of Δ, $e \in E(P_C)$ and $\mu(e) = u_n^k, k \in \mathbb{Z}$ then there exists a unique label reduced path p in $\Delta(n-1)$ such that $o(p) = o(e), t(p) = t(e), \mu(p) = \pi(u_n)^k$;

(iv) if C is a u_n-component of Δ and $v \in V(C) \cap V(\Delta(n-1))$ then there exists a unique label reduced path p in $\Delta(n-1)$ such that $o(p) = t(p) = v, \mu(p) = \pi(u_n)^k, k \in \mathbb{Z}$ and $H_{u_n}(v) \cap \langle u_n \rangle = \langle u_n^k \rangle$;

(v) if C is a u_n-component of Δ and $v_1, v_2 \in V(C)$ are connected by a reduced path p in P_C then either p consists only of edges labeled by finite exponents of u_n or there exists no number $k_p \in \mathbb{Z}$ such that $\overline{\mu(p)} * u^{-k_p} \in H_{u_n}(v_1)$;

(vi) for any u_n-component C of Δ and two of its vertices $v_1, v_2, v_1 \neq v_2$ which are joined by some path p in P_C with $o(p) = v_1, t(p) = v_2$ there exists no reduced path r in $\Delta(n-1)$ such that $o(r) = v_1, t(r) = v_2, \overline{\mu(r)} = u_n^k, k \in \mathbb{Z}$ and $\overline{\mu(p)} * \overline{\mu(r)}^{-1} \notin H_{u_n}(v_1)$;

(vii) for any u_n-component C of Δ, its vertex v and a reduced path p in $\Delta(n-1)$ such that $o(p) = v, \overline{\mu(p)} = u_n^k, k \in \mathbb{Z}$ it follows that $t(p) \in V(C)$;

(viii) **(a)** for any u_n-component C of Δ, its vertex v and a label reduced path p in $\Delta(n-1)$ such that $o(p) = v, \overline{\mu(p)} = w, w = u_n^{\delta} \circ c, \delta \in \{1, -1\}$, there exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v, t(q) = t(q_2) = t(p), \mu(q_1) = \pi(u_n)^{\delta}, t(q_1) \in V(C)$;

(b) for any u_n-component C of Δ, its vertex v and a label reduced path p in $\Delta(n)$ such that $p = z_1 z_2, o(p) = v, z_1 \in \Delta(n-1), \overline{\mu(z_1)} = w_1, \overline{\mu(z_2)} = w_2 \circ c = u_n^{\gamma} \circ c_1, u_n^{\delta} = w_1 \circ w_2,$, there exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v, t(q) = t(q_2) = t(p), \mu(q_1) = \pi(u_n)^{\delta}, t(q_1) \in V(C)$;

(c) for any u_n-component C of Δ, its vertex v and a path p in $\Delta(n-1)$ such that $o(p) = v, \overline{\mu(p)} = w_1, u_n^{\delta} = w_1 \circ w_2, w_2 \neq \varepsilon, \delta \in \{1, -1\}$, if there exists an edge e' in C such that $o(e') = v, \mu(e') = u^{\gamma}, \gamma\delta > 0$ then there exists a label reduced path p' in $\Delta(n-1)$ such that $o(p') = v, \overline{\mu(p')} = u_n^{\delta}, t(p') \in V(C)$ and p is an initial subpath of p';

(ix) for any reduced path p in Δ with $\overline{\mu(p)} = w$ there exists a unique label reduced path q such that $o(q) = o(p), t(q) = t(p), \mu(q) = \pi(w)$;

(x) for the standard decomposition $\pi(g)$ of any $g \in F^{\mathbb{Z}[t]}$ and any $v \in V(\Delta)$ either there exists a unique label reduced path p in Δ starting at v such that $\mu(p) = \pi(g)$ or for any path q in Δ starting at v it follows that $\overline{\mu(q)} \neq g$.

PROPOSITION 2. [**16**] *Let Γ be a finite connected $(\mathbb{Z}[t], X)$-graph. Then there exists a U-folded $(\mathbb{Z}[t], X)$-graph Δ, which is obtained from Γ by a finite sequence of partial and u-foldings. Moreover Δ can be found effectively.*

PROPOSITION 3. [**16**] *Let H be a finitely generated subgroup of $F^{\mathbb{Z}[t]}$. Then there exists a U-folded $(\mathbb{Z}[t], X)$-graph Γ and a vertex v of Γ such that $L(\Gamma, v) = H$.*

PROPOSITION 4. [**16**] *There is an algorithm which, given finitely many standard decompositions of elements $h_1, \ldots, h_k$ from $F^{\mathbb{Z}[t]}$, constructs a U-folded $(\mathbb{Z}[t], X)$-graph Γ, such that $L(\Gamma, v) = \langle h_1, \ldots, h_k \rangle$.*

The properties of U-folded graphs make it possible to solve the membership problem in finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

PROPOSITION 5. [**16**] *Every finitely generated subgroup of $F^{\mathbb{Z}[t]}$ has a solvable membership problem. That is, there exists an algorithm which, given finitely many standard decompositions of elements $g, h_1, \ldots, h_k$ from $F^{\mathbb{Z}[t]}$, decides whether or not g belongs to the subgroup $H = \langle h_1, \ldots, h_n \rangle$ of $F^{\mathbb{Z}[t]}$.*

3. Intersection of two finitely generated subgroups of $F^{\mathbb{Z}[t]}$

In [**8**] a very convenient and simple way to compute an intersection of two subgroups of a free group was shown, which used the notion of a *product-graph.* Recall that if Θ_1, Θ_2 are graphs labeled by some alphabet A, then the product-graph $\Theta_1 \times \Theta_2$ is defined as follows:

(1) the vertex set of $\Theta_1 \times \Theta_2$ is the set $V(\Theta_1) \times V(\Theta_2)$;
(2) for a pair of vertices (s,t), $(s',t') \in V(\Theta_1 \times \Theta_2)$ (so that s, $s' \in V(\Theta_1)$, t, $t' \in V(\Theta_2)$) and a letter $z \in A$ an edge labeled by z with origin (s,t) and terminus (s',t') is introduced, provided there is an edge labeled by z from s to s' in Θ_1 and there is an edge labeled by z from t to t' in Θ_2.

In Subsection 3.1 we introduce a similar notion, adjusted to the case of U-folded $(\mathbb{Z}[t], X)$-graphs and then in Subsection 3.2 we show how to find an intersection of two finitely generated subgroups of $F^{\mathbb{Z}[t]}$ pretty much in the same way as in a free group.

3.1. Product-graph of U-folded $(\mathbb{Z}[t], X)$-graphs.

Observe that any finite $(\mathbb{Z}[t], X)$-graph Γ is characterized by its u-components where $u \in U(\Gamma)$, and any u-component is associated with a free abelian group of a finite rank, its subgroup and a finite set of coset representatives of this subgroup. Thus, in order to realize the idea of a product-graph in the case of $(\mathbb{Z}[t], X)$-graphs we have to show how to construct the product-graph of two u-components and then introduce the notion in general.

Notice that if H and K are subgroups of a group G and $u, v \in G$, then either $uH \cap vK = w(H \cap K)$ for some $w \in G$ or the intersection of these two cosets is empty. We frequently use this fact below.

Recall from [**16**] that if Θ is a finite U-folded $(\mathbb{Z}[t], X)$-graph, $v \in V(\Theta)$ and $K = Comp_u(v)$, where $u \in U(\Theta)$ then K is finite and by Lemma 2, $H_u(v)$ is isomorphic to a subgroup H of $\mathbb{Z}^{n(K)}$, where $n(K) = deg_u(K) + 1$. Moreover, there exist finitely many positive subpaths $p_0, p_1, \ldots, p_n$ of P_K, all starting at z_K, which form a set of *path representatives* $Rep(K)$, such that any $w \in V(K)$ is associated with $p_w \in Rep(K)$, where $\overline{\mu(p_w)} = u^{h_w}$ and h_w is a coset representative of H in $\mathbb{Z}^{n(K)}$. Thus, any $w \in V(K)$ can be associated with a coset representative of H in $\mathbb{Z}^{n(K)}$. On the other hand, let $w \in V(K)$ be fixed and let $\overline{\mu(p_i)} * \overline{\mu(p_w)}^{-1} = u^{h_i - h_w}, i \in [1, n]$. Then by Lemma 4 it follows that $h_0 - h_w,\ h_1 - h_w,\ \dots,\ h_n - h_w$ are coset representatives in $\mathbb{Z}^{n(K)}$ by H.

Definition 1. *Let Θ_1, Θ_2 be finite U-folded $(\mathbb{Z}[t], X)$-graphs. We define a* partial product-graph $\Theta_1 \diamond \Theta_2$ *as follows:*

(1) *the vertex set of $\Theta_1 \diamond \Theta_2$ is the set $V(\Theta_1) \times V(\Theta_2)$;*
(2) *for a pair of vertices $(s_1, t_1), (s_2, t_2) \in V(\Theta_1 \diamond \Theta_2)$ (so that $s_1, s_2 \in V(\Theta_1), t_1, t_2 \in V(\Theta_2)$) and a letter $x \in X$, we add an edge labeled by x with origin (s_1, t_1) and terminus (s_2, t_2), provided there is an edge labeled by x from s_1 to s_2 in Θ_1 and there is an edge labeled by x from t_1 to t_2 in Θ_2;*
(3) *for a pair of vertices $(s_1, t_1), (s_2, t_2) \in V(\Theta_1 \diamond \Theta_2)$ (so that $s_1, s_2 \in V(\Theta_1), t_1, t_2 \in V(\Theta_2)$) and $u \in U(\Theta_1) \cap U(\Theta_2)$, we add an edge labeled by u^f with origin (s_1, t_1) and terminus (s_2, t_2), provided*
   a) *$s_1 \sim_u s_2,\ t_1 \sim_u t_2$ (denote $K_1 = Comp_u(s_1), K_2 = Comp_u(t_1)$);*
   b) *h_1, h_2 are coset representatives of $H_u(s_1)$ in $\mathbb{Z}^{n(K_1)}$ corresponding to s_1, s_2;*
   c) *g_1, g_2 are coset representatives of $H_u(t_1)$ in $\mathbb{Z}^{n(K_2)}$ corresponding to t_1, t_2;*
   d) *$(h_2 - h_1 + H_u(s_1)) \cap (g_2 - g_1 + H_u(t_1)) = f + H_u(s_1) \cap H_u(t_1)$, where f is a coset representative of $H_u(s_1) \cap H_u(t_1)$ in $\mathbb{Z}^n$ for $n = \max\{n(K_1), n(K_2)\}$.*

Observe that in general $K_1 \diamond K_2$ consists of several u-components of $\Theta_1 \diamond \Theta_2$ because $K_1 \diamond K_2$ can be disconnected.

Lemma 7. *Let Θ_1, Θ_2 be finite U-folded $(\mathbb{Z}[t], X)$-graphs, $u \in U(\Theta_1) \cap U(\Theta_2)$ and let K_1, K_2 be non-empty u-components of Θ_1, Θ_2 correspondingly. Denote $H_i = H_u(v_i)$, where $v_i \in V(K_i), i = 1, 2$ and $H = H_u(v_1) \cap H_u(v_2)$. Then*

a) *if there exists an edge $e_1 \in E(K_1 \diamond K_2)$ such that $o(e_1) = (s_1, t_1), t(e_1) = (s_2, t_2)$ and $\mu(e_1) = u^{p_1}$ then there exists also an edge $e_2 \in E(K_1 \diamond K_2)$ such that $o(e_2) = (s_2, t_2), t(e_2) = (s_1, t_1)$ and $\mu(e_2) = u^{p_2}$;*
b) *if $p = e_1 \cdots e_k$ is a loop in $K_1 \diamond K_2$ such that $\mu(e_i) = u^{f_i}$ then $f_1 + \cdots + f_k \in H$;*
c) *if $p = e_1 \cdots e_k$ is a simple path in $K_1 \diamond K_2$ such that $\mu(e_i) = u^{f_i}$ and $f_1 + \cdots + f_k \neq 0$ then $f_1 + \cdots + f_k \notin H$.*

Proof. Let $n = \max\{n(K_1), n(K_2)\}$.

a) The existence of e_1 means that

$$(h_2 - h_1 + H_1) \cap (g_2 - g_1 + H_2) = f + H,$$

where h_1, h_2 are coset representatives of H_1 in $\mathbb{Z}^n$ corresponding to s_1, s_2 , g_1, g_2 are coset representatives of H_2 in $\mathbb{Z}^n$ corresponding to t_1, t_2 and f is a coset representative of H in $\mathbb{Z}^n$. Thus we have

$$h_2 - h_1 + a_1 = g_2 - g_1 + a_2 = f + c,$$

where $a_i \in H_i, c \in H$. So

$$h_1 - h_2 + b_1 = g_1 - g_2 + b_2 = -f - c = f' + c'$$

for $b_i \in H_i, c' \in H, -f \in f' + H$, which means that

$$(h_1 - h_2 + H_1) \cap (g_1 - g_2 + H_2)$$

is not empty, therefore there exists an edge $e_2 \in E(K_1 \times K_2)$ with $o(e_2) = (s_2, t_2)$, $t(e_2) = (s_1, t_1)$ and $\mu(e_2) = u^{f'}$.

b) Suppose we have a cycle $p = e_1 \cdots e_k, \mu(e_i) = u^{f_i}$ in $K_1 \times K_2$. By a) we can assume all f_i to be positive. We have $o(e_i) = (s_i, t_i), t(e_i) = (s_{i+1}, t_{i+1}), (s_1, t_1) = (s_{k+1}, t_{k+1}), i \in [1, k]$. By Definition 1, there exist a coset representative g_i of H_1 in $\mathbb{Z}^n$ corresponding to s_i and a coset representative h_i of H_2 in $\mathbb{Z}^n$ corresponding to t_i such that for an edge e_i we have

$$(h_{i+1} - h_i + H_1) \cap (g_{i+1} - g_i + H_2) = f_i + H,$$

where $i \in [1, k]$. Thus,

$$h_{i+1} - h_i + a_i = g_{i+1} - g_i + b_i = f_i + c_i,$$

for some $a_i \in H_1, b_i \in H_2, c_i \in H$. So, summing up the above equalities for all $i \in [1, k]$ we obtain

$$a_1 + \cdots + a_k = b_1 + \cdots + b_k = f_1 + \cdots + f_k + c,$$

for $c \in H$ and it follows $f_1 + \cdots + f_k \in H$.

c) Suppose we have a simple path $p = e_1 \cdots e_k, \mu(e_i) = u^{f_i}$ in $K_1 \times K_2$. We have $o(e_i) = (s_i, t_i), t(e_i) = (s_{i+1}, t_{i+1}), i \in [1, k]$. By Definition 1, there exist a coset representative g_i of H_1 in $\mathbb{Z}^n$ corresponding to s_i and a coset representative h_i of H_2 in $\mathbb{Z}^n$ corresponding to t_i such that for an edge e_i we have

$$(h_{i+1} - h_i + H_1) \cap (g_{i+1} - g_i + H_2) = f_i + H,$$

where $i \in [1, k]$. That is,

$$h_{i+1} - h_i + a_i = g_{i+1} - g_i + b_i = f_i + c_i,$$

for some $a_i \in H_1, b_i \in H_2, c_i \in H$. We sum up the above equalities for all $i \in [1, k]$ and we obtain

$$h_{k+1} - h_1 + a = g_{k+1} - g_1 + b = f_1 + \cdots + f_k + c,$$

where $a \in H_1, b \in H_1, c \in H$. Hence,

$$(h_{k+1} - h_1 + H_1) \cap (g_{k+1} - g_1 + H_2)$$

is not empty, so there exists an edge $e \in E(K_1 \times K_2)$ from (s_1, t_1) to (s_{k+1}, t_{k+1}) labeled by u^α and $\alpha \notin H$. Since $q = e_1 \cdots e_k e^{-1}$ is a cycle in $K_1 \times K_2$, it follows from b) that $f_1 + \cdots + f_k - \alpha \in H$, so, $f_1 + \cdots + f_{k+1} \notin H$.

