G-constructible groups

Olga Kharlampovich[*] Alexei Myasnikov[†]
Vladimir Remeslennikov[‡] Denis Serbin

August 19, 2005

Abstract

In this paper we study G-constructible groups which are finitely generated subgroups of $\mathbb{Z}[t]$-completion $G^{\mathbb{Z}[t]}$ of a given CSA-group G. Using Bass-Serre theory we prove that G-constructible groups can be obtained from subgroups of G by free constructions of a special type. As an application of this technique we compute cohomological and homological dimensions of fully residually free groups which can be viewed as G-constructible groups, where G is a free group.

1 Introduction

Let $\mathcal{K}$ be a class of groups. By $\mathcal{K}$-constructible groups we denote groups which can be obtained from groups from $\mathcal{K}$ by finitely many operations of a certain type: free products, extensions of centralizers, free products with amalgamation along abelian subgroups one of which is maximal, and HNN-extensions with abelian associated subgroups one of which is maximal. We call these operations *elementary operations*.

In particular, for a group G one can consider $\mathcal{K}$-constructible groups, where $\mathcal{K} = Sub(G)$ is the class of all subgroups of G. These groups play an important part in algebraic geometry over groups and theory of quasi-varieties (see [18]). Our interest to such groups originates from the following open problem:

> For a given torsion-free hyperbolic group G describe finitely generated G-groups which are G-universally equivalent to G.

It is known [1, 17] that finitely generated G-groups G-universally equivalent to G are precisely the coordinate groups of irreducible algebraic sets over G, or equivalently, the finitely generated groups discriminated by G. In [17] the authors, following Lyndon [14], introduced a $\mathbb{Z}[t]$-completion $G^{\mathbb{Z}[t]}$ of a given CSA-group G. In paper [2] it was shown that finitely generated subgroups of

[*]Supported by NSERC grant
[†]Supported by NSERC grant
[‡]Supported by RFFI grant

$G^{\mathbb{Z}[t]}$ are G-universally equivalent to G. There are reasonable indications that the reverse is also true.

Conjecture. *A finitely generated G-group H is G-universally equivalent to a given torsion-free hyperbolic group G if and only if H is embeddable into $G^{\mathbb{Z}[t]}$.*

Observe, that this conjecture holds when G is a free non-abelian group [9, 10].

Finitely generated subgroups of $G^{\mathbb{Z}[t]}$ we call *G-constructible groups.*

Notice, the group $G^{\mathbb{Z}[t]}$ is a union of an ascending chain of extensions of centralizers of the group G (see [17]), so every G-constructible group H is also a subgroup of a finite chain of extensions of centralizers of G. Therefore, H is a subgroup of the fundamental group of a very particular graph of groups. Now, Bass-Serre theory tells one that H is itself the fundamental group of an induced graph of groups, hence (by induction) it can be obtained by free constructions from subgroups of G and it can be shown that free constructions applied are in fact elementary operations.

In fact, Bass-Serre theory gives one of many possible ways to obtain H from subgroups of G by elementary operations (which are described by *construction trees*, see Section 2). Every construction tree for H gives rise to the corresponding *$Sub(G)$-decomposition* of H. Clearly, $Sub(G)$-decompositions of H are closely related to very powerful JSJ-decompositions of H (see [19]) and hierarchy theorems (see [7]). But in many cases it is much easier to construct $\mathbb{Z}$-splittings via construction trees, which are as robust as any other decompositions of H.

In this paper we show how one can refine a given construction tree for H to obtain a particularly nice $Sub(G)$-decomposition of H. As an example, we would like to mention the following result from Section 4.

Theorem 4 (p.16). *Let G be a non-abelian CSA-group. Then any G-constructible group H can be obtained from $\mathbb{Z}$ and finitely many subgroups of G by finitely many operations of the following type: free products, extensions of centralizers, free products with amalgamation along maximal abelian subgroups in both factors and separated HNN-extensions with maximal abelian associated subgroups.*

Moreover, in the case of finitely generated fully residually free groups their $\mathbb{Z}$-decompositions as above can be constructed effectively (Corollary 3 in Section 4).

In Section 5 we apply this technique to study cohomological [homological] dimension $cd(H)$ [$hd(H)$] of a fully residually free group H

Mayer-Vietoris sequences allow one to derive a very simple formula for $cd(H)$ in terms of the ranks of centralizers of H. Namely, the following result holds (in fully residually free groups all centralizers are free abelian of finite rank, so here $rank_C(G)$ is the maximal rank of centralizers of G):

Theorem 5 (p.19). *Let G be a fully residually free group. Then*

1) if $rank_C(G) \geq 2$ then $hd(G) = cd(G) = rank_C(G)$,

2) *if* $rank_C(G) = 1$ *then* $hd(G) = cd(G) \leq 2$ *and* $hd(G) = cd(G) = 2$ *if and only if* G *is not free.*

Moreover, one can compute $cd(G)$ effectively.