□

DEFINITION 2. *Let Θ_1, Θ_2 be finite U-folded $(\mathbb{Z}[t], X)$-graphs. We define a* product-graph $\Theta_1 \times \Theta_2$ *as follows:*

(1) *the vertex set of $\Theta_1 \times \Theta_2$ is $V(\Theta_1 \diamond \Theta_2)$;*
(2) *for any u-component K of $\Theta_1 \diamond \Theta_2$ choose a single vertex $v_K \in V(K)$ and let $V(u)$ be the set of all chosen vertices in $\Theta_1 \diamond \Theta_2$ for a fixed $u \in U(\Theta_1) \cap U(\Theta_2)$;*

(3) *for any* $v = (v_1, v_2) \in V(u)$ *let* $H_u(v_1) \cap H_u(v_2) = \langle h_1, \ldots, h_k \rangle$ *and let* $E(v)$ *be a bouquet of edge-loops* $e_1, \ldots, e_k$ *labeled by* $u^{h_i}, i \in [1, k]$*;*
(4) *the edge set of* $\Theta_1 \times \Theta_2$ *is obtained from* $E(\Theta_1 \diamond \Theta_2)$ *by attaching to every* $v \in V(u), u \in U(\Theta_1) \cap U(\Theta_2)$ *a graph* $E(v)$*.*

LEMMA 8. *Let* Θ_1, Θ_2 *be finite* U*-folded* $(\mathbb{Z}[t], X)$*-graphs,* $u \in U(\Theta_1) \cap U(\Theta_2)$ *and let* K_i *be a* u*-component of* $\Theta_i, i = 1, 2$*. Then*

a) *if* $v = (v_1, v_2) \in V(K_1 \times K_2)$ *and* $H_i = H_u(v_i), i = 1, 2$ *then* $H_u(v) \simeq H_u(v_1) \cap H_u(v_2)$*;*
b) *if* $u^{h_1}, \ldots, u^{h_k}$ *are reduced labels of all simple paths starting at* $v \in V(K_1 \times K_2)$ *in* K *then* $h_1, \ldots, h_k$ *is a system of coset representatives of* $H_u(v)$ *in* $\mathbb{Z}^n$*, where* $n = \max\{n(K_1), n(K_2)\}$*.*

Proof. a) $H_u(v)$ is generated by all loops at v and from the definition of $\Theta_1 \times \Theta_2$ it follows that $H_u(v_1) \cap H_u(v_2) \subseteq H_u(v)$. On the other hand, in view of Lemma 7.b) we have $H_u(v) \subseteq H_u(v_1) \cap H_u(v_2)$.

b) Follows from Lemma 7.c).

□

Observe that $\Theta_1 \times \Theta_2$ is partially folded but not necessarily U-folded. However, all u-components in $\Theta_1 \times \Theta_2$ are complete and u-folded.

LEMMA 9. *Let* Θ_1, Θ_2 *be finite* U*-folded* $(\mathbb{Z}[t], X)$*-graphs and let* $g \in F^{\mathbb{Z}[t]}$ *be such that there exist label reduced paths* p_1, p_2 *in* Θ_1, Θ_2 *correspondingly with* $\mu(p_1) = \mu(p_2) = \pi(g)$*. Then there exists a path* p *in* $\Theta_1 \times \Theta_2$ *such that* $o(p) = (o(p_1), o(p_2)), t(p) = (t(p_1), t(p_2))$ *and* $\mu(p) = \pi(g)$*.*

Proof. Let
$$\pi(g) = \pi(g_1) w^{\alpha_1} \pi(g_2) \cdots w^{\alpha_k} \pi(g_{k+1}).$$
and let $l_i = \min\{m \mid z_i \in \Theta_i(m)\}, i = 1, 2$. Observe that $l_1 = l_2$ and we denote $L = n_1$.

We use the induction on L.

If $L = 0$, that is, $g \in F(X)$ then the existence of p follows from the definition of $\Theta_1 \times \Theta_2$.

Assume the statement to be true for $L < n$ and let $L = n$.

Since Θ_1, Θ_2 are U-folded, there are unique paths $y_1 \in \Theta_1(L)$ and $z_1 \in \Theta_2(L)$ such that $o(y_1) = o(p_1), o(z_1) = o(p_2), \mu(y_1) = \mu(z_1) = \pi(g_1)$ such that $t(y_1) \in V(K_1), t(z_1) \in V(K_2)$ for some w-components K_1, K_2 in Θ_1, Θ_2 correspondingly. By the induction hypothesis there exists a path q_1 in $\Theta_1 \times \Theta_2$ such that $o(q_1) = (o(y_1), o(z_1)), t(q_1) = (t(y_1), t(z_1))$ and $\mu(q_1) = \pi(g_1)$. Observe that $t(q_1)$ belongs to some connected w-component $K_1 \times K_2$ in $\Theta_1 \times \Theta_2$.

Since p_i is a path in $\Theta_i, i = 1, 2$, there are continuations of y_1 in K_1 and z_1 in K_2. These continuations are paths $y_2 \in K_1, z_2 \in K_2$ (not unique) such that $\mu(y_2) = \mu(z_2) = w^{\alpha_1}$ with fixed terminal vertices $t(y_2)$ and $t(z_2)$ which are completely determined by w^{α_1}. Let $n = \max\{n(K_1), n(K_2)\}$ and H_1, H_2 be subgroups of $\mathbb{Z}^n$ such that $H_1 = H_w(o(y_2)), H_2 = H_w(o(z_2))$ and denote $H = H_1 \cap H_2$.

There exist coset representatives β_1, γ_1 of H_1 in $\mathbb{Z}^n$ which correspond to $o(y_2)$, $t(y_2)$ respectively and coset representatives β_2, γ_2 of H_2 in $\mathbb{Z}^n$ which correspond to $o(z_2), t(z_2)$ respectively, such that
$$\alpha_1 \in (\gamma_1 - \beta_1 + H_1) \cap (\gamma_2 - \beta_2 + H_2) \neq \emptyset.$$

By definition of $\Theta_1 \times \Theta_2$, there exists a coset representative δ_1 of H in $\mathbb{Z}^n$ and a path $q_2, \overline{\mu(q_2)} = u^{\alpha_1}$ from $(o(y_2), o(z_2))$ to $(t(y_2), t(z_2))$ such that

$$(\gamma_1 - \beta_1 + H_1) \cap (\gamma_2 - \beta_2 + H_2) = \delta_1 + H = \alpha_1 + H.$$

Thus, we have a continuation q_2 of q_1 which has a fixed terminus in $K_1 \times K_2$ determined by w^{α_1}.

Now, the induction on k, the number of entries of infinite exponents of w, produces the required path p

□

COROLLARY 1. *Let Θ_1, Θ_2 be finite U-folded $(\mathbb{Z}[t], X)$-graphs and $v_i \in V(\Theta_i)$ for $i = 1, 2$. Then*

$$L(\Theta_1 \times \Theta_2, (v_1, v_2)) = L(\Theta_1, v_1) \cap L(\Theta_2, v_2).$$

Proof.

$$L(\Theta_1 \times \Theta_2, (v_1, v_2)) \subseteq L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$$

follows directly from the definition of $\Theta_1 \times \Theta_2$ and

$$L(\Theta_1, v_1) \cap L(\Theta_2, v_2) \subseteq L(\Theta_1 \times \Theta_2, (v_1, v_2))$$

follows from Lemma 9.

□

It is easy to see that in fact a more general result holds. Let Θ be a finite U-folded $(\mathbb{Z}[t], X)$-graph and $v, w \in V(\Theta)$. Then by

$$L(\Theta, v, w) = \{\overline{\mu(p)} \mid p \text{ is a path in } \Theta \text{ such that } o(p) = v, t(p) = w\}$$

we denote the *language of Θ with respect to the initial vertex v and the terminal vertex w*. Observe that if $v = w$ then $L(\Theta, v, w)$ coincides with $L(\Theta, v)$. The following result is proved in the same way as Corollary 1.

COROLLARY 2. *Let Θ_1, Θ_2 be finite U-folded $(\mathbb{Z}[t], X)$-graphs and $v_i, w_i \in V(\Theta_i)$ for $i = 1, 2$. Then*

$$L(\Theta_1 \times \Theta_2, (v_1, v_2), (w_1, w_2)) = L(\Theta_1, v_1, w_1) \cap L(\Theta_2, v_2, w_2).$$

Finally, one can get further generalization defining the language of a finite U-folded $(\mathbb{Z}[t], X)$-graph taking instead of single initial and terminal vertices some finite sets of vertices. In this case reformulation and proof of Corollary 2 is straightforward.

3.2. Finding intersection of two finitely generated subgroups of $F^{\mathbb{Z}[t]}$. Using the construction of a product-graph one can find effectively the intersection of two finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

From the definition of a product-graph the following result follows immediately.

LEMMA 10. *Let Θ_1, Θ_2 be finite U-folded $(\mathbb{Z}[t], X)$-graphs. Then $\Theta_1 \times \Theta_2$ can be constructed effectively.*

THEOREM 4. *There exists an algorithm which, given finitely many standard decompositions of elements $h_1, \ldots, h_k, g_1, \ldots, g_m$ from $F^{\mathbb{Z}[t]}$, finds the generators of $H \cap K$ which is finitely generated, where $H = \langle h_1, \ldots, h_k \rangle, K = \langle g_1, \ldots, g_m \rangle$.*

Proof. By Proposition 4 there exists an algorithm which constructs U-folded $(\mathbb{Z}[t], X)$-graphs Γ_1 and Γ_2 such that $L(\Gamma_1, v_1) = H, L(\Gamma_2, v_2) = K, v_i \in V(\Gamma_i)$. Then by Lemma 10 one can construct $\Gamma_1 \times \Gamma_2$ effectively and by Proposition 2 there exists a U-folded $(\mathbb{Z}[t], X)$-graph Γ_3 such that

$$L(\Gamma_3, (v_1, v_2)) = L(\Gamma_1 \times \Gamma_2, (v_1, v_2))$$

and Γ_3 can be found effectively. Finally, by Corollary 1 $L(\Gamma_3, (v_1, v_2)) = L(\Gamma_1, v_1) \cap L(\Gamma_2, v_2) = H \cap K$. Since Γ_3 is finite, one can find all simple loops at (v_1, v_2) and their reduced labels generate $L(\Gamma_3, (v_1, v_2)) = H \cap K$.

□

The following result follows directly from Theorem 4.

Corollary 3. *(Howson Property) The intersection of two finitely generated subgroups of $F^{\mathbb{Z}[t]}$ is finitely generated.*

Corollary 2 makes it possible to find intersections of cosets by finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Theorem 5. *There exists an algorithm which, given finitely many standard decompositions of elements $h_1, \ldots, h_k,\ f_1, \ldots f_m,\ w_1, w_2$ from $F^{\mathbb{Z}[t]}$, finds the intersection*

$$H \cap (w_1 * K * w_2),$$

where $H = \langle h_1, \ldots, h_k \rangle,\ K = \langle f_1, \ldots, f_m \rangle$.

Proof. Take a path p labeled by the standard decomposition of w_1, a path q labeled by the standard decomposition of w_2 and $\Gamma(H), \Gamma(K)$ such that $H = L(\Gamma(H), 1_H), K = L(\Gamma(K), 1_K)$. Identify $t(p), o(q)$ and 1_K. Denote the obtained $(\mathbb{Z}[t], X)$-graph by Δ' and notice that

$$w_1 * K * w_2 = \{\overline{\mu(r)} \mid r \text{ is a path in } \Delta' \text{ with } o(r) = o(p), t(r) = t(q)\}.$$

By Proposition 2, one can obtain effectively a U-folded $(\mathbb{Z}[t], X)$-graph Δ from Δ' and with abuse of notation we call the vertices of Δ corresponding to $o(p), t(q) \in \Delta'$ again by $o(p)$ and $t(q)$.

By Lemma 10, one can construct effectively $\Gamma(H) \times \Delta$ and we have

$$H \cap (w_1 * K * w_2) = \left\{ \overline{\mu(r)} \;\middle|\; \begin{array}{l} r \text{ is a path in } \Gamma(H) \times \Delta \text{ such that} \\ o(r) = (1_H, o(p)), t(r) = (1_H, t(q)) \end{array} \right\}.$$

□

Theorem 4 and Corollary 3 can be reformulated for finitely generated fully residually free groups.

Theorem 6. *Let $H = \langle h_1, \ldots, h_k \rangle, K = \langle g_1, \ldots, g_m \rangle$ be finitely generated subgroups of a finitely generated fully residually free group G. There exists an algorithm which finds the generators of $H \cap K$, which is finitely generated.*

Proof. By Theorem 3 one can effectively obtain generators of G, H and K viewed as infinite words and by Proposition 8.3 [**15**] compute their standard decompositions.

By Proposition 4 one can effectively find finite U-folded $(\mathbb{Z}[t], X)$-graphs $\Gamma(G)$, $\Gamma(H)$ and $\Gamma(K)$, such that $G = L(\Gamma(G), 1_G), H = L(\Gamma(H), 1_H),\ K = L(\Gamma(K), 1_K)$

for some $1_G \in V(\Gamma(G)), 1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$ and the result follows from Theorem 4.

□

COROLLARY 4. *The intersection of two finitely generated subgroups of a finitely generated fully residually free group G is finitely generated in G.*

4. Properties of intersections of finitely generated subgroups of $F^{\mathbb{Z}[t]}$

In this section we investigate properties of U-folding graphs in the connection with the *malnormality problem.* Recall that a subgroup H of a group G is called *malnormal* if for any $g \in G - H$

$$gHg^{-1} \cap H = 1.$$

Correspondingly, the malnormality problem is decidable in G if there exists an algorithm which, given finitely many elements $h_1, \ldots, h_n \in G$ decides if $H = \langle h_1, \ldots, h_n \rangle$ is malnormal in G.

From now on if $g \in F^{\mathbb{Z}[t]}$ then by $C(g)$ we denote the centralizer of g in $F^{\mathbb{Z}[t]}$, that is,

$$C(g) = C_{F^{\mathbb{Z}[t]}}(g) \simeq \bigoplus_{n=0}^{\infty} \langle g^n \rangle.$$

The following two results are analogous to respectively Lemma 7.5 and Proposition 9.8 from [**8**].

LEMMA 11. *Let Γ be a connected U-folded $(\mathbb{Z}[t], X)$-graph and $u, v \in V(\Gamma)$ be connected by a reduced path q such that $o(q) = v, t(q) = u, \overline{\mu(q)} = g$. If $H = L(\Gamma, v)$, $K = L(\Gamma, u)$ then $H = g * K * g^{-1}$.*

Proof. Let p be a reduced loop at u in Γ. We have automatically $\overline{\mu(p)} = k \in K$. Then the path $p' = qpq^{-1}$ is a loop at v such that $\mu(p') = \mu(q)\mu(p)\mu(q)^{-1}$. Thus we have $\overline{\mu(p')} = \overline{\mu(q)} * \overline{\mu(p)} * \overline{(\mu(q))}^{-1} = g * k * g^{-1}$. The path p' may be not reduced, so it can be transformed by finitely many path-reductions to a reduced path p'' such that $o(p'') = o(p') = t(p'') = t(p') = v, \overline{\mu(p'')} = \overline{\mu(p')}$. Hence $\overline{\mu(p'')} \in H$ and $\overline{\mu(p'')} = g * k * g^{-1} \in H$. That is, $g * K * g^{-1} \subseteq H$. A symmetric argument shows that $g^{-1} * H * g \subseteq K$, thst is, $H \subseteq g * K * g^{-1}$ and therefore $H = g * K * g^{-1}$, as required.

□

LEMMA 12. *Let $H, K \leq F^{\mathbb{Z}[t]}$ be finitely generated and $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H)$, $K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$. Then for any vertex (v, u) of $\Gamma(H) \times \Gamma(K)$ the subgroup $L(\Gamma(H) \times \Gamma(K), (v, u))$ is conjugate to a subgroup of the form $g * H * g^{-1} \cap K$ for some $g \in F^{\mathbb{Z}[t]}$. Moreover, if (v, u) does not belong to the connected component of $(1_H, 1_K)$, then the element g can be chosen so that $K * g * H \neq K * H$.*

Proof. Let p_v be a label reduced path in $\Gamma(H)$ from 1_H to v such that $\overline{\mu(p_v)} = w_1$. Similarly, let p_u be a label reduced path in $\Gamma(K)$ from 1_K to u such that $\overline{\mu(p_u)} = w_2$. By Lemma 11, $L(\Gamma(H), v) = w_1^{-1} * H * w_1$ and $L(\Gamma(K), u) = w_2^{-1} * K * w_2$.

Therefore, by Corollary 1

$$L(\Gamma(H) \times \Gamma(K), (v, u)) = w_1^{-1} * H * w_1 \cap w_2^{-1} * K * w_2$$

which is conjugate to

$$(w_2 * w_1^{-1}) * H * (w_1 * w_2^{-1}) \cap K$$

and $g = w_2 * w_1^{-1}$ satisfies the requirement of the proposition.

Suppose that (v, u) does not belong to the connected component of $(1_H, 1_K)$ in $\Gamma(H) \times \Gamma(K)$ but $g = w_2 * w_1^{-1} \in K * H$. Thus, $w_2 * w_1^{-1} = k * h$ for some $k \in K$, $h \in H$ and therefore

$$w = k^{-1} * w_2 = h * w_1.$$

Since $k \in K, h \in H$ then there exists a loop p_1 at 1_K in $\Gamma(K)$ such that $\overline{\mu(p_1)} = k$ and a loop p_2 at 1_H in $\Gamma(H)$ such that $\overline{\mu(p_2)} = h$.

Then $\overline{\mu(p_1 p_u)} = w, \overline{\mu(p_2 p_v)} = w$ and there are a unique label reduced path p_1' in $\Gamma(K)$ from 1_K to u with label $\pi(w)$ and a unique label reduced path p_2' in $\Gamma(H)$ from 1_H to v with label $\pi(w)$.

Now by Lemma 9, there exists a path in $\Gamma(H) \times \Gamma(K)$ from $(1_H, 1_K)$ to (v, u) with the label $\pi(w)$. However, this contradicts our assumption that (v, u) does not belong to the connected component of $(1_H, 1_K)$.

Thus, $g \notin K * H$ and $K * g * H \neq K * H$, as required.

□

At first we prove several auxiliary results which will be used in proofs of main technical results of this section.

Lemma 13. *Let $H, K \leq F^{\mathbb{Z}[t]}$ be finitely generated and $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H)$, $K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$. Let $g \in F(X), h \in H \cap F(X), f \in K \cap F(X)$ be such that $g * h * g^{-1} = f$. Then g can be represented as a product $g = y * z$, so that, there exists a path p in $\Gamma(H)$ starting at 1_H with $\overline{\mu(p)} = z^{-1}$ and there exists a path q in $\Gamma(K)$ starting at 1_K with $\overline{\mu(q)} = y$.*

Proof. Consider two cases.

1. h does not cancel completely in $g * h * g^{-1}$.

Then, $h = a \circ h_1 \circ b$, $g = g_1 \circ a^{-1} = g_2 \circ b$ and $f = g_1 \circ h_1 \circ g_2^{-1}$. $h \in H$, so, there exists a loop at 1_H in $\Gamma(H)$ which is labeled by h and since $h \in F(X)$, this loop has an initial subpath labeled by a. On the other hand, $f \in K$, so, there exists a loop at 1_K in $\Gamma(H)$ which is labeled by f and since $h \in F(X)$, this loop has an initial subpath labeled by g_1. So, $g = g_1 \circ a^{-1}$ is the required representation of g.

2. h cancels completely in $g * h * g^{-1}$.

We use the induction on $|g|$.

If $|g| = 1$, that is, $g \in X$ then the statement is obviously true. Assume that the statement is proved for any g such that $|g| < m$ and any subgroups $H, K \leq F^{\mathbb{Z}[t]}$, which satisfy the conditions of the lemma. Let $|g| = m$.

a) h cancels completely in $g * h$ (similarly in $h * g^{-1}$).

We have $g = g_1 \circ h^{-1}$ so $g * h * g^{-1} = g_1 * (h \circ g_1^{-1}) = f \in K$. By the induction hypothesis, since $|g_1| < |g|$, g_1 can be represented as a product $g_1 = y_1 * z_1$, so that there exist a path p_1 in $\Gamma(H)$ starting at 1_H and a path q_1 in $\Gamma(K)$ starting at 1_K, such that $\overline{\mu(p_1)} = z_1^{-1}$ and $\overline{\mu(q_1)} = y_1$. Thus, $g = g_1 \circ h^{-1} = (y_1 * z_1) \circ h^{-1} = y_1 * (z_1 * h^{-1})$. Since $h \in H$, there exists a cycle p' at 1_H in $\Gamma(H)$ such that $\overline{\mu(p')} = h$ and for the concatenation $p' p_1$ we have $o(p' p_1) = 1_H, \overline{\mu(p' p_1)} = h * z_1^{-1}$.

Thus, the required product decomposition for g is $g = y_1 * (z_1 * h^{-1})$.

b) h cancels completely in $g * h * g^{-1}$ but not in $g * h$ or $h * g^{-1}$.

We have $h = h_1 \circ h_2, g = g_1 \circ h_1^{-1} = g_2 \circ h_2$. Observe that $|g_1| \neq |g_2|$ and without loss of generality we can assume $|g_1| > |g_2|$. Then $|h_1| < |h_2|, h_2 = c \circ h_1^{-1}$ and $h = h_1 \circ c \circ h_1^{-1}$. Thus we have

$$g * h * g^{-1} = (g_1 \circ h_1^{-1}) * (h_1 \circ c \circ h_1^{-1}) * (h_1 \circ g_1^{-1}) = g_1 * c * g_1^{-1} \in K.$$

Since $h = h_1 \circ c \circ h_1^{-1} \in H$, there exists a label reduced path p' such that $o(p') = 1_H, t(p') = v_1 \in V(\Gamma(H))$ and $\mu(p') = h_1$. If $H_1 = L(\Gamma(H), v_1), H = L(\Gamma(H), 1_H)$ then by Lemma 11 we have $H = h_1 * H_1 * h_1^{-1}$.

Now, let us consider the triple g_1, H_1, K. Observe that $\varepsilon \neq c \in H_1, g_1 * c * g_1^{-1} \in K$. Since $|g_1| < |g|$, by the induction hypothesis g_1 can be represented as a product $g_1 = y_1 * z_1$ so that there exist a path p_1 in $\Gamma(H)$ starting at v_1 and a path q_1 in $\Gamma(K)$ starting at 1_K such that $\overline{\mu(p_1)} = z_1^{-1}$ and $\overline{\mu(q_1)} = y_1$. Thus, $g = g_1 \circ h_1^{-1} = (y_1 * z_1) \circ h_1^{-1} = y_1 * (z_1 * h_1^{-1})$. We claim that this is the required product decomposition for g. Indeed, q_1 is a path in $\Gamma(K)$ starting at 1_K such that $\overline{\mu(q_1)} = y_1$ and $p'p_1$ is a path in $\Gamma(H)$ starting at 1_H such that $\overline{\mu(p'p_1)} = h_1 * z_1^{-1}$. □

LEMMA 14. *Let $H, K \leq F^{\mathbb{Z}[t]}$ be finitely generated and $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H)$, $K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$. Let $g \in F^{\mathbb{Z}[t]}, h \in H, f \in K$ be such that $g * h * g^{-1} = f$ and $\max\{U(g)\} < \max\{U(h)\}$. Then g can be represented as a product*

$$g = y * u^{\alpha} * z, \ \alpha \in \mathbb{Z},$$

where there exist paths p in $\Gamma(H)$ and q in $\Gamma(K)$, such that $o(p) = 1_H, o(q) = 1_K$ and $\overline{\mu(p)} = z^{-1}$, $\overline{\mu(q)} = y$, and one of the following holds

(1) $\alpha = 0$;
(2) $\alpha \neq 0$, $u \in U(\Gamma(H)) \cap U(\Gamma(K))$ *and either $\alpha = \pm 1$ or $t(p)$ and $t(q)$ belong to some u-components of $\Gamma(H)$ and $\Gamma(K)$ respectively, so that,*

$$H_u(t(p)) \cap \langle u \rangle = H_u(t(q)) \cap \langle u \rangle = \varepsilon$$

and

$$C(u) \cap H^z \cap K^{y^{-1}} \subseteq u^{\alpha} * H^z * u^{-\alpha} \cap K^{y^{-1}}.$$

Proof. Let $u = \max\{U(g)\}, w_1 = \max\{U(h)\}, w_2 = \max\{U(f)\}$. Then we have the u-reduced form for g

$$g = g_1 \circ u^{\alpha_1} \circ g_2 \circ \cdots \circ u^{\alpha_m} \circ g_{m+1},$$

the w_1-reduced form f or h

$$h = h_1 \circ w_1^{\beta_1} \circ h_2 \circ \cdots \circ w_1^{\beta_r} \circ h_{r+1},$$

and the w_2-reduced form for f

$$f = f_1 \circ w_2^{\gamma_1} \circ f_2 \circ \cdots \circ w_2^{\gamma_l} \circ f_{l+1}.$$

Since $g * h * g^{-1} = f$ and $u < w_1$, it follows that $w_1 = w_2$ and $l = r$. Denote $w = w_1 = w_2$.

Consider the following cases.

1. $r > 1$.

In this case

$$g * h * g^{-1} = (g * h_1) \circ w^{\beta'_1} \circ h_2 \circ \cdots \circ w^{\beta'_r} \circ (h_{r+1} * g^{-1})$$

is the w-reduced form for $g * h * g^{-1}$, where $\beta_1 - \beta_1',\ \beta_r - \beta_r' \in \mathbb{Z}$. Thus, it follows from the uniqueness of w-reduced forms that $g * (h_1 \circ w^{\beta_1}) = f_1 \circ w^{\gamma_1}$ and

$$g = (f_1 \circ w^{\gamma_1}) * (h_1 \circ w^{\beta_1})^{-1}$$

is the required decomposition of g since there exist a path p in $\Gamma(H)$ starting at 1_H and a path q in $\Gamma(K)$ starting at 1_K such that $\overline{\mu(p)} = h_1 \circ w^{\beta_1}$ and $\overline{\mu(q)} = f_1 \circ w^{\gamma_1}$.

2. $r = 1$.

In this case $h = h_1 \circ w^{\beta_1} \circ h_2,\ f = f_1 \circ w^{\gamma_1} \circ f_2$.

By Lemma 6.9 [**15**], $f_1^{-1} * g * h_1 = w^k$ for some $k \in \mathbb{Z}$. Thus, $g = f_1 * w^k * h_1^{-1}$ and without loss of generality we can assume $k > 0$. Observe that there exist a path p in $\Gamma(H)$ starting at 1_H with $\overline{\mu(p)} = h_1$ and a path q in $\Gamma(K)$ starting at 1_K with $\overline{\mu(q)} = f_1$.

If $H_w(t(p)) \cap \langle w\rangle \neq \varepsilon$ (or $H_w(t(p)) \cap \langle w\rangle \neq \varepsilon$) then by the property **(iv)** of U-folded graphs there exists a loop z at $t(p)$ labeled by $w^n, n \in \mathbb{Z}, n > 0$. By the property of elements from U we have $\pi(w^2) = \pi(w)\pi(w)$, that is, for any $0 < n_1 < n$ there exists a subpath z_1 of z such that $\mu(z_1) = \pi(w^{n_1}) = \underbrace{\pi(w) \cdots \pi(w)}_{n_1 \text{times}}$. Thus, the existence of an initial path z' with $o(z') = t(p), \mu(z') = \pi(w^k)$ follows.

Now, assume $H_w(t(p)) \cap \langle w\rangle = H_w(t(p)) \cap \langle w\rangle = \varepsilon$.

We set

$$H_1 = h_1^{-1} * H * h_1, \quad K_1 = f_1^{-1} * K * f_1$$

and

$$h' = (w^{\beta_1} \circ h_2) * h_1 = w^{\beta} \circ d_1, \quad f' = (w^{\gamma_1} \circ f_2) * f_1 = w^{\gamma} \circ d_2,$$

where $\beta_1 - \beta,\ \gamma_1 - \gamma \in \mathbb{Z}$. Then $h' \in H_1, f' \in K_1$ and

$$w^k * h' * w^{-k} = f'.$$

By Lemma 11,

$$H_1 = L(\Gamma(H), t(p)),\ K_1 = L(\Gamma(K), t(q))$$

and we can consider the triple w^k, H_1, K_1 instead of g, H, K.