Theorem 8 (p.20). *There exists an algorithm which for every finitely generated fully residually free group* G *computes* $cd(G)$.

2 $\mathcal{K}$-constructible groups

In this section we consider groups that can be obtained from a given collection of groups by finitely many free products with amalgamation and HNN-extensions of a very particular type.

The following constructions are called *elementary operations*:

1) a free product $G * H$ of groups G and H;

2) a free product with amalgamation of groups G and H of the type

$$G *_A H = \langle G * H \mid a = a^\phi \ (a \in A) \rangle,$$

where A is a maximal abelian subgroup in G and $\phi : A \to H$ is an embedding;

3) an HNN-extension of a group G (so-called *separated HNN-extension*)

$$\langle G, t \mid a^t = a^\phi (a \in A) \rangle,$$

where A is a maximal abelian subgroup of G, $\phi : A \to G$ is a monomorphism such that $g^{-1}Ag \cap A^\phi = 1$ for every $g \in G$;

4) a free extension of a centralizer of a group G, that is, the following free product with amalgamation

$$G(u, A) = G *_{C(u)} (C(u) \times A),$$

where $u \in G, u \neq 1,\ C(u)$ is the centralizer of u in G, A is a free abelian group of finite rank and the same elements from $C(u)$ are identified.

Notice that we do not assume in 2) and 3) that A^ϕ is a maximal abelian subgroup in the ambient group.

Definition 1 *Let* $\mathcal{K}$ *be a class of groups. By* $\mathcal{K}^c$ *we denote the minimal class of groups which contains all groups from* $\mathcal{K}$ *and is closed under elementary operations and isomorphisms. Groups from* $\mathcal{K}^c$ *are termed* $\mathcal{K}$-constructible.

If $\mathcal{K}$ is a class of all finitely generated free groups then $\mathcal{K}$-constructible groups are called *free constructible*. Notice that every non-abelian finitely generated group which is discriminated by free groups is free constructible [9]. Finitely generated non-abelian groups which are universally equivalent to free groups provide another type of examples of free constructible groups.

Recall that a group G is called a *CSA-group* if every maximal abelian subgroup M of G is malnormal, that is, $gMg^{-1} \cap M = 1$ for every $g \in G - M$. It has been proven in [8] that elementary operations 1)–4) preserve the CSA property. Since free groups are CSA it follows that every free constructible group is CSA.

Operations 1)–3) preserve the hyperbolicity of groups ([10]), therefore a free constructible group is hyperbolic if and only if all its centralizers of non-trivial elements are cyclic.

Every $\mathcal{K}$-constructible group G can be associated with a directed labelled binary tree $T(G)$ (*a construction tree*) which reflects the particular way of building G from groups of $\mathcal{K}$ by elementary operations. Vertices of $T(G)$ are $\mathcal{K}$-constructible groups, each vertex has at most two incoming edges and at most one outgoing edge. Arrows and labels show how a given vertex group is constructed from the preceding group (in the case of an extension of a centralizer or an HNN-extension) or two groups (in the case of free products with amalgamation). The group G is in the root of $T(G)$ and leaves vertices are groups from $\mathcal{K}$. We label edges as follows:

1. if H is obtained from P and Q by a free product, then we label each of the edges (P, H) and (Q, H), directed into H, by the identity element 1;
2. if H is obtained from P and Q by a free product with amalgamation of two abelian subgroups $A = \langle a_i \mid i \in I\rangle$ and $B = \langle b_i \mid i \in I\rangle$ under an isomorphism $\phi : a_i \to b_i$, then we label the edge (P, H) by the indexed set $\{a_i \mid i \in I\}$ and the edge (Q, H) by $\{b_i \mid i \in I\}$ (both edges are directed towards H);
3. if H is obtained from the preceding group P as an HNN-extension, where abelian groups $A = \langle a_i \mid i \in I\rangle$ and $B = \langle b_i \mid i \in I\rangle$ are identified under an isomorphism $\phi : a_i \to b_i$, then we label the edge (P, H) by an indexed set of pairs $\{(a_i, b_i) \mid i \in I\}$;
4. if H is obtained from the preceding group P as an extension of a centralizer $C_P(u) = \langle u_i, i \in I\rangle$ by a free abelian group A with a basis $S = \{a_1, \ldots, a_n\}$ then the label of the edge (P, H), directed from P to H, is the pair (U, S).

Observe that knowing leaves groups and labels of all edges, one can write down particular presentations of all vertex groups, including G. Such presentation of a group G is called the presentation of G *related* to the tree $T(G)$.