We have $w^k * (w^{\beta} \circ d_1) * w^{-k} = w^{\gamma} \circ d_2$. Consider the following cases.

a) $d_1 = \varepsilon$

It follows that $d_2 = \varepsilon$ and $h_2 * h_1 = w^m$. We have a loop $p_h = pp_1p_2$ at 1_H in $\Gamma(H)$ such that $\mu(p) = \pi(h_1), \mu(p_1) = w^{\beta_1}, \mu(p_2) = \pi(h_2)$. If $m \neq 0$ then in $\Gamma(H)$ we have a path p_1' such that $o(p_1') = t(p_1), t(p_1') = o(p_1), \overline{\mu(p_1')} = w^m$. So, either we have a contradiction with the fact that $\Gamma(H)$ is U-folded (the property **(vi)**, minimality of a w-component containing $t(p)$ breaks) or $H_w(t(p)) \cap \langle w\rangle \neq \varepsilon$ and we have a contradiction with our assumption. Hence, $m = 0$ and we have $h_1 = h_2^{-1}, f_1 = f_2^{-1}$ from, which follows

$$C(w) \cap H_1 \cap K_1 \subseteq w^k * H_1 * w^{-k} \cap K_1.$$

b) $d_1 \neq \varepsilon$

Suppose d_1 does not cancel completely in $(w^{\beta} \circ d_1) * w^{-k}$ then we have $d_1 = c \circ w_2 \circ w^n$, where $w^{-1} = w_2^{-1} \circ w_1^{-1}, n \leq k$ and $w^k * (w^{\beta} \circ d_1) * w^{-k} = w^{\beta+k} \circ c \circ w_1^{-1} \circ w^{-k+n+1}$. Observe that in $\Gamma(H)$ there exists a path p_1 such that $o(p_1) = t(p), \overline{\mu(p_1)} = (c \circ w_2 \circ w^n)^{-1}$. Since $o(p_1)$ belongs to w-component, by the property **(viii.a)** of U-folded graphs there exists a path p_2 in $\Gamma(H)$ such that $o(p_2) =$

$t(p), \overline{\mu(p_2)} = w^{-n}$. On the other hand, in $\Gamma(K)$ there exists a path q_1 such that $o(q_1) = t(q), \overline{\mu(q_1)} = (w^{\beta+k} \circ c \circ w_1^{-1} \circ w^{-k+n+1})^{-1}$. By the property **(viii.b)** of U-folded graphs there exists a path q_2 in $\Gamma(K)$ such that $o(q_2) = t(q), \overline{\mu(q_2)} = w^{k-n-1}$. Thus, we have

$$w^k = w^{k-n-1} * w * w^n,$$

which is the required product decomposition.

Now, assume d_1 cancels completely in $(w^\beta \circ d_1) * w^{-k}$ and without loss of generality we can assume $\beta > 0$. We have $w^\beta \circ d_1 = w^{\beta-n_1} \circ w^{n_1} \circ d_1 = w^{\beta-n_1} \circ w_1 \circ w^{n_2}$, where $w = w_1 \circ w_2, n_2 \leq k$ so that $f' = w^k * (w^\beta \circ d_1) * w^{-k} = (w^{\beta+k-n_1} \circ w_1) * w^{-k+n_2}$ and the length of the cancellation is less than $|w|$, that is, f' has w^{-k+n_2+1} as a terminal segment. Hence we have a path p_1 in $\Gamma(H)$ such that $o(p_1) = t(p), \overline{\mu(p_1)} = (w^{\beta-n_1} \circ w_1 \circ w^{n_2})^{-1}$ and a path q_1 in $\Gamma(K)$ such that $o(q_1) = t(q), \overline{\mu(q_1)} = ((w^{\beta+k-n_1} \circ w_1) * w^{-k+n_2})^{-1}$, and by the property **(viii.b)** of U-folded graphs there exist a path p_2 in $\Gamma(H)$ such that $o(p_2) = t(p), \overline{\mu(p_2)} = w^{-n_2}$ and a path q_2 in $\Gamma(K)$ such that $o(q_2) = t(q), \overline{\mu(q_2)} = w^{k-n_2-1}$. Thus, we have

$$w^k = w^{k-n_2-1} * w * w^{n_2},$$

which is the required product decomposition.

□

LEMMA 15. *Let $H, K \leq F^{\mathbb{Z}[t]}$ be finitely generated and $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H)$, $K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$. Let $g \in F^{\mathbb{Z}[t]}, h \in H, f \in K$ be such that $g * h * g^{-1} = f$ and $\max\{U(g)\} = \max\{U(h)\}$. Then g can be represented as a product*

$$g = y * u^\alpha * z, \ \alpha \in \mathbb{Z}[t],$$

where there exist paths p in $\Gamma(H)$ and q in $\Gamma(K)$, such that $o(p) = 1_H, o(q) = 1_K$ and $\overline{\mu(p)} = z^{-1}$, $\overline{\mu(q)} = y$, and one of the following holds

(1) $\alpha = 0$;
(2) $\alpha \in \mathbb{Z}[t] - \mathbb{Z}, \ u \in U(\Gamma(H)) \cap U(\Gamma(K))$;
(3) $0 \neq \alpha \in \mathbb{Z}, \ u \in U(\Gamma(H)) \cap U(\Gamma(K))$ *and either* $\alpha = \pm 1$ *or* $t(p)$ *and* $t(q)$ *belong to some u-components of $\Gamma(H)$ and $\Gamma(K)$ respectively, so that,*

$$H_u(t(p)) \cap \langle u \rangle = H_u(t(q)) \cap \langle u \rangle = \varepsilon$$

and

$$C(u) \cap H^z \cap K^{y^{-1}} \subseteq u^\alpha * H^z * u^{-\alpha} \cap K^{y^{-1}}.$$

Proof. Let $u = \max\{U(g)\} = \max\{U(h)\}$ and $w = \max\{U(f)\}$. Then we have the u-reduced forms for g

$$g = g_1 \circ u^{\alpha_1} \circ g_2 \circ \cdots \circ u^{\alpha_m} \circ g_{m+1},$$

$$h = h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_r} \circ h_{r+1},$$

and the w-reduced form for f

$$f = f_1 \circ w^{\gamma_1} \circ f_2 \circ \cdots \circ w^{\gamma_l} \circ f_{l+1}.$$

We prove the lemma by the induction on the number of syllables g_i or u^{α_i} in g, which is $M = 2m + 1$.

If $M = 1$ then $g = u_1^{\alpha}$, $\alpha_1 \in \mathbb{Z}[t] - \mathbb{Z}$ ($g = g_1$ is not possible because $u = \max\{U(g)\}$) and this is already the required product decomposition of g since $u \in U(\Gamma(H)) \cap U(\Gamma(K))$.

Suppose the statement is proved for $M < N$ and let $M = N$.

Consider

$$g * h * g^{-1} = g * (h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_r} \circ h_{r+1}) * g =$$
$$= f_1 \circ w_1^{\gamma_1} \circ f_2 \circ \cdots \circ w_1^{\gamma_l} \circ f_{l+1} = f.$$

Depending on the cancellation in $(u^{\alpha_m} \circ g_{m+1}) * h * (g_{m+1}^{-1} \circ u^{-\alpha_m})$ we have the following cases.

Case I. $u^{\alpha_m} * g_{m+1} * h_1 \circ u^{\beta_1} \in \langle u \rangle$ (similarly $u^{\beta_r} * h_{r+1} * g_{m+1}^{-1} * u^{-\alpha_m} \in \langle u \rangle$).

By Lemma 6.9 [**15**], $g_{m+1} * h_1 = u^{k_1}$ and $\alpha_m + \beta_1 \in \mathbb{Z}$. Without loss of generality we can assume $\alpha_m > 0, \beta_1 < 0$. Then we have

$$u^{\alpha_m} = u^{-\beta_1} * u^{k_2}, \quad g_{m+1} = u^{k_1} * h_1^{-1}$$

and

$$u^{\alpha_m} * g_{m+1} = u^{-\beta_1} * u^{k_2} * u^{k_1} * h_1^{-1} = u^{k_1+k_2} * (u^{-\beta_1} * h_1^{-1}).$$

Denote $g' = g_1 \circ u^{\alpha_1} \circ g_2 \circ \cdots \circ g_m$, so,

$$g = g' \circ (u^{\alpha_m} \circ g_{m+1}) = (g' * u^{k_1+k_2}) * (u^{-\beta_1} * h_1^{-1}).$$

Observe that there exists a path p' in $\Gamma(H)$ such that $o(p') = 1_H, t(p') = v_1 \in V(\Gamma(H))$ and $\overline{\mu(p')} = h_1 \circ u^{\beta_1}$. If $H_1 = L(\Gamma(H), v_1), H = L(\Gamma(H), 1_H)$ then by Lemma 11 we have

$$H = (h_1 \circ u^{\beta_1}) * H_1 * (h_1 \circ u^{\beta_1})^{-1}$$

and we consider the triple $g'' = (g' * u^{k_1+k_2})$, H_1, K instead of g, H, K. By the induction hypothesis we have the product decomposition

$$g'' = y_1 * u_1^{\alpha} * z_1, \quad u_1 \in U(\Gamma(H)) \cap U(\Gamma(K)),$$

so that there exist a path p_1 in $\Gamma(H)$ starting at v_1 and a path q_1 in $\Gamma(K)$ starting at 1_K such that $\overline{\mu(p_1)} = z_1^{-1}$ and $\overline{\mu(q_1)} = y_1$, and if α is finite then the conditions listed in the statement of the lemma hold. Thus we have the required product decomposition

$$g = (y_1 * u_1^{\alpha} * z_1) * (u^{-\beta_1} * h_1^{-1}) = y_1 * u_1^{\alpha} * (z_1 * u^{-\beta_1} * h_1^{-1}),$$

where $p'p_1$ starts at 1_H and $\overline{\mu(p'p_1)} = h_1 * u^{\beta_1} * z_1^{-1}$.

Case II. $u^{\alpha_m} * g_{m+1} * h_1 \circ u^{\beta_1}$, $u^{\beta_r} * h_{r+1} * g_{m+1}^{-1} * u^{-\alpha_m} \notin \langle u \rangle$.

In this case $w = u$ and from the uniqueness of u-reduced forms we have $f_1 = g_1$.

1. $f_1 \neq \varepsilon$

There exists a path q' in $\Gamma(K)$ such that $o(q') = 1_K, t(q') = v_1 \in V(\Gamma(K))$ and $\overline{\mu(q')} = f_1$. If $K_1 = L(\Gamma(K), v_1)$ then by Lemma 11 we have

$$K = f_1 * K_1 * f_1^{-1}$$

and we consider the triple g', H, K_1, where

$$g' = u^{\alpha_1} \circ g_2 \circ \cdots \circ u^{\alpha_m} \circ g_{m+1}$$

instead of g, H, K. By the induction hypothesis we have the product decomposition

$$g' = y_1 * u_1^{\alpha} * z_1, \quad u_1 \in U(\Gamma(H)) \cap U(\Gamma(K)),$$

so that there exist a path p_1 in $\Gamma(H)$ starting at 1_H and a path q_1 in $\Gamma(K)$ starting at v_1 such that $\overline{\mu(p_1)} = z_1^{-1}$ and $\overline{\mu(q_1)} = y_1$, and if α is finite then the conditions listed in the statement of the lemma hold. Thus we have the required product decomposition

$$g = g_1 \circ g' = f_1 \circ g' = (f_1 * y_1) * u_1^{\alpha} * z_1,$$

where $q'q_1$ starts at 1_K and $\overline{\mu(q'q_1)} = f_1 * y_1$.

2. $f_1 = \varepsilon$

it follows immediately that $f_{l+1} = \varepsilon$.

If $m > 1$ then from the uniqueness of u-forms it follows that $u^{\alpha_1} = u^{\gamma_1}$ and, hence, exists a path q' in $\Gamma(K)$ such that $o(q') = 1_K, t(q') = v_1 \in V(\Gamma(K))$ and $\overline{\mu(q')} = u^{\alpha_1}$. Following the argument presented in **1.** one can get the required product decomposition for g.

Assume $m = 1$, so we have $g = u^{\alpha_1} \circ g_2$.

a) $r > 1$, that is, syllables u^{β_1} and u^{β_r} do not coincide in h.

If $[g_2 * h_1, u] \neq \varepsilon$ then by Lemma 6.9 [**15**] there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$u^{\alpha_1} * (g_2 * h_1) * u^{\beta_1} = u^{\alpha_1 - k_1} \circ (u^{k_1} * g_2 * h_1 * u^{k_2}) \circ u^{\beta_1 - k_2}.$$

From $g * h * g^{-1} = f$ we have

$$u^{\alpha_1-k_1} \circ (u^{k_1} * g_2 * h_1 * u^{k_2}) \circ u^{\beta_1-k_2} \circ h_2 \cdots \circ (u^{\beta_r} * (h_{r+1} * g_2^{-1}) * u^{-\alpha_1}) = u^{\gamma_1} \circ f_2 \circ \cdots \circ u^{\gamma_l}$$

and

$$u^{\alpha_1-k_1} \circ (u^{k_1} * g_2 * h_1 * u^{k_2}) \circ u^{\beta_1-k_2} = u^{\gamma_1} \circ f_2 \circ u^{\gamma_2}.$$

Thus,

$$g = u^{\alpha_1} \circ g_2 = (u^{\gamma_1} \circ f_2 \circ u^{\gamma_2}) * (h_1 \circ u^{\beta_1})^{-1},$$

which is the required product decomposition of g since there exist paths p in $\Gamma(H)$ and q in $\Gamma(K)$, such that $o(p) = 1_H, o(q) = 1_K$ and $\overline{\mu(p)} = h_1 \circ u^{\beta_1}$, $\overline{\mu(q)} = u^{\gamma_1} \circ f_2 \circ u^{\gamma_2}$.

If $[g_2 * h_1, u] = \varepsilon$ then $g_2 * h_1 = u^k$ and $u^{\gamma_1} = u^{\alpha_1+\beta_1+k}$. Thus, $u^{\alpha_1} = u^{\gamma_1} * u^{-\beta_1} * u^{-k}, g_2 = u^k * h_1^{-1}$ and

$$g = u^{\alpha_1} \circ g_2 = (u^{\gamma_1} * u^{-\beta_1} * u^{-k}) * (u^k * h_1^{-1}) = u^{\gamma_1} * (u^{-\beta_1} * h_1^{-1})$$

is the required product decomposition of g since there exist a path p in $\Gamma(H)$ starting at 1_H and a path q in $\Gamma(K)$ starting at 1_K such that $\overline{\mu(p)} = h_1 \circ u^{\beta_1}$ and $\overline{\mu(q)} = u^{\gamma_1}$.

b) $r = 1$, that is, $h = h_1 \circ u^{\beta_1} \circ h_2$.

b1) $[g_2 * h_1, u] \neq \varepsilon \quad ([h_2 * g_2^{-1}, u] \neq \varepsilon)$.

By Lemma 6.9 [**15**] there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$u^{\alpha_1} * (g_2 * h_1) * u^{\beta_1} = u^{\alpha_1 - k_1} \circ (u^{k_1} * g_2 * h_1 * u^{k_2}) \circ u^{\beta_1 - k_2}.$$

Thus we have $u^{\gamma_1} = u^{\alpha_1 - k_1}, u^{\alpha_1} = u^{\gamma_1} * u^{k_1}$ and $g = u^{\gamma_1} * (u^{k_1} * g_2)$. Observe that there exists a path q' in $\Gamma(K)$ such that $o(q') = 1_K, t(q') = v_1 \in V(\Gamma(K))$ and $\overline{\mu(q')} = u^{\gamma_1}$. If $K_1 = L(\Gamma(K), v_1)$ then by Lemma 11 we have

$$K = u^{\gamma_1} * K_1 * u^{-\gamma_1}$$

and we can consider the triple g', H, K_1, where $g' = u^{k_1} * g_2$, instead of g, H, K because $g' * h * g'^{-1} = f'$, where $f' = u^{-\gamma_1} * f * u^{\gamma_1}$. Since $\max\{U(g')\} < \max\{U(h)\} = \max\{U(f')\}$, the required product decomposition of g follows from Lemma 14.

b2) $[g_2 * h_1, u] = [h_2 * g_1^{-1}, u] = \varepsilon$.

Then $g_2 * h_1 = u^{k_1}, h_2 * g_1^{-1} = u^{k_2}$ and $g * h * g^{-1} = f = u^{\beta_1} * u^{k_1+k_2}$. We have $g_2 = u^{k_1} * h_1^{-1}$ and

$$g = (u^{\alpha_1} * u^{k_1}) * h_1^{-1}$$

is the required product decomposition of g since there exists a path p' in $\Gamma(H)$ such that $o(p') = 1_H$ and $\overline{\mu(p')} = h_1$.

□

LEMMA 16. *Let $g \in F^{\mathbb{Z}[t]}$ and $u \in U$ be such that $g * u = g \circ u$ and $\max\{U(g)\} < u$. Then there exists $n \in \mathbb{N}$ such that $\pi(g \circ u^{n+1}) = \pi(g \circ u^n)\pi(u)$.*

Proof. We prove by the induction on $|U(g)|$.

Let $|U(g)| = 0$, that is, $g \in F(X)$. If $u \in F(X)$ then obviously $\pi(g \circ u) = \pi(g)\pi(u)$, so, the lemma holds for $n = 0$. If $u \notin F(X)$ then let $w = \max\{U(u)\}$ and

$$u = h_1 \circ w^{\alpha_1} \circ h_2 \circ \cdots \circ w^{\alpha_r} \circ h_{r+1}$$

be the w-reduced form for u. Observe that $r \geq 1$ and

$$g \circ u = h_1' \circ w^{\alpha_1'} \circ h_2 \circ \cdots \circ w^{\alpha_r} \circ h_{r+1}$$

is the w-reduced form for $g \circ u$, where $g \circ h_1 = h_1' \circ w^k, k \in \mathbb{Z}$ and $\alpha_1' = \alpha_1 + k$. Now, since $\pi(u \circ u) = \pi(u)\pi(u)$, we have $\pi(g \circ u^2) = \pi(g \circ u)\pi(u)$ and we can choose $n = 1$.

Assume the lemma to hold for any g with $|U(g)| < N$ and let $|U(g)| = N$.

Let $w = \max\{U(u)\}, z = \max\{U(g)\}$ so that

$$u = h_1 \circ w^{\alpha_1} \circ h_2 \circ \cdots \circ w^{\alpha_r} \circ h_{r+1}$$

is the w-reduced form for u and

$$g = g_1 \circ z^{\beta_1} \circ g_2 \circ \cdots \circ z^{\beta_l} \circ g_{l+1}$$

is the z-reduced form for g. By the assumption of the lemma we have $z \leq w$.

1. If $z < w$ then

$$g \circ u = h_1' \circ w^{\alpha_1'} \circ h_2 \circ \cdots \circ w^{\alpha_r} \circ h_{r+1}$$

is the w-reduced form for $g \circ u$, where $(g \circ h_1) = h_1' \circ w^k, k \in \mathbb{Z}$ and $\alpha_1' = \alpha_1 + k$. Now, since $\pi(u \circ u) = \pi(u)\pi(u)$, we have $\pi(g \circ u^2) = \pi(g \circ u)\pi(u)$ and we can choose $n = 1$.

2. Let $z = w$. Observe that the induction hypothesis holds for g_{l+1} since $|U(g_{l+1})| < N$. So, there exists $k \in \mathbb{N}$ such that $\pi(g_{l+1} \circ u^{k+1}) = \pi(g_{l+1} \circ u^k)\pi(u)$. We have

$$\pi(g) = \pi(g_1) w^{\beta_1} \pi(g_2) \cdots w^{\beta_l} \pi(g_{l+1}).$$

Observe that if $g_{l+1} \circ u^{k+1}$ contains $w^\gamma, \gamma \in \mathbb{Z}[t]$ as an initial segment then w^γ is an initial segment of $g_{l+1} \circ u$, which follows from Lemma 6.9 [**15**] and the definition of w-reduced forms. Hence, we have

$$\pi(g \circ u^{k+1}) = \pi(g_1) w^{\beta_1} \pi(g_2) \cdots w^{\beta_l} \pi(g_{l+1} \circ u^k)\pi(u)$$

and so

$$\pi(g \circ u^{k+1}) = \pi(g \circ u^k)\pi(u).$$

□

Combining Lemmas 13, 14, 15 and 16 we obtain the following result.

Proposition 6. *Let $H, K \leq F^{\mathbb{Z}[t]}$ be finitely generated and $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H)$, $K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$. Let $g \in F^{\mathbb{Z}[t]}$ be such that $g * H * g^{-1} \cap K \neq \varepsilon$. Then g can be represented as a product*

$$g = y * u^{\alpha} * z, \ \alpha \in \mathbb{Z}[t],$$

so that there exist a path p in $\Gamma(H)$ starting at 1_H with $\overline{\mu(p)} = z^{-1}$ and a path q in $\Gamma(K)$ starting at 1_K with $\overline{\mu(q)} = y$, and one of the following holds:

(1) $\alpha = 0$;
(2) $\alpha \in \mathbb{Z}[t] - \mathbb{Z}, \ u \in U(\Gamma(H)) \cap U(\Gamma(K))$;
(3) $0 \neq \alpha \in \mathbb{Z}, \ u \in U(\Gamma(H)) \cap U(\Gamma(K))$ *and either $\alpha = \pm 1$ or $t(p)$ and $t(q)$ belong to some u-components of $\Gamma(H)$ and $\Gamma(K)$ respectively, so that,*

$$H_u(t(p)) \cap \langle u \rangle = H_u(t(q)) \cap \langle u \rangle = \varepsilon$$

and

$$C(u) \cap H^z \cap K^{y^{-1}} \subseteq u^{\alpha} * H^z * u^{-\alpha} \cap K^{y^{-1}};$$

(4) $\alpha \in \mathbb{Z}[t] - \mathbb{Z}, \ u \notin U(\Gamma(H)) \cap U(\Gamma(K))$ *but $u \in U(g)$ and*

$$u^{\alpha} * H^z * u^{-\alpha} \cap K^{y^{-1}} = \langle u \rangle \cap H^z \cap K^{y^{-1}}$$

Proof. Since $g * H * g^{-1} \cap K \neq \varepsilon$, there exist $h \in H, f \in K$ such that

$$g * h * g^{-1} = f \neq \varepsilon.$$

We prove the lemma by the induction on $|U(g)|$.

Suppose $|U(g)| = 0$, that is, $g \in F(X)$. If $h \in F(X)$ then $f \in F(X)$ and the required result follows from Lemma 13. If $h \notin F(X)$ then the required result follows from Lemma 14.

Assume that the proposition holds for g with $|U(g)| < n$. Let $|U(g)| = n, \ u = \max\{U(g)\}$ and

$$g = g_1 \circ u^{\alpha_1} \circ g_2 \circ \cdots \circ u^{\alpha_m} \circ g_{m+1}$$

be the u-reduced from for g. Let $w_1 = \max\{U(h)\}, \ w_2 = \max\{U(f)\}$, so that,

$$h = h_1 \circ {w_1}^{\beta_1} \circ h_2 \circ \cdots \circ {w_1}^{\beta_r} \circ h_{r+1}$$

is the w_1-reduced from for h and

$$f = f_1 \circ {w_2}^{\gamma_1} \circ f_2 \circ \cdots \circ {w_2}^{\gamma_l} \circ f_{l+1}$$

is the w_2-reduced from for f. Without loss of generality we can assume $w_1 \geq w_2$.

1. $u < w_1$

Then $w_1 = w_2, \ l = r$ and the required result follows from Lemma 14.

2. $u = w_1$

The required result follows from Lemma 15

3. $u > w_1$

In this case all exponents $u^{\alpha_i}, i \in [1, m]$ cancel in $g * h * g^{-1}$. Thus, we have $g_{m+1} * h * g_{m+1}^{-1} = u^{k_1}$ where $k_1 \in \mathbb{Z}$. If $m > 1$ then $g_m * u^{k_1} * g_m^{-1} = u^{k_2}, k_2 \in \mathbb{Z}$ and since u is cyclically reduced we have either $g_m * u^{k_1} = g_m \circ u^{k_1}$ or $u^{k_1} * g_m^{-1} = u^{k_1} \circ g_m^{-1}$. In both cases we have a contradiction with the properties of u-reduced forms. Thus, $m = 1$ and

$$g = g_1 \circ u^{\alpha_1} \circ g_2, \ h = g_2^{-1} * u^{k_1} * g_2, \ f = g_1 * u^{k_1} * g_1^{-1}.$$

Observe that $u \in U(g)$. Since u is cyclically reduced then either $g_2^{-1} * u^{k_1} = g_2^{-1} \circ u^{k_1}$ or $u^{k_1} * g_2 = u^{k_1} \circ g_2$. Assume the former.

Notice that $h = g_2^{-1} * u^{k_1} * g_2 = g_2^{-1} \circ u^{k_1 - 1} \circ u_1 \circ g_2'$, where $u = u_1 \circ u_2, |u_2| > 0$ and $g_2 = u_2^{-1} \circ g_2'$. Without loss of generality we can assume $k_1 > 0$. By Lemma 16, there exists $n_1 \in \mathbb{Z}$ such that

$$\pi(g_2^{-1} \circ u^{2n_1+1} \circ u_1 \circ g_2') = \pi(g_2^{-1} \circ u^{n_1})\pi(u)\pi(u^{n_1} \circ u_1 \circ g_2').$$

Observe that we can always assume $k_1 > 2n_1 + 1$ because we can consider the tuple $h^k,\ f^k$ for some appropriate integer k instead of $h,\ f$ since $g * h^k * g^{-1} = h^k$. Thus we have

$$\pi(h) = \pi(g_2^{-1} \circ u^{k_1-1} \circ u_1 \circ g_2') = \pi(g_2^{-1} \circ u^{n_1})\pi(u^{k_1-2n_1-1})\pi(u^{n_1} \circ u_1 \circ g_2').$$

Since $h \in H$, there exists a loop p at 1_H in $\Gamma(H)$ such that $\mu(p) = \pi(h)$, hence, from the above equality it follows that p has a subpath p_1 such that $o(p_1) = o(p),\ \mu(p_1) = \pi(g_2^{-1} \circ u^{n_1})$.

The same argument can be applied to $f = g_1 * u^{k_1} * g_1^{-1}$, that is, one can find a path q_1 starting at 1_K such that $\mu(q_1) = g_1 \circ u^{n_2}, n_2 \in \mathbb{Z}$. Finally

$$g = g_1 \circ u^{\alpha_1} \circ g_2 = (g_1 \circ u^{n_2}) * u^{\alpha_1 - n_1 - n_2} * (u^{n_1} \circ g_2),$$

where $\delta = \alpha_1 - n_1 - n_2 \in \mathbb{Z}[t] - \mathbb{Z}$.

If $u \in U(\Gamma(H)) \cup U(\Gamma(K))$ then it is easy to see that $u \in U(\Gamma(H)) \cap U(\Gamma(K))$ and we are done. Suppose $u \notin U(\Gamma(H)) \cap U(\Gamma(K))$. Hence, $t(p_1)$ and $t(q_1)$ do not belong to any u-components of $\Gamma(H)$ and $\Gamma(K)$ respectively and we prove

$$u^\delta * H^z * u^{-\delta} \cap K^{y^{-1}} = \langle u \rangle \cap H^z \cap K^{y^{-1}}.$$

Indeed, from the product decomposition of g it follows that

$$\langle u \rangle \cap H^z \cap K^{y^{-1}} \subseteq u^\delta * H^z * u^{-\delta} \cap K^{y^{-1}}.$$

On the other hand, let $a \in u^\delta * H^z * u^{-\delta} \cap K^{y^{-1}}$. Then

$$a = u^\delta * (u^{n_1} \circ g_2) * h' * (g_2^{-1} \circ u^{-n_1}) * u^{-\delta} = (g_1 \circ u^{n_2})^{-1} * f' * (g_1 \circ u^{n_2}),$$

where $h' \in H, f' \in K$. Conjugating both sides by $g_1 \circ u^{n_2}$ we get

$$g * h' * g^{-1} = f'.$$

If h' and f' fall into cases **1.** or **2.** then from Lemmas 14, 15 it follows that $u \in U(\Gamma(H)) \cap U(\Gamma(K))$ and we have a contradiction with our assumption. Hence, h' and f' fall into the case **3.** and by the same argument as for h and f we get

$$h' = g_2^{-1} * u^{k_2} * g_2,\ f' = g_1 * u^{k_2} * g_1^{-1},$$

and $a = u^{k_2} \in \langle u \rangle \cap H^z \cap K^{y^{-1}}$. So,

$$u^\delta * H^z * u^{-\delta} \cap K^{y^{-1}} \subseteq \langle u \rangle \cap H^z \cap K^{y^{-1}}.$$

□

LEMMA 17. *Let $H, K \le F^{\mathbb{Z}[t]}$ be finitely generated and $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H),\ K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$. Let $g = u^\alpha,\ u \in U(\Gamma(H)) \cap U(\Gamma(K)),\ \alpha \in \mathbb{Z}[t] - \mathbb{Z}$ be such that $g * H * g^{-1} \cap K \neq \varepsilon$. Then one of the following holds:*

(1) *g can be represented as a product*

$$g = y * u^n * z, \ n \in \mathbb{Z},$$

where there exist paths p in $\Gamma(H)$ and q in $\Gamma(K)$, such that $o(p) = 1_H, o(q) = 1_K$ and $\overline{\mu(p)} = z^{-1}$, $\overline{\mu(q)} = y$, and if $n \neq 0$ then $t(p)$ and $t(q)$ belong to some u-components of $\Gamma(H)$ and $\Gamma(K)$ respectively, so that,

$$H_u(t(p)) \cap \langle u \rangle = H_u(t(q)) \cap \langle u \rangle = \varepsilon$$

and

$$C(u) \cap H^z \cap K^{y^{-1}} \subseteq u^n * H^z * u^{-n} \cap K^{y^{-1}};$$

(2) $g * H * g^{-1} \cap K = C(u) \cap H \cap K.$

Proof. Consider two cases.

1. $1_H \in V(C)$, $1_K \in V(D)$, where C and D are u-components of $\Gamma(H)$ and $\Gamma(K)$ correspondingly.

Suppose there exist paths p and q in C and D correspondingly such that $o(p) = 1_H, o(q) = 1_K, \overline{\mu(p)} = u^\beta, \overline{\mu(q)} = u^\gamma$ and $\alpha = \beta + \gamma + n$ for some $n \in \mathbb{Z}$. Without loss of generality we can assume n to be minimal positive with the property above. Thus we have a product decomposition

$$g = u^\gamma * u^n * u^\beta.$$

Observe that if $n \neq 0$ then

$$H_u(t(p)) \cap \langle u \rangle = H_u(t(q)) \cap \langle u \rangle = \varepsilon.$$

Indeed, if $H_w(t(p)) \cap \langle w \rangle \neq \varepsilon$ (for $H_w(t(p)) \cap \langle w \rangle \neq \varepsilon$ the same argument) then by the property **(iv)** of U-folded graphs there exists a loop z at $t(p)$ labeled by $w^r, r \in \mathbb{Z}, r > 0$. By the property of elements from U we have $\pi(w^2) = \pi(w)\pi(w)$, that is, for any $0 < r_1 < r$ there exists a subpath z_1 of z such that $\mu(z_1) = \pi(w^{r_1}) = \underbrace{\pi(w) \cdots \pi(w)}_{r_1 \text{times}}$. It follows that if $H_w(t(p)) \cap \langle w \rangle \neq \varepsilon$ then p can be continued in C to a path p' so that $o(p') = 1_H, \overline{\mu(p)} = u^{\beta+n}$ and we have a contradiction with the choice of n. Finally

$$C(u) \cap H^z \cap K^{y^{-1}} \subseteq u^n * H^z * u^{-n} \cap K^{y^{-1}}$$

is obvious.