Notice, that a $\mathcal{K}$-constructible group G can be obtained from groups of $\mathcal{K}$ by many different sequences of elementary operations, so G may have different trees $T(G)$ and, therefore, different related presentations. Let $\mathcal{T}_G$ be the class of all construction trees for G. If $T(G)$ is a construction tree for G then by $\chi(T(G))$

we denote the number of vertices in $T(G)$ which are not leaves. Clearly, $\chi(T(G))$ is the number of elementary operations needed to build G according to $T(G)$. The following number is called *the complexity* of G

$$\chi(G) = min\{\chi(T(G)) \mid T(G) \in \mathcal{T}_G\}.$$

Thus, one needs at least $\chi(G)$ elementary operations to produce G. We call $T(G)$ *minimal* if $\chi(T(G)) = \chi(G)$. In the case of a finitely generated group G which is discriminated by free groups we consider only trees $T(G)$ in which one uses only operations 1)–4). In this event, the minimal trees are defined in a similar way.

Lemma 1 *Let G be a free constructible group. Then:*

1) each proper centralizer of G is a free abelian group of finite rank;

2) $Spec(G) = \{rank(C_G(g)) \mid 1 \neq g \in G\}$ is finite.

Proof. Let G be a free constructible group. It has been noticed above that elementary operations preserve the CSA property, hence G is a CSA-group and all its centralizers are maximal abelian. On the other hand, since a free group is torsion-free, by the facts on torsion in free products, amalgamated free products and HNN-extensions (see [16],[15]) it follows that G is torsion-free and hence all its centralizers are free abelian.

Now we proceed by the induction on $\chi(G)$. Let $T(G)$ be a construction tree for G such that $\chi(T(G)) = \chi(G)$.

If $\chi(T(G)) = 0$ then G is a free group and the result is trivially true in this case.

Suppose the lemma holds for $\chi(T(G)) < n$ and we assume $\chi(T(G)) = n > 0$. By definition it means that G is obtained from some free constructible groups by operations 1)–4) described above.

a) $G = G_1 * G_2$ and the lemma holds for both G_1 and G_2.

By Corollary 4.1.6 [16], two elements of G commute either if they belong to the same conjugate of G_1 or G_2, or if they are both powers of the same element.

Let $g \in G$. If all elements of $C_G(g)$ are powers of some element h then $C_G(g) \simeq \mathbb{Z}$ and

$$rank(C_G(g)) \leq \max_{i=1,2}\{\max\{Spec(G_1)\}, \max\{Spec(G_2)\}\}.$$

Now, suppose there exist $g_1, g_2 \in C_G(g)$, which are not powers of the same element. Hence, we can assume $g_1, g_2 \in G_1^h, h \in G$ and we have $C_g(g) \leq G_1^h$ is a free abelian of finite rank and

$$rank(C_G(g)) \leq \max\{Spec(G_1)\}.$$

Thus, both statements of the lemma hold for G.

b) $G = G_1 *_A G_2$, where A is a maximal abelian subgroup in G_1, and the lemma holds for both G_1 and G_2.

Let $g \in G$. We use the characterization of commuting elements in amalgamated free products given in Theorem 4.5 [16].

If $g \in A^h$ for some $h \in G$ then $C_G(g) = A^h$. Indeed, let $f \in C_G(g)$. Then we have

$$g(f_1 f_2 \cdots f_l a) = (f_1 f_2 \cdots f_l a)g,$$

where $f_1 f_2 \cdots f_l a$ is a normal form for $h^{-1}fh$, that is, $f_j, j \in [1, l]$ are non-trivial representatives of the left cosets of $G_i, i = 1, 2$ by A, no two adjacent ones belong to the same factor and $a \in A$. Without loss of generality we can assume $f_1 \in G_1$ and we have $gf_1 = f_1 a_1, a_1 \in A$. Hence, $f_1^{-1} g f_1 \in A$. Since A is maximal abelian and G_1 is CSA, it follows that $f_1 \in A$ - a contradiction. Hence, $C_G(g) \simeq \mathbb{Z}^k$ for some natural k and

$$rank(C_G(g)) \leq \max_{i=1,2}\{\max\{Spec(G_1)\}, \max\{Spec(G_2)\}\}.$$

Suppose there exists $h \in G$ such that $f \in C_G(g) - A^h$ belongs to a conjugate of a factor. If $f \in G_1^h$ then $C_G(g) \leq G_1^h$ and since A^h is a maximal abelian subgroup of G_1^h then $C_G(g) = A^h \simeq \mathbb{Z}^k, k \in \mathbb{N}$. If $f \in G_2^h$ then $C_G(g) \leq G_2^h$ and the induction step applies. In both cases we have

$$rank(C_G(g)) \leq \max_{i=1,2}\{\max\{Spec(G_1)\}, \max\{Spec(G_2)\}\}.$$

Finally, we can assume that no element of $C_G(g)$ belongs to a conjugate of a factor. In this case any $f \in C_G(g)$ can be represented as a product $f = ha_f h^{-1} \cdot w^{k_f}$, where $h, w \in G, a_f \in A, k_f \in \mathbb{N}$ and $[ha_f h^{-1}, w] = 1$. Suppose there exists $f \in C_G(g)$ such that $a_f \neq 1$. Then $[a_f, h^{-1}wh] = 1$ and we have