Now we assume that there exist no paths p and q with the above property. It follows that there exists no path p in C such that $o(p) = 1_H, \overline{\mu(p)} = u^\beta$ and $\alpha = \beta + k$ for any $k \in \mathbb{Z}$.

Consider the graph Δ, which is obtained from $\Gamma(H)$ by attaching a single edge e labeled by u^α in the following way.

$$V(\Delta) = V(\Gamma(H)) \cup \{v'\}, \ E(\Delta) = E(\Gamma(H)) \cup \{e\}$$

and $o(e) = v, t(e) = 1_H, \mu(e) = u^\alpha$. It is easy to see that $L(\Delta, v) = H_1$, where $H_1 = u^{-\alpha} * H * u^\alpha$.

Observe that Δ is not U-folded because the u-component $C' = C \cup \{e\}$ is not u-folded. Let $\Gamma(H_1)$ be $(\mathbb{Z}[t], X)$-graph, which is obtained from Δ by a sequence of u-foldings of e with P_C and let C'' be a u-folded u-component of $\Gamma(H_1)$ corresponding to C'. It is easy to see that $\Gamma(H_1)$ is U-folded. Indeed, from our assumption it follows that $Rep(C) \subsetneq Rep(C'')$ because after e is folded with P_C, the vertex v

defines a new point in P_C, which corresponds to a new representative in the finite set of coset representatives in $\mathbb{Z}^{n(C)}$ by $H_C(1_H)$ associated with C. Moreover, let $v' \in V(\Gamma(H_1))$ be the vertex which corresponds to $v \in V(\Delta)$, then v' has no outgoing edges which do not belong to C''. Thus the property **(iv)** of U-folded holds for C'' and other conditions trivially follow since they hold for $\Gamma(H)$.

Notice that for any path p' in C'' such that $o(p') = v'$ we have $\overline{\mu(p')} = u^\gamma, |\gamma| >> 0$. It follows that if $h \in H_1$ and $[h, u] \neq \varepsilon$ then h has the following form

$$h = u^{\beta_1} \circ h' \circ u^{\beta_2},$$

where $|\beta_1|, |\beta_m| >> 0$ and $\pi(h) = u^{\beta_1}\pi(h')u^{\beta_2}$.

Now, suppose $h \in g * H * g^{-1} \cap K = H_1 \cap K$ and $[h, u] \neq \varepsilon$. Then there exist a label reduced loop p in $\Gamma(H_1)$ at v and a label reduced loop q in $\Gamma(K)$ at 1_K such that $\mu(p) = \mu(q) = \pi(h) = u^{\beta_1}\pi(h')u^{\beta_2}$. Hence, there exist a path p_1 in C'' starting at v' and a path q_1 in D starting at 1_K such that $\overline{\mu(p_1)} = \overline{\mu(q_1)} = u^{\beta_1}$. On the other hand we have a path p' in $P_{C''}$ such that $o(p') = v', t(p') = 1_H, \overline{\mu(p')} = u^\alpha$ and a path p'' in $P_{C''}$ such that $o(p'') = 1_H, t(p'') = t(p_1), \overline{\mu(p'')} = u^\gamma$. Thus, $u^{\beta_1} * u^{-\gamma} * u^{-\alpha} = u^\delta \in H_{C''}(1_H) = H_C(1_H)$ and we have the product decomposition

$$g = u^\alpha = u^{\beta_1} * u^{-\gamma-\delta},$$

where u^{β_1} is the label of the path q_1 in $\Gamma(K)$ starting at 1_K and $u^{\gamma+\delta}$ is the label of a path in $\Gamma(H)$ starting at 1_H - a contradiction. Hence, it follows that $g * H * g^{-1} \cap K \subseteq C(u)$. So

$$g * H * g^{-1} \cap K \subseteq C(u) \cap H \cap K$$

and the inverse inclusion is obvious.

2. Either 1_H or 1_K, or both do not belong to any u-components.

If both 1_H and 1_K do not belong to any u-components then as above we easily get a contradiction. Indeed, any element $h \in H_1 = u^{-\alpha} * H * u^\alpha$ such that $[h, u] \neq \varepsilon$ has the following form

$$h = u^{\beta_1} \circ h' \circ u^{\beta_2},$$

where $|\beta_1|, |\beta_m| >> 0$ and $\pi(h) = u^{\beta_1}\pi(h')u^{\beta_2}$. On the other hand, since 1_K does not belong to any u-component then for any $f \in K$ there can be no infinite exponent of u as an initial letter in $\pi(f)$. Hence $H_1 \cap K \subseteq C(u) \cap H_1 \cap K = C(u) \cap H \cap K$ and

$$g * H * g^{-1} \cap K = C(u) \cap H \cap K$$

follows.

If 1_H belongs to some u-component of $\Gamma(H)$ but 1_K does not belong to any u-component of $\Gamma(K)$ then using the same argument as in 1. one gets the required result.

$\square$

LEMMA 18. *Let $H, K \leq F^{\mathbb{Z}[t]}$ be finitely generated. Let $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H)$, $K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$ and*

1. *$1_H \in V(C)$, $1_K \in V(D)$, where C and D are u-components of $\Gamma(H)$ and $\Gamma(K)$ respectively;*
2. *$H_u(1_H) \cap \langle u \rangle = H_u(1_K) \cap \langle u \rangle = \varepsilon$;*

(3) $u^k * H * u^{-k} \cap K \neq \varepsilon$ *for* $k \in \mathbb{Z}$ *and*

$$C(u) \cap H \cap K \subseteq u^k * H * u^{-k} \cap K.$$

Then one of the following holds

(1) g *can be represented as a product* $g = y * z$, *where there exist paths* p *in* $\Gamma(H)$ *and* q *in* $\Gamma(K)$, *such that* $o(p) = 1_H, o(q) = 1_K$ *and* $\overline{\mu(p)} = z^{-1}$, $\overline{\mu(q)} = y$;
(2) $u^k * H * u^{-k} \cap K = C(u) \cap H \cap K$;
(3) $|k| \leq N(H, K)$, *where* $N(H, K)$ *depends only on* H *and* K *and can be found effectively.*

Proof. Without loss of generality we can assume $k > 0$.

Claim. There exists a finite set $E_H = \{n_1, \ldots, n_s\} \subset \mathbb{Z}$, such that, for any $h \in H$ if $h = u^\alpha \circ h'$, where $\alpha \in \mathbb{Z}$ and h' does not have $u^{\pm 1}$ as an initial segment then $\alpha \in E_H$.

Indeed, suppose $h = u^\alpha \circ h'$, where $\alpha \in \mathbb{Z}$ and h' does not have $u^{\pm 1}$ as an initial segment. Since $\Gamma(H)$ is U-folded, by the properties **(vii)** and **(viii)** of U-folded graphs there exists a path p_h in $\Gamma(H)$ such that $o(p_h) = 1_H, \overline{\mu(p_h)} = u^\alpha$ and $t(p_h) \in C$. Hence $u^\alpha = \overline{\mu(p_i)} * c$, where $p_i \in Rep(C)$ and $c \in H_u(1_H)$, that is, $u^\alpha \in \overline{\mu(p_i)} * H_u(1_H)$. Finally, since $H_u(1_H) \cap \langle u \rangle = \varepsilon$ then $\overline{\mu(p_i)} * H_u(1_H)$ contains not more than one finite exponent u^{n_i} of u and finiteness of $Rep(C)$ completes the proof of the claim. From the proof it is easy to see that E_H can be found effectively.

Denote

$$n_H = \max\{|n| \mid n \in E_H\}.$$

Observe that Claim holds also for $\Gamma(K)$ and D.

Suppose $C(u) \cap H \cap K \subsetneq u^k * H * u^{-k} \cap K$, that is, there exist $h \in H, f \in K$ such that $u^k * h * u^{-k} = f$ and $[h, u] \neq \varepsilon, [f, u] \neq \varepsilon$. Let $w_1 = \max\{U(h)\}$, $w_2 = \max\{U(f)\}$ and

$$h = h_1 \circ w_1^{\beta_1} \circ h_2 \circ \cdots \circ w_1^{\beta_r} \circ h_{r+1}$$

be the w_1-reduced form for h,

$$f = f_1 \circ w_2^{\gamma_1} \circ f_2 \circ \cdots \circ w_2^{\gamma_l} \circ f_{l+1},$$

be the w_2-reduced form for f. Consider the following cases.

I. $w_1 < u$

Since $[h, u] \neq \varepsilon$, by Lemma 6.9 [**15**] there exists $M_h \in \mathbb{N}$ (which can be found effectively), such that

$$u^{M_h+n} * h * u^{-M_h-n} = u^n \circ (u^{M_h} * h * u^{-M_h}) \circ u^{-n}$$

for any $n > 0$. Hence, if $k > M_h + n_K$ then

$$u^k * h * u^{-k} = u^{k-M_h} \circ (u^{M_h} * h * u^{-M_h}) \circ u^{-k+M_h}$$

has u^{k-M_h}, $k - M_h > n_K$ as an initial segment and, by Claim, can not be an element of K - a contradiction. Thus,

$$k \leq \min\{M_h + n_K \mid h \in H\}$$

and obviously this minimum exists and depends only on H and K.

II. $w_1 = u$

It follows that $w_1 = w_2$. Consider

$$u^k * (h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_r} \circ h_{r+1}) * u^{-k}.$$

a) Either $h_1 \neq \varepsilon$ or $h_{r+1} \neq \varepsilon$.

Without loss of generality we can assume $h_1 \neq \varepsilon$ and $\beta_1 > 0$. Since $[h_1, u] \neq \varepsilon$, by Lemma 6.9 [**15**] there exists $M_{h_1} \in \mathbb{N}$ (which can be found effectively), such that

$$u^{M_{h_1}+n_1} * h_1 * u^{M_{h_1}+n_2} = u^{n_1} \circ (u^{M_{h_1}} * h_1 * u^{M_{h_1}}) \circ u^{n_2}$$

for any $n_1, n_2 > 0$. Moreover, M_{h_1} can be chosen so that $u^{M_{h_1}} * (h_1 \circ w_1^{\beta_1})$ does not have u as an initial segment. Hence, if $k > M_{h_1} + n_K$ then

$$u^k * (h_1 \circ u^{\beta_1}) = u^{k-M_{h_1}} \circ (u^{M_{h_1}} * h_1 * u^{M_{h_1}}) \circ u^{\beta_1 - M_{h_1}}$$

and f has $u^{k-M_{h_1}},\ k - M_h > n_K$ as an initial segment - a contradiction because in this case, by Claim, f can not be an element of K. Thus,

$$k \leq \min \left\{ \begin{array}{c|l} M_{h_1} + n_K & \begin{array}{l}[h_1, u] \neq \varepsilon \text{ and there exists a path } p \text{ in } \Gamma(H) \\ \text{such that } o(p) = 1_H, \overline{\mu(p)} = h_1\end{array} \end{array} \right\}.$$

Clearly this minimum exists and depends only on H and K.

b) $h_1 = h_{r+1} = \varepsilon$

It follows immediately that $f_1 = f_{l+1} = \varepsilon$. If $r = 0$ then $[h, u] = \varepsilon$ and we have a contradiction. Thus, $r > 0$ and $u^{\gamma_1} = u^{\beta_1 + k}$, that is,

$$u^k = u^{\gamma_1} * u^{-\beta_1},$$

where there exist paths p in $\Gamma(H)$ and q in $\Gamma(K)$, such that $o(p) = 1_H, o(q) = 1_K$ and $\overline{\mu(p)} = u^{\beta_1},\ \overline{\mu(q)} = u^{\gamma_1}$.

III. $w_1 > u$

If $h_1 \neq \varepsilon$ then without loss of generality we can assume that $u = \max\{U(h_1)\}$. If

$$h_1 = g_1 \circ u^{\delta_1} \circ g_2 \circ \cdots \circ u^{\delta_m} \circ g_{m+1}$$

is the u-reduced form of h_1 then considering

$$u^k * (g_1 \circ u^{\delta_1} \circ g_2 \circ \cdots \circ u^{\delta_m} \circ g_{m+1})$$

and applying the argument from **II.** one gets the required result.

Suppose $h_1 = \varepsilon$. Since $[u, w_1] \neq \varepsilon$ then it follows that $w_2 = w_1$ and $f_1 \neq \varepsilon$, so, considering $u^{-k} * f * u^k$ and applying the argument from **II.** one gets the required result.

Finally, if

$$N_1 = \min\{M_h + n_K \mid h \in H\}, \quad N_2 = \min\{M_f + n_H \mid f \in K\},$$

$$N_3 = \min \left\{ \begin{array}{c|l} M_h + n_K & \begin{array}{l}[h, u] \neq \varepsilon \text{ and there exists a path } p \text{ in } \Gamma(H) \\ \text{such that } o(p) = 1_H, \overline{\mu(p)} = h\end{array} \end{array} \right\},$$

$$N_4 = \min \left\{ \begin{array}{c|l} M_f + n_H & \begin{array}{l}[f, u] \neq \varepsilon \text{ and there exists a path } q \text{ in } \Gamma(K) \\ \text{such that } o(q) = 1_K, \overline{\mu(q)} = f\end{array} \end{array} \right\},$$

then $N_3 \geq N_1,\ N_4 \geq N_2$ and we can set

$$N(H, K) = \max\{N_3, N_4\}.$$

□

Definition 3. *Let Γ be a finite $(\mathbb{Z}[t], X)$-graph and $v_0 \in V(\Gamma)$. We denote by S_{Γ,v_0} a finite set of paths in Γ such that:*

(1) *for any $p \in S_{\Gamma,v_0}$, $o(p) = v_0$;*
(2) *for any $v \in V(\Gamma)$ there exists $p \in S_{\Gamma,v_0}$ such that $t(p) = v$;*
(3) *for any $p_1, p_2 \in S_\Gamma$ we have $t(p_1) \neq t(p_2)$.*

Now, combining Proposition 6 with Lemmas 17 and 18 we prove the main technical result about intersections of finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Proposition 7. *Let $H, K \leq F^{\mathbb{Z}[t]}$ be finitely generated and $\Gamma(H), \Gamma(K)$ be U-folded $(\mathbb{Z}[t], X)$-graphs such that $H = L(\Gamma(H), 1_H)$, $K = L(\Gamma(K), 1_K)$ for some $1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$. Let*

$$S_{\Gamma(H),1_H} = \{p_1, \ldots, p_n\},\ S_{\Gamma(K),1_K} = \{q_1, \ldots, q_m\}.$$

If $g \in F^{\mathbb{Z}[t]}$ is such that $g * H * g^{-1} \cap K \neq \varepsilon$. then one of the following holds:

1.
$$g * H * g^{-1} \cap K = f * (H^{y_j * {z_i}^{-1}} \cap K) * f^{-1},$$
where z_i, y_j are labels of some p_i, q_j correspondingly, $f \in K$ and $g \in K * (y_j * {z_i}^{-1}) * H$;

2.
$$g * H * g^{-1} \cap K = f * (C(u)^{y_j} \cap H^{y_j * {z_i}^{-1}} \cap K) * f^{-1},$$
where $u \in U(\Gamma(H)) \cap U(\Gamma(K))$, z_i, y_j are labels of some p_i, q_j correspondingly, $f \in K$, $g \in K * (y_j * u^k * {z_i}^{-1}) * H$, $k \in \mathbb{Z}$, $g \notin KH$ and $C(u)^{y_j} \cap K \neq \varepsilon$;

3.
$$g * H * g^{-1} \cap K = f * (H^{y_j * u^k * {z_i}^{-1}} \cap K) * f^{-1},$$
where $u \in U(\Gamma(H)) \cap U(\Gamma(K))$, $|k| < N(H, K)$ ($N(H, K)$ depends only on H, K and can be found effectively), z_i, y_j are labels of some p_i, q_j correspondingly, $f \in K$ and $g \in K * (y_j * u^k * {z_i}^{-1}) * H$;

4.
$$g * H * g^{-1} \cap K = f * (\langle u\rangle^{y_j} \cap H^{y_j * {z_i}^{-1}} \cap K) * f^{-1},$$
where $u \in U(g)$, z_i, y_j are labels of some p_i, q_j correspondingly, $f \in K$, $g \in K * (y_j * u^\alpha * {z_i}^{-1}) * H$, $deg(\alpha) \leq deg_u(g)$, $g \notin KH$ and $\langle u\rangle^{y_j} \cap K \neq \varepsilon$.

Proof. The claim below follows directly from the definition of $S_H = S_{\Gamma(H),1_H}$ and $S_K = S_{\Gamma(K),1_K}$.

Claim. If p is a path in $\Gamma(H)$, q is a path in $\Gamma(K)$, such that $o(p) = 1_H$, $o(q) = 1_K$ and $\overline{\mu(p)} = z^{-1}$, $\overline{\mu(q)} = y$ then there exist $p_i \in S_H$, $q_j \in S_K$ such that

$$z = z_i^{-1} * h,\ y = f * y_j,$$

where $z_i = \overline{\mu(p_i)}$, $y_j = \overline{\mu(q_j)}$ and $h \in H$, $f \in K$.

Now, we are ready to prove the proposition.

By Proposition 6, g can be represented as a product

$$g = y * u^\alpha * z,\ \alpha \in \mathbb{Z}[t],$$

so that there exist a path p in $\Gamma(H)$ starting at 1_H with $\overline{\mu(p)} = z^{-1}$ and a path q in $\Gamma(K)$ starting at 1_K with $\overline{\mu(q)} = y$, and one of the following four cases holds.

Case 1. $\alpha = 0$.

Since $g = y * z$ we have

$$g * H * g^{-1} \cap K = (y * z) * H * (y * z)^{-1} \cap K$$

and the required result follows from Claim.

Observe that in all the following cases we can assume $g \notin KH$ because otherwise we get into **Case 1.**

Case 2. $\alpha \in \mathbb{Z}[t] - \mathbb{Z},\ u \in U(\Gamma(H)) \cap U(\Gamma(K))$.

Let $H_1 = L(\Gamma(H), t(p)),\ K_1 = L(\Gamma(K), t(q))$, then by Lemma 11, we have

$$H_1 = z * H * z^{-1}, \quad K_1 = y^{-1} * K * y$$

and

$$u^\alpha * H_1 * u^{-\alpha} \cap K_1 \neq \varepsilon.$$

By Lemma 17 we have one of the following two cases.

a) u^α can be represented as a product

$$u^\alpha = y' * w^k * z',\ k \in \mathbb{Z},$$

where there exist paths p' in $\Gamma(H)$ and q' in $\Gamma(K)$, such that $o(p') = t(p), o(q') = t(q)$ and $\overline{\mu(p')} = {z'}^{-1}$, $\overline{\mu(q')} = y'$, and if $k \neq 0$ then $t(p')$ and $t(q')$ belong to some w-components of $\Gamma(H)$ and $\Gamma(K)$ respectively, so that,

$$H_w(t(p')) \cap \langle w \rangle = H_w(t(q')) \cap \langle w \rangle = \varepsilon$$

and

$$C(w) \cap H_1^{z'} \cap K_1^{{y'}^{-1}} \subseteq w^k * H_1^{z'} * w^{-k} \cap K_1^{{y'}^{-1}}.$$

If $k = 0$ then $g = (y * y') * (z' * z)$, where $y * y'$ is the label of qq' and $z^{-1} * {z'}^{-1}$ is the label of pp', and this case reduces to **Case 1.**

Suppose $k \neq 0$. Let $H_2 = L(\Gamma(H), t(p')),\ K_2 = L(\Gamma(K), t(q'))$. Then by Lemma 11, we have

$$H_2 = z' * H_1 * {z'}^{-1}, \quad K_2 = {y'}^{-1} * K_1 * y'$$

and

$$w^k * H_2 * w^{-k} \cap K_2 \neq \varepsilon.$$

By Lemma 18 we have one of the following three possibilities.

a1) w^k can be represented as a product $w^k = y'' * z''$, where there exist paths p'' in $\Gamma(H)$ and q'' in $\Gamma(K)$, such that $o(p'') = t(p'),\ o(q'') = t(q'),\ \overline{\mu(p'')} = {z''}^{-1}$, $\overline{\mu(q'')} = y''$ - in this case $g = (y * y' * y'') * (z'' * z' * z)$ and we have a reduction to **Case 1.**.

a2) $w^k * H_2 * w^{-k} \cap K_2 = C(w) \cap H_2 \cap K_2$.

In this case we have

$$w^k * (z' * z * H * z^{-1} * {z'}^{-1}) * w^{-k} \cap {y'}^{-1} * y^{-1} * K * y * y' =$$
$$= C(w) \cap (z' * z * H * z^{-1} * {z'}^{-1}) \cap ({y'}^{-1} * y^{-1} * K * y * y'),$$

where $g = y * y' * w^k * z' * z$, so

$$g * H * g^{-1} \cap K = C(w)^{y * y'} \cap H^{y * y' * z' * z} \cap K,$$

where $y * y'$ is the label of qq' and $z^{-1} * {z'}^{-1}$ is the label of pp'. Hence, the required result follows from Claim.

a3) $|k| \leq N(H,K)$, where $N(H,K)$ depends only on H and K and can be found effectively,

So $g = y * y' * w^k * z' * z$ and

$$g * H * g^{-1} \cap K = H^{y*y'*w^k*z'*z} \cap K,$$

where $|k| \leq N(H,K)$, and since $y * y'$ is the label of qq' and $z^{-1} * z'^{-1}$ is the label of pp', the required result follows from Claim.

b) $u^\alpha * H_1 * u^{-\alpha} \cap K_1 = C(u) \cap H_1 \cap K_1$.

In this case we have

$$(u^\alpha * z) * H * (u^\alpha * z)^{-1} \cap y^{-1} * K * y = C(u) \cap z * H * z^{-1} \cap y^{-1} * K * y,$$

and since $g = y * u^\alpha * z$ then

$$g * H * g^{-1} \cap K = C(u)^y \cap H^{y*z} \cap K,$$

and the required result follows from Claim.

Case 3. $0 \neq \alpha \in \mathbb{Z},\ u \in U(\Gamma(H)) \cap U(\Gamma(K))$ and either $\alpha = \pm 1$ or $t(p)$ and $t(q)$ belong to some u-components of $\Gamma(H)$ and $\Gamma(K)$ respectively, so that,

$$H_u(t(p)) \cap \langle u \rangle = H_u(t(q)) \cap \langle u \rangle = \varepsilon$$

and

$$C(u) \cap H^z \cap K^{y^{-1}} \subseteq u^\alpha * H^z * u^{-\alpha} \cap K^{y^{-1}}.$$

If $\alpha = \pm 1$ then

$$g * H * g^{-1} \cap K = H^{y*u^k*z} \cap K,$$

where $|k| = 1 \leq N(H,K)$, and since y is the label of q and z^{-1} is the label of p, the required result follows from Claim.

If $\alpha \neq \pm 1$ then the required result follows by the argument presented in **Case 2.a)**.

Case 4. $\alpha \in \mathbb{Z}[t] - \mathbb{Z},\ u \notin U(\Gamma(H)) \cap U(\Gamma(K))$ but $u \in U(g)$ and

$$u^\alpha * H^z * u^{-\alpha} \cap K^{y^{-1}} = \langle u \rangle \cap H^z \cap K^{y^{-1}}.$$

Observe that $deg(\alpha) \leq deg_u(g)$. In this case it follows that

$$g * H * g^{-1} \cap K = \langle u \rangle^y \cap H^{y*z} \cap K$$

and the required result follows from Claim.

□

From Proposition 7 one can derive the following important corollaries.

Theorem 7. *Let H, K be finitely generated subgroups of a finitely generated fully residually free group G. Then one can effectively find a finite family $\mathcal{J}_G(H,K)$ of non-trivial finitely generated subgroups of G (given by finite generating sets), such that*

(1) *every $J \in \mathcal{J}_G(H,K)$ is of one of the following types*

$$H^{g_1} \cap K, \quad H^{g_1} \cap C_K(g_2),$$

where $g_1 \in G - H,\ g_2 \in K$, moreover $g_1,\ g_2$ can be found effectively;

(2) *for any non-trivial intersection* $H^g \cap K,\ g \in G - H$ *there exists* $J \in \mathcal{J}_G(H,K)$ *and* $f \in K$ *such that*

$$H^g \cap K = J^f,$$

moreover J *and* f *can be found effectively.*

Proof. By Theorem 3 one can effectively obtain generators of G, H and K viewed as infinite words and by Proposition 8.3 [**15**] compute their standard decompositions.

By Proposition 4 one can effectively find finite U-folded $(\mathbb{Z}[t], X)$-graphs $\Gamma(G)$, $\Gamma(H)$ and $\Gamma(K)$, such that $G = L(\Gamma(G), 1_G), H = L(\Gamma(H), 1_H),\ K = L(\Gamma(K), 1_K)$ for some $1_G \in V(\Gamma(G)), 1_H \in V(\Gamma(H)), 1_K \in V(\Gamma(K))$.

Let $g \in G$ be such that $H^g \cap K \neq \varepsilon$. From Proposition 7 it follows that $H^g \cap K = J^f$ where $f \in K$ and one of the following holds.

1. $J = H^{y*u^k*z^{-1}} \cap K$, where $u \in U(\Gamma(H)) \cap U(\Gamma(K)),\ |k| < N(H,K)$ ($N(H,K)$ can be found effectively), z, y are labels of some $p \in S_{\Gamma(H),1_H},\ q \in S_{\Gamma(K),1_K}$ and $y * u^k * z^{-1} \in G$.

2. $J = C(u)^y \cap H^{y*z^{-1}} \cap K$, where $u \in U(\Gamma(H)) \cap U(\Gamma(K)),\ z, y$ are labels of some $p \in S_{\Gamma(H),1_H},\ q \in S_{\Gamma(K),1_K}$. In this case $g \in K * (y * u^k * z^{-1}) * H,\ k \in \mathbb{Z},\ g \notin KH$ and $C(u)^y \cap K \neq \varepsilon$. We have

$$C(u)^y \cap H^{y*z^{-1}} = (C(u) \cap H^{z^{-1}})^y = (C(u) \cap H^{z^{-1}})^{(y*u^\alpha)} = C(u)^y \cap H^{y*u^\alpha*z^{-1}}$$

for any $\alpha \in \mathbb{Z}[t]$, so

$$C(u)^y \cap H^{y*z^{-1}} \cap K = C(u)^y \cap H^{y*u^k*z^{-1}} \cap K,$$

where $y * u^k * z^{-1} \in G$. Finally, since $C(u)^y \cap K \neq \varepsilon$ then

$$C(u)^y \cap K = C(u^\alpha)^y \cap K = C_K(y * u^\alpha * y^{-1}),$$

where $y * u^\alpha * y^{-1} \in K$.

3. $J = \langle u \rangle^y \cap H^{y*z^{-1}} \cap K$, where $u \in U(\Gamma(G)),\ z, y$ are labels of some $p \in S_{\Gamma(H),1_H},\ q \in S_{\Gamma(K),1_K}$. In this case $g \in K * (y_j * u^\beta * z_i{}^{-1}) * H,\ deg(\beta) \leq deg_u(g),\ g \notin KH$ and $\langle u \rangle^{y_j} \cap K \neq \varepsilon$ and we have

$$\langle u \rangle^y \cap H^{y*z^{-1}} = (\langle u \rangle \cap H^{z^{-1}})^y = (\langle u \rangle \cap H^{z^{-1}})^{(y*u^\alpha)} = \langle u \rangle^y \cap H^{y*u^\alpha*z^{-1}}$$

for any $\alpha \in \mathbb{Z}[t]$. Thus,

$$\langle u \rangle^y \cap H^{y*z^{-1}} \cap K = \langle u \rangle^y \cap H^{y*u^\beta*z^{-1}} \cap K,$$

where $y * u^\beta * z^{-1} \in G$. Finally, since $\langle u \rangle^y \cap K \neq \varepsilon$ then

$$\langle u \rangle^y \cap K = \langle u^y \rangle \cap K = C_K(y * u^k * y^{-1}),$$

where $y * u^k * y^{-1} \in K$.

Let $\mathcal{J}_G(H,K)$ be composed by all non-trivial J which have the conditions listed in 1)-3) above. Observe that this is a finite family of subgroups of G, which has the required properties listed in the statement of the theorem.

$\mathcal{J}_G(H,K)$ can be found effectively. Indeed, $U(\Gamma(H)),\ U(\Gamma(K)),\ S_{\Gamma(H),1_H}$, $S_{\Gamma(K),1_K}$ are finite, moreover

1) $|k| < N(H,K)$, so there are only finitely many elements of the form $y * u^k * z^{-1}$, and we need only those of them which belong to $G - H$ - this can be checked effectively;

2) at first, observe that

$$C_K(u^y) = C(u)^y \cap K = (\bigoplus_{n=0}^{M} \langle u^n \rangle)^y \cap K,$$

where $M = deg_u(\Gamma(K)) + 1$, and on the other hand for each triple $y,\ u,\ z$ we have to find $y * u^k * z^{-1}$, which belongs to $G - H$, this can be done effectively - the required element exists if and only if

$$y * \langle u \rangle * z^{-1} \cap G \neq \emptyset$$

and

$$(y * \langle u \rangle * z^{-1} \cap G) \cap H \neq (y * \langle u \rangle * z^{-1} \cap G),$$

where both can be checked effectively by Theorem 5 and if the answer is positive the required element can be found checking step by step $y * u^k * z^{-1}$ for all $k \in \mathbb{N}$.