$$a_f(w_1 w_2 \cdots w_p a) = (w_1 w_2 \cdots w_p a)a_f,$$

where $w_1 w_2 \cdots w_p a$ is a normal form for $h^{-1}wh$, that is, $w_j, j \in [1, p]$ are non-trivial representatives of the left cosets of $G_i, i = 1, 2$ by A, no two adjacent ones belong to the same factor and $a \in A$. Without loss of generality we can assume $w_1 \in G_1$ and we have $a_f w_1 = w_1 a_1, a_1 \in A$. Hence, $w_1^{-1} a_f w_1 \in A$. Since, A is maximal abelian and G_1 is CSA, it follows that $w_1 \in A$ - a contradiction. Thus, $C_G(g) = \langle w \rangle \simeq \mathbb{Z}$ and

$$rank(C_G(g)) \leq \max_{i=1,2}\{\max\{Spec(G_1)\}, \max\{Spec(G_2)\}\}.$$

c) $G = H(u, A) = H *_{C(u)} (C(u) \times A)$, where A is a free abelian group of finite rank and the lemma holds for H.

The same argument as in b).

d) $G = \langle H, t \mid a^t = a^\phi \ (a \in A) \rangle$, where A is a maximal abelian subgroup of H, $\phi : A \to G$ is a monomorphism such that $g^{-1}Ag \cap A^\phi = 1$ for every $g \in G$ and the lemma holds for H.

Observe that $A = mal_H(A)$ and $B \trianglelefteq B_1 = mal_H(B)$, where B_1 is maximal abelian subgroup of H. Also, since G is CSA then centralizers of G are exactly maximal abelian subgroups of G. By Lemma 2 [8] it follows that any centralizer of G either is a conjugate of B_1, or belongs to a conjugate of H, or is cyclic. In all these cases the statement of the lemma is straightforward.

□

3 Subgroups of $\mathcal{K}$-constructible CSA-groups

In this section we consider subgroups of $\mathcal{K}$-constructible CSA-groups. It turns out that these subgroups can be constructed from subgroups of groups from $\mathcal{K}$ and an infinite cyclic group $\mathbb{Z}$ by elementary operations 1)–4). In particular, if the class $\mathcal{K}$ consists of CSA-groups, contains $\mathbb{Z}$, and is closed under taking subgroups then subgroups of $\mathcal{K}$-constructible CSA-groups are $\mathcal{K}$-constructible. This is a corollary of Bass-Serre theory and the results of [8].

We begin with a discussion of a particular type of $\mathcal{K}$-constructible groups which are *fundamental groups* of graphs of groups.

A graph X consists of a set of vertices $V(X)$, a set of edges $E(X)$ (here $V(X) \cap E(X) = \emptyset$) and three maps

$$\sigma : E(X) \to V(X), \quad \tau : E(X) \to V(X), \quad ^{-} : E(X) \to E(X),$$

which satisfy the following conditions:

$$\sigma(\bar{e}) = \tau(e), \ \tau(\bar{e}) = \sigma(e), \ \bar{\bar{e}} = e, \ \bar{e} \neq e.$$

Recall, that a graph of groups $(\mathcal{G}, X)$ consists of a connected graph X and an assignment $G_x \in \mathcal{G}$ to every $x \in V(X) \cup E(X)$, such that for every $e \in E(X)$, $G_e = G_{\bar{e}}$, and there exists a boundary monomorphism $i_e : G_e \to G_{\sigma(e)}$.

Let $(\mathcal{G}, X)$ be a graph of groups and T be a maximal subtree of X. Recall that the fundamental group $\pi(\mathcal{G}, X, T)$ of a graph of groups $(\mathcal{G}, X, T)$ is the group with the following presentation:

$$\langle G_v \ (v \in V(X)), \ t_e \ (e \in E(X)) \mid rel(G_v), \ t_e i_e(g) t_e^{-1} = i_{\bar{e}}(g) \ (g \in G_e),$$

$$t_e t_{\bar{e}} = 1, \ t_e = 1 \ (e \in T) \rangle.$$

The fundamental group $\pi(\mathcal{G}, X, T)$ can be obtained from vertex groups by a sequence (in general, infinite) of free products with amalgamation and HNN-extensions. To show this we need the following definition. Let $\Gamma = (\mathcal{G}, X, T)$ be a graph of groups and $Z \subset X$ be a connected subgraph of X such that $Z \cap T$ is a maximal subtree of Z. Denote by $\Gamma_Z = (\mathcal{G}|_Z, Z, Z \cap T)$ the subgraph of groups which rises from Z (the group assignment $\mathcal{G}|_Z$ is the restriction of $\mathcal{G}$ to Z). The identical maps

$$G_v \to G_v, \ t_e \to t_e, \ (v \in V(Z), e \in E(Z))$$

extend to the canonical monomorphism

$$\phi : \pi(\Gamma_Z) \to \pi(\Gamma).$$

The *collapse* of the graph of groups Γ along the subgraph Z is the graph of groups Γ/Z which is defined as follows. We replace the subgraph Z by a new point z in X, that is, for each edge $e \in X - Z$ with an endpoint v in Z we replace v by z and assign to the vertex z the group $G_z = \pi(\Gamma_Z)$. Finally, we define the boundary monomorphism from G_e into G_z equal to the old boundary monomorphism $G_e \to G_v$ (this is possible since G_v is a subgroup of G_z under the canonical embedding). Notice that $\pi(\Gamma/Z) = \pi(\Gamma)$. The operation which is inverse of a collapse is called a *refinement* of the vertex z by the graph Γ_Z. So, collapses and refinements do not change the fundamental group (up to isomorphism). It follows that the fundamental group $G = \pi(\Gamma)$ is isomorphic to the fundamental group $\pi(\Gamma/T)$ (collapse along the maximal subtree T) which has only one vertex. So, G can be obtained from $\pi(\Gamma/T)$ by a sequence of HNN-extensions. Observe, that the vertex group of the graph of groups Γ/T is isomorphic to $\pi(\Gamma_T)$ which can be obtained from vertex groups associated with T by a sequence of free products with amalgamation. Now it is clear how to construct a tree $T(G)$ for the fundamental group G starting with a graph of groups $(\mathcal{G}, X, T)$.

Our description of subgroups of $\mathcal{K}$-constructible CSA-groups is based on the Bass-Serre technique. We consider here in detail only the case of free products with amalgamation. The case of HNN-extensions can be treated similarly.

Let

$$G = A *_U B$$

be a free product of groups A and B with amalgamation along an abelian subgroup U. Observe, that G is isomorphic to the fundamental group of the graph of groups

$$A \xrightarrow{U} B. \tag{1}$$

By the standard procedure (see for example [6]) one can construct a directed tree X on which G acts without inversions in such a way that the quotient graph X/G is isomorphic to the initial graph of groups (1) for G. In our case the tree X is the following: $V(X)$ consists of all cosets gA and gB $(g \in G)$; $E(X)$ consists of all cosets gU $(g \in G)$, the maps σ and τ defined by

$$\sigma(gU) = gA, \ \tau(gU) = gB.$$

Now we can convert the directed graph X into non-oriented graph, adding, as usual, inverse edges and the involution $e \to \bar{e}$ $(e \in E(X))$. Notice, that due to the chosen orientation an edge gU goes from gA to gB. It is easy to check that X is a tree and G acts on X without inversions by the left multiplication. Hence the subgroup H also acts on X without inversions. Let $Y = X/H$ and T be a maximal subtree of Y. Following Bass-Serre theory we define a graph of groups $(\mathcal{G}, Y, T)$ with the fundamental group isomorphic to H.

Denote by $p : X \to X/H = Y$ the canonical projection of X onto its quotient, so $p(v) = Hv$ and $p(e) = He$. There exists an injective morphism of graphs $j : T \to X$ such that $pj = id_T$ (see, for example [6]), in particular jT is a subtree of X. One can extend j to a map (which we again denote by j) $j : Y \to X$ such that j maps vertices into vertices, edges into edges, and such that $pj = id_Y$. Notice, that in general j is not a graph morphism. To this end choose an orientation O of the graph Y. Let $e \in O - T$. Then there exists an edge $e' \in X$ such that $p(e') = e$. Clearly, $\sigma(e')$ and $j\sigma(e)$ are in the same H-orbit. Hence $h\sigma(e') = j\sigma(e)$ for some $h \in H$. Define $je = he'$ and $j\bar{e} = \overline{je}$. Notice that vertices $j\tau(e)$ and $\tau(je)$ are in the same H-orbit. Hence there exists an element $\gamma_e \in H$ such that $\gamma_e\tau(je) = j\tau(e)$.

Now we are in the position to define a graph of groups $(\mathcal{G}, Y, T)$. Put

$$G_v = Stab_H(jv),\ G_e = Stab_H(je),$$

and define boundary monomorphisms as inclusion maps $G_e \hookrightarrow G_{\sigma(e)}$ for edges $e \in T \cup O$ and as conjugations by $\gamma_{\bar{e}}$ for edges $e \notin T \cup O$, that is,

$$i_e(g) = \begin{cases} g, & \text{if } e \in T \cup O, \\ \gamma_{\bar{e}} g \gamma_{\bar{e}}^{-1}, & \text{if } e \notin T \cup O. \end{cases}$$

According to the Bass-Serre structure theorem we have $H \simeq \pi(\mathcal{G}, Y, T)$. Observe, that

$$Stab_H(gA) = H \cap gAg^{-1},\ Stab_H(gB) = H \cap gBg^{-1},\ Stab_H(gU) = H \cap gUg^{-1}.$$

So H can be obtained (up to isomorphism) from an infinite cyclic group $\mathbb{Z}$ and subgroups of A and B by free products, free products with amalgamation and HNN-extensions in which amalgamated and associated subgroups are abelian.

Similar argument provides the following result for HNN-extensions. Let G be an HNN-extension of a group G_0 with associated abelian subgroups A and B. Then a subgroup H of G can be obtained (up to isomorphism) from $\mathbb{Z}$ and subgroups of G_0 by free products, free products with amalgamation and HNN-extensions in which amalgamated and associated subgroups are abelian.

Based on these results, we prove the following theorem. For a class of groups $\mathcal{K}$ denote by $Sub(\mathcal{K})$ the class of all subgroups of groups from $\mathcal{K}$.

Theorem 1 *Let $\mathcal{K}$ be a class of groups such that $Sub(\mathcal{K}) = \mathcal{K}$ and $\mathbb{Z} \in \mathcal{K}$. Then every finitely generated subgroup H of a CSA $\mathcal{K}$-constructible group G is $\mathcal{K}$-constructible.*

Proof. Let G be a CSA $\mathcal{K}$-constructible group and H be a subgroup of G. We prove the theorem by induction on $\chi(G)$. If $\chi(G) = 0$ then $G \in \mathcal{K}$ and we have nothing to prove. In all other cases G is obtained from $\mathcal{K}$-constructible groups of lesser complexity by one of the elementary operations 1)–4). By the discussion preceding the theorem we can assume that H is obtained from $\mathbb{Z}$ and subgroups of some $\mathcal{K}$-constructible groups by free products with amalgamation and HNN-extensions with abelian amalgamated and associated subgroups. By induction these subgroups are $\mathcal{K}$-constructible. Indeed, they are subgroups of $\mathcal{K}$-constructible groups of lesser complexity and every subgroup of a CSA-group is a CSA-group. Notice, that among free products with amalgamation and HNN-extensions in which amalgamated and associated subgroups are abelian only elementary operations 1)–4) preserve the CSA property by Theorems 4 and 6 in [8]. Hence H is constructed from $\mathcal{K}$-constructible groups by operations 1)–4), therefore H is $\mathcal{K}$-constructible, as desired.

$\square$

Since elementary operations 1)–4) preserve the CSA property, the following result follows directly from Theorem 1.

Corollary 1 *Let $\mathcal{K}$ be a class of CSA-groups such that $Sub(\mathcal{K}) = \mathcal{K}$ and $\mathbb{Z} \in \mathcal{K}$. Then $Sub(\mathcal{K}^c) = \mathcal{K}^c$. In particular, subgroups of free constructible groups are free constructible.*

4 Subgroups of extensions of centralizers of CSA-groups

In this section we consider finitely generated subgroups of extensions of centralizers of CSA-groups. If the CSA-group G is fixed then we call them *G-constructible groups*.

These subgroups play a key role in the study of fully residually free groups. Indeed, it has been proven in [9] that every finitely generated fully residually free group is a subgroup of a group obtained from a free group by finitely many extensions of centralizers.

Now let A be a CSA-group, $U = C_A(u)$ be the centralizer of a non-trivial element $u \in A$, and $B = U \times C$ be a direct product of U and a torsion-free abelian group C. Notice that proper centralizers in CSA-groups are abelian, it follows that U, and hence B, is an abelian group. Denote by

$$G = A *_U B$$

the extension of the centralizer U by B. Let H be a subgroup of G. We avail ourselves to the technique and notations developed in the previous sections.

As we have seen in the previous section the group H is the fundamental group of the graph of groups $\pi(\mathcal{G}, Y, T)$. Notice, that the vertex groups and the edge group corresponding to an edge e of the type $gA \to gB$ are

$$H \cap gAg^{-1},\ H \cap gBg^{-1},\ H \cap gUg^{-1}.$$

Let $e \in O$. Then $je = gU$ for some $g \in G$. If $G_e = H \cap gUg^{-1} = 1$ then we have a free product of vertex groups.

If $e \in T$ then $\sigma(je) = gA, \tau(je) = gB$, and boundary monomorphisms for e and $\bar{e}$ are inclusions. Obviously, $i_e(G_e)$ is maximal abelian in $G_{\sigma(e)}$ (at least one of the subgroups must be maximal abelian). On the other hand, the image $i_{\bar{e}}(G_e) = H \cap gUg^{-1}$ is a direct factor in the group $G_{\tau(e)} = H \cap gBg^{-1}$. Indeed, it suffices to show that $H \cap gUg^{-1}$ is a pure subgroup of $H \cap gBg^{-1}$, that is, the quotient

$$H \cap gBg^{-1}/H \cap gUg^{-1}$$

is torsion free. Let $h \in H \cap gBg^{-1}$ then $h = gucg^{-1}$, where $u \in U, c \in C$ (recall that $B = U \times C$). If $h^m \in gUg^{-1}$, then $c^m = 1$ and hence $c = 1$, consequently, $h \in H \cap gUg^{-1}$.

Now suppose, $e \notin T$. Then $j\sigma(e) = gA$ and $j\tau(e) = \gamma_e gB$. As before the boundary monomorphism i_e of $G_e = Stab_H(je)$ into $G_{\sigma(e)} = Stab_H(j\sigma(e))$ is an inclusion. But the boundary monomorphism $i_{\bar{e}}$ of G_e into the group $G_{\tau(e)} = Stab_H(j\tau(e)) = H \cap \gamma_e gBg^{-1}\gamma_e^{-1}$ is the conjugation by γ_e. Hence, we have

$$i_{\bar{e}}(G_e) = \gamma_e G_e \gamma_e^{-1} = \gamma_e(H \cap gUg^{-1})\gamma_e^{-1}.$$

Since $\gamma_e \in H$ we have

$$i_{\bar{e}}(G_e) = H \cap \gamma_e gUg^{-1}\gamma_e^{-1} \leq H \cap \gamma_e gBg^{-1}\gamma_e^{-1}.$$

Denote $y = g^{-1}\gamma_e^{-1}$. So the image of G_e in $G_{\tau(e)}$ under the boundary monomorphism $i_{\bar{e}}$ is equal to $H \cap U^y$. As we saw before in this event $H \cap U^y$ is a direct factor of $H \cap B^y = G_{\tau(e)}$.

The discussion above shows that the following result holds.

Lemma 2 *Let $G = A *_U B$ be an extension of a centralizer U of a CSA-group A by an abelian group $B = U \times C$, where C is torsion free. Suppose a subgroup H of G is the fundamental group of the graph of groups $(\mathcal{G}, Y, T)$ described above. Then for each $e \in E(Y)$ the edge group G_e is either trivial or a maximal abelian subgroup of $G_{\sigma(e)}$, and the image of G_e under the boundary map $i_{\bar{e}}$ is a direct summand of the abelian group $G_{\tau(e)}$.*

Lemma 3 *Let $G = A *_U B$ be a extension of a centralizer U of a CSA-group A by an abelian group $B = U \times C$, where C is torsion free. Then for every maximal abelian subgroup K of A there exists the unique maximal abelian subgroup M of G such that $M = K \times C_M$, where C_M is either trivial or torsion-free abelian.*

Proof. Let M be the maximal abelian subgroup of G such that $K \leq M$. By Lemma 2 [17] we have the following cases.

1. $M \leq A^g,\ g \in G$.

Hence, $K \leq A \cap A^g$ and $g = g_1\ b_1\ g_2 \cdots g_n\ b_n\ g_{n+1}$ is the normal form of g, where $g_i \in A,\ i \in [1, n+1],\ b_i \in B,\ i \in [1, n]$. Now, for any $f \in K$ we have

$$(g_1\ b_1\ g_2 \cdots g_n\ b_n\ g_{n+1})\ f\ (g_{n+1}^{-1}\ b_n^{-1}\ g_n \cdots g_2^{-1}\ b_1^{-1}\ g_1^{-1})\ \in\ A.$$

It follows that either $g = g_1 \in A$ and we are done because $K \leq M \leq A$ and $K = M$ since K is maximal in A, or $g_{n+1}\ f\ g_{n+1}^{-1} \in U$. In the latter case $K \leq U^{g_{n+1}} \leq A$ and since $U^{g_{n+1}}$ is abelian then $K = U^{g_{n+1}}$. Now, observe that

$$B^{g_{n+1}} = U^{g_{n+1}} \times C^{g_{n+1}}$$

and $C^{g_{n+1}}$ is torsion free. Finally, $B^{g_{n+1}}$ is a maximal abelian subgroup of G if and only if B is, hence the proof follows from the Claim below.

Claim. B is a maximal abelian subgroup of G.

Let $B < B_1 \leq G$, where B_1 is abelian. Hence, there exists $b \in B_1 - B$ and we have the normal form $b = w_1\ w_2 \cdots w_{k+1}$, where $w_i,\ i \in [1,k]$ are representatives of the left cosets of A and B by U such that $w_i,\ w_{i+1},\ i \in [1, k-1]$ do not belong to the same factor while w_{k+1} is any element of A or B. Without loss of generality we can assume $w_1 \in A - U$. Since commutation is transitive in G then $[b,u] = 1$ for any $1 \neq u \in U$. Thus, $w_1\ w_2 \cdots (w_{k+1}u)$ is the normal form for bu and

$$u\ (w_1\ w_2 \cdots w_{k+1}) = w_1\ w_2 \cdots (w_{k+1}u).$$

It follows that $u\ w_1 = w_1\ u_1,\ u_1 \in U$ and $w_1^{-1}\ u\ w_1 \in U$ which is possible only when $w_1 \in B$ - contradiction.

2. $M \leq B^g,\ g \in G$.

Hence $M = B^g$ because M is maximal and $K \leq A \cap B^g$. Let $g = g_1\ b_1\ g_2 \cdots g_n\ b_n\ g_{n+1}$, hence for any $f \in K$ we have

$$(g_1\ b_1\ g_2 \cdots g_n\ b_n\ g_{n+1})\ f\ (g_{n+1}^{-1}\ b_n^{-1}\ g_n \cdots g_2^{-1}\ b_1^{-1}\ g_1^{-1})\ \in\ B.$$

Like in 1, it follows that either $g = g_1 = g_{n+1} \in A$ or $g_{n+1}\ f\ g_{n+1}^{-1} \in U$. In the former case we have $K^{g_{n+1}} \in A \cap B = U$, hence, $K = U^{g_{n+1}}$ while in the latter one we have $K \leq U^{g_{n+1}} \leq A$ and again $K = U^{g_{n+1}}$. Now the proof follows from Claim above like in 1.

3. $M = \langle z \rangle$, where $z \notin A^g,\ B^g$ for any $g \in G$.

But then $A \cap \langle z \rangle \neq 1$ which is impossible because of the assumption about z. This completes the proof of the lemma.

□

Lemma 4 *Let $G = A *_U B$ be an extension of a centralizer U of a CSA-group A by an abelian group $B = U \times C$, where C is torsion free. Suppose a subgroup H of G is the fundamental group of the graph of groups $(\mathcal{G}, Y, T)$ described above. Then for each $v \in V(Y)$ the maximal abelian subgroup K of the vertex group G_v is a direct summand of the unique maximal abelian subgroup M of H.*

Proof. Consider two cases.

1. $G_v = H \cap gAg^{-1} = H \cap A^{g^{-1}}$.

Since K is maximal in $H \cap A^{g^{-1}}$ then $K = H \cap K_1^{g^{-1}}$, where K_1 is a maximal abelian subgroup of A. By Lemma 3 there exists the unique maximal abelian subgroup M_1 of G such that $M_1 = K_1 \times C_1$, where C_1 is either trivial or torsion-free abelian.

Let M be maximal abelian in H such that $K = H \cap K_1^{g^{-1}} < M$. Since commutation is transitive in G it follows that $[h, f^{g^{-1}}] = 1$ for any $h \in M,\ f \in K_1$ and then $[h^g, f] = 1$. Hence, $[h^g, f_1] = 1$ for any $f_1 \in M_1$ and then $h^g \in M_1$ for any $h \in M$. Now, $M \leq M_1^{g^{-1}}$ and $M \leq H$, thus $M \leq H \cap M_1^{g^{-1}}$ and since M is maximal abelian in H then it follows that

$$M = H \cap M_1^{g^{-1}} = H \cap (K_1 \times C_1)^{g^{-1}}.$$

Observe that $H \cap K_1^{g^{-1}} \trianglelefteq H \cap (K_1 \times C_1)^{g^{-1}}$ and it is left to check if

$$H \cap (K_1 \times C_1)^{g^{-1}} / \ H \cap K_1^{g^{-1}}$$

is torsion free. Take any $z \in H \cap (K_1 \times C_1)^{g^{-1}}$. Then $z = gz_1z_2g^{-1} \in H$, where $z_1 \in K_1,\ z_2 \in C_1$. If $z^k \in H \cap K_1^{g^{-1}}$ then it follows that $z_2^k = 1$ because C_1 is torsion free. Hence, $z = gz_1g^{-1} \in H \cap K_1^{g^{-1}}$.

Thus, K is a direct summand of M.

2. $G_v = H \cap gBg^{-1} = H \cap B^{g^{-1}}$.

In this case G_v is abelian and $K = H \cap B^{g^{-1}}$. Moreover, since $B^{g^{-1}}$ is maximal abelian in G then $H \cap B^{g^{-1}}$ is maximal abelian in H, so the result follows immediately.

□

Let $(\mathcal{G}, Y, T)$ be a graph of groups for H so that $H = \pi(\mathcal{G}, Y, T)$ and let $e \in E(Y)$. Observe that we have two cases.

1. $Y - \{e\}$ is connected.

Then

$$\pi(\mathcal{G}, Y, T) = \langle H(e), t_e \mid t_e^{-1} G_e t_e = G_e^{\phi_e} \rangle,$$

where $H(e) = \pi(\mathcal{G}', Y', T')$, $Y' = Y - \{e\}$, $T' \subseteq T$ is the maximal subtree of Y', $\mathcal{G}'$ is a restriction of $\mathcal{G}$ on Y' and $\phi_e = i_{\bar{e}} \circ \phi$, where ϕ is a canonical embedding of $G_{\tau(e)}$ into $H(e)$. Hence, we say that H *splits over* e *as an HNN-extension.*

2. $Y - \{e\}$ is not connected.

Then

$$\pi(\mathcal{G}, Y, T) = H_1(e) *_{G_e} H_2(e),$$

where $H_i(e) = \pi(\mathcal{G}_i, Y_i, T_i),\ i