3) at first, from $g \in G$ it follows that $U(g) \subseteq U(\Gamma(G))$ and since G is finitely generated then $U(\Gamma(G))$ is finite, and on the other hand for each triple $y,\ u,\ z$ we have to find $y * u^\beta * z^{-1},\ deg(\beta) \leq deg_u(g)$, which belongs to $G - H$, this can be done effectively - the required element exists if and only if

$$y * (\bigoplus_{n=0}^{M} \langle u^n \rangle) * z^{-1} \cap G \neq \emptyset,$$

for $M = deg_u(\Gamma(G)) + 1$ and

$$(y * (\bigoplus_{n=0}^{M} \langle u^n \rangle) * z^{-1} \cap G) \cap H \neq y * (\bigoplus_{n=0}^{M} \langle u^n \rangle) * z^{-1} \cap G,$$

both can be checked effectively by Theorem 5 and if the answer is positive the required element can be found checking step by step $y * u^\beta * z^{-1}$ for all $M + 1$-tuples β.

Finally, suppose H, K and g are fixed and such that $H^g \cap K \neq 1$. Compose $\mathcal{J}_G(H,K)$. It is finite and its elements can be enumerated. Enumerate effectively elements of K taking formal products of generators. Thus, all J^f, where $J \in \mathcal{J}_G(H,K),\ f \in K$ can be effectively enumerated. Since $H^g \cap K \neq 1$ it follows that there exists $J \in \mathcal{J}_G(H,K)$ and $f \in K$ such that $H^g \cap K = J^f$, and comparing step by step $H^g \cap K$ with enumerated J^f eventually one obtains the required.

□

Corollary 5. *Let H, K be finitely generated subgroups of a finitely generated fully residually free group G. Then up to conjugation by elements from K there are only finitely many subgroups of G of the type $H^g \cap K$.*

Corollary 6. *Let H be a finitely generated subgroup of a finitely generated fully residually free group G. Then there is an algorithm which decides if H is malnormal in G or not.*

Proof. Observe that H is malnormal in G if and only if for any $g \in G - H$, $H^g \cap H = 1$. This is equivalent to $\mathcal{J}_G(H,H) = \emptyset$. By Theorem 7 one can compute effectively $\mathcal{J}_G(H,H)$ and check if it is empty. □

Corollary 7. *Let H, K be finitely generated subgroups of a finitely generated fully residually free group G. Then one can effectively check if there exists $g \in G$ such that*

(1) $H^g = K$,
(2) $H^g \leq K$.

Moreover, g can be found effectively.

Proof. 1) Observe that $H^h = K$ for some $h \in H$ if and only if $H = K$. One can verify this algorithmically since the membership problem in G is decidable (Proposition 5).

Claim. H and K are conjugate in G by $g \in G - H$ if and only if $\mathcal{J}_G(H, K) \neq \emptyset$ and there exists $J \in \mathcal{J}_G(H, K)$ either of the form $H^{g_1} \cap K$ or $H^{g_1} \cap C_K(g_2)$, where $g_1 \in G - H,\ g_2 \in K$, such that $H^{g_1} = K$.

Suppose there exists $g \in G - H$ for which $H^g = K$. By Theorem 7 there exists $J \in \mathcal{J}_G(H, K) \neq \emptyset$ such that $H^g \cap K = J^f,\ f \in K$ and by the construction of $\mathcal{J}_G(H, K)$ one of the following holds.

a) $J = H^{g_1} \cap K,\ g_1 \in G - H$ and $g \in K * g_1 * H$.

Thus, we have $J = K$ and $J = H^g = H^{g_1}$, so $J = H^{g_1} = K$.

b) $J = H^{g_1} \cap C_K(g_2),\ g_1 \in G - H, g_2 \in K$ and $H^{g_1} \cap C_K(g_2) = H^g \cap C_K(g_2)$.

Since $J = K = H^g$ we have $K = K \cap C_K(g_2)$, so $K = C_K(g_2)$ and $H^g = J = H^{g_1} \cap K$. It follows that $H^g \leq H^{g_1}$. But since K is free abelian of a finite rank, so are $H^g,\ H^{g_1}$. We have $H^{g_1 g} \leq H$. Observe that H is contained in some maximal abelian subgroup H_1 of G, which is malnormal. Thus $g_1 g \in H_1$ and it follows that $[g_1 g, h] = 1$ for all $h \in H$. Hence $H^{g_1 g} = H,\ H^g = H^{g_1}$ and $J = H^{g_1} = K$.

Conversely, if $\mathcal{J}_G(H, K) \neq \emptyset$ and there exists $J \in \mathcal{J}_G(H, K)$ either of the form $H^{g_1} \cap K$ or $H^{g_1} \cap C_K(g_2)$, where $g_1 \in G - H,\ g_2 \in K$ and $H^{g_1} = K$ then immediately H and K are conjugate in G. This completes the proof of the claim.

From Claim and Theorem 7 it follows that one can verify effectively if $H^g = G,\ g \in G - H$ and in case of positive answer effectively find such g.

2) If $g \in H$, then $H^g \leq K$ is equivalent to $H \leq K$. This is algorithmically decidable.

If $g \in G - H$ and $H^g \leq K$ then $H^g = H^g \cap K$ and hence is a conjugate of some $J \in \mathcal{J}_G(H, K)$, or, equivalently, H is a conjugate of J. Observe, that the converse is also true, i.e., if $H^g = J$ for some $J \in \mathcal{J}_G(H, K)$, then $H^g \leq K$. Finally, the conjugacy of H and elements of $\mathcal{J}_G(H, K)$ can be verified algorithmically by Theorem 7 and Statement 1) above.

□

Corollary 8. *Let H be a finitely generated subgroup of a finitely generated fully residually free group G and $h \in G$. Then one can effectively check if there exists $g \in G$ such that $h^g \in H$.*

Proof. Let $K = \langle h \rangle$. By Theorem 7 one can compute effectively $\mathcal{J}_G(H, K)$ such that if $H^g \cap K \neq 1$ then there exists $J \in \mathcal{J}_G(H, K)$ for which we have

$$H^g \cap K = J^f$$

for some $f \in K$.

It follows that

$$\exists\, g \in G :\ h^g \in H \Longleftrightarrow \langle h \rangle \in \mathcal{J}_G(H, K).$$

□

5. Centralizers in finitely generated fully residually free groups

Let G be a finitely generated fully residually free group. By Theorem 1, for a given finite presentation of G one can effectively construct an embedding of G into $F^{\mathbb{Z}[t]}$. Moreover, by Theorem 2 one can effectively find a finite series

$$F = G_0 < G_1 < G_2 < \cdots < G_n$$

for G, such that G_{i+1} is obtained from G_i by a centralizer extension of a single element $u_i \in G_i$ and $G \leq G_n,\ G \not\leq G_{n-1}$.

Recall that if H is a group then the centralizers extension of a single element $u \in H$ is an HNN-extension of the form

$$H(u,s) = H *_{\langle u \rangle} (\langle u \rangle \times \langle s \rangle) = \langle H, s \mid u^s = u \rangle.$$

Every $h \in H(u,s)$ can be represented (non-uniquely) as

$$h = h_0 s^{n_1} h_1 \cdots h_{k-1} s^{n_k} h_k,$$

where $k \geq 1$ and $[h_i, u] \neq 1, i \in [0, k-1],\ n_i \neq 0, i \in [1, k]$. The *syllable length* of h in this case is $2k+1$. h is called *cyclically reduced* if $[h_k h_0, u] \neq 1$. $\|h\|$ denotes the syllable length of a cyclically reduced element which is conjugate to h in $H(u,s)$.

LEMMA 19. [**13**] *Let H be a CSA-group and $G = H(u,s)$ a centralizer extension of $u \in H$. If $h \in G$ then one of the following holds:*

1) *if $h \in H^g$ for some $g \in G$, then $C_G(h) = C_H(h)^g \leq H^g$;*
2) *if $h \in (\langle u \rangle \times \langle s \rangle)^g$ for some $g \in G$, then $C_G(h) = (\langle u \rangle \times \langle s \rangle)^g$;*
3) *if $\|h\| \geq 2$ then $C_G(h) = \langle z \rangle$, where $h = z^m$ for some $m \in \mathbb{N}$.*

LEMMA 20. *Let G be a finitely generated fully residually free group. Then:*

1) *each proper centralizer of G is a free abelian group of finite rank;*
2) *the set $Spec(G) = \{rank(C_G(g)) \mid 1 \neq g \in G\}$ is finite.*

Proof. Follows from Lemma 19.

□

Lemma 19 provides a tool for computing centralizers in all G_n.

LEMMA 21. *Let*

$$F = G_0 < G_1 < G_2 < \cdots < G_n$$

be a series of groups where every G_{i+1} is obtained from G_i by a centralizer extension of a single element $u_i \in G_i$. Then for any $h \in G_n$ one can effectively find a finite set of generators of the centralizer $C_{G_n}(h)$.

Proof. We use the induction on n. If $n = 0$ then $G_n = F$ and for a non-trivial $h \in F$ the centralizer $C_F(h)$ is cyclic generated by the maximal root g of h. Notice, that one can find the root g effectively (see, for example, [**15**]).

Assume that the centralizers of elements of G_{n-1} can be found effectively and let $h \in G_n$. By Lemma 19 there are three cases to consider:

Case 1. $h \in G_{n-1}^g$ for some $g \in G_n$.

Then $h = z^g$ for some $z \in G_{n-1}$ and $C_{G_n}(h) = C_{G_{n-1}}(z)^g$. Observe that z and g can be found effectively.

Case 2. $h \in C_{G_n}(u_n)^g$ for some $g \in G_n$.

Observe, that such g can be found effectively. In this case $C_{G_n}(h) = C_{G_n}(u_n)^g$.
Case 3. $\|h\| \geq 2$.

Then $C_{G_n}(h) = \langle z \rangle$, where z is the maximal root of h. As we have mentioned above such z can be found effectively.

□

THEOREM 8. *For any finitely generated fully residually free group G and a finite non-empty subset $M \subset G$, one can effectively find a finite set of generators of $C_G(M)$.*

Proof. We may assume, as above, that G is a finitely generated subgroup of a group G', which is obtained from a free group F by finitely many centralizer extensions. Observe that by Theorem 2 one can find G' effectively.

Let $M = \{h\} \subset G$. Then $C_G(h) = C_{G'}(h) \cap G$ and in this case the required result follows from Lemma 21 and Theorem 6.

Let $M = \{h_1, \ldots, h_n\}$. Then

$$C_G(M) = \bigcap_{i=1}^{n} C_G(h_i)$$

and the result follows from the result above and Theorem 6. This proves the theorem.

□

THEOREM 9. *For any finitely generated fully residually free group G one can find the set $Spec(G)$ effectively.*

Proof. By Theorem 2, G can be effectively embedded into a group G' which is obtained from a free group by finitely many centralizer extensions. We have

$$F = G_0 < G_1 < G_2 < \cdots < G_n = G',$$

where every G_{i+1} is obtained from G_i by a centralizer extension of a single element $u_i \in G_i$. Observe that any non-cyclic centralizer in G' is a conjugate of $C_{G'}(u_i)$ for some $i \in [1, n]$. Thus, a non-cyclic centralizer of $h \in G$ is of the form

$$C_G(h) = C_{G'}(u_i)^g \cap G$$

for some $g \in G'$.

We can view G and G' as subgroups of $F^{\mathbb{Z}[t]}$. For each $u_i, i \in [1, n]$ compose $\mathcal{J}_{G'}(C_{G'}(u_i), G)$ which by definition consists of subgroups of G' isomorphic to abelian groups of finite ranks. Finally we have

$$Spec(G) = \bigcup_{i=1}^{n} \{rank(J) \mid J \in \mathcal{J}_{G'}(C_{G'}(u_i), G)\}.$$

To finish the proof it suffices to show that for a finitely generated subgroup J of G, given by a finite generating set, one can effectively find the rank of J. Indeed by [**9**] we can find the presentation of J by generators and relations. Using this presentation and the structure theorem for finitely generated abelian groups we can find the rank of J. □

6. Conjugacy problem in finitely generated subgroups of $F^{\mathbb{Z}[t]}$

Effectiveness of construction of the intersection of two finitely generated subgroups of $F^{\mathbb{Z}[t]}$ makes it possible to solve the conjugacy problem for finitely generated subgroups of $F^{\mathbb{Z}[t]}$ and hence, for finitely generated fully residually free groups.

THEOREM 10. *Any finitely generated subgroup of $F^{\mathbb{Z}[t]}$ has a solvable conjugacy problem. That is, there exists an algorithm which, given standard decompositions of elements $f, g \in H = \langle h_1, \ldots, h_k \rangle$, decides if f is conjugate to g in H, and if yes, generates an element $c \in H$ such that $c^{-1}fc = g$.*

Proof. H can be effectively embedded into a group G which is obtained from a free group by finitely many centralizer extensions. We have

$$F = G_0 < G_1 < G_2 < \cdots < G_n = G,$$

where every G_{i+1} is obtained from G_i by a centralizer extension of a single element $u_i \in G_i$ and $H \leq G_n, H \not\leq G_{n-1}$.

By Corollary 8.6 [**15**], one can find effectively cyclic decompositions

$$f = u_f^{-1} \circ \bar{f} \circ u_f, \; g = u_g^{-1} \circ \bar{g} \circ u_g,$$

where $u_f, \; \bar{f}, \; u_g, \; \bar{g} \in G$. Then by Lemma 8.9 [**15**], one can determine if $x^{-1} * \bar{f} * x = \bar{g}$ for some $x \in F^{\mathbb{Z}[t]}$.

If there exists no such $x \in F^{\mathbb{Z}[t]}$ then f is not conjugate to g in H and we are done.

Suppose such x exists, then it follows from Lemmas 8.8 and 8.9 [**15**] that it can be found effectively and $x \in G$. Moreover, by Lemma 8.7 [**15**] for any x_1 such that $x_1^{-1} * \bar{f} * x_1 = \bar{g}$ we have $x_1 \in C(\bar{f}) * x$. Since x exists, it follows that there exists $y \in G$ such that $y^{-1} * f * y = g$ and

$$(u_g * y^{-1} * u_f^{-1}) * \bar{f} * (u_f * y * u_g^{-1}) = \bar{g}.$$

Hence, $u_f * y * u_g^{-1} \in C(\bar{f}) * x$ and $y \in u_f^{-1} * C(\bar{f}) * x * u_g$. Finally, we have

$$f \text{ is conjugate to } g \text{ in } H \iff H \cap (u_f^{-1} * C(\bar{f}) * x * u_g) \neq \emptyset$$

or equivalently

$$f \text{ is conjugate to } g \text{ in } H \iff H \cap (C(f) * (u_f^{-1} * x * u_g)) \neq \emptyset.$$

In fact it is enough to take $C_G(f)$ instead of $C(f)$ because we need only to check elements of $C(f) * (u_f^{-1} * x * u_g)$, which belong to G. Thus,

$$f \text{ is conjugate to } g \text{ in } H \iff H \cap (C_G(f) * (u_f^{-1} * x * u_g)) \neq \emptyset.$$

By Theorem 5 one can effectively check if $H \bigcap (C_G(f) * (u_f^{-1} * x * u_g))$ is empty. □

Theorem 10 can be reformulated for finitely generated fully residually free groups.

THEOREM 11. *Any finitely generated fully residually free group has solvable conjugacy problem.*

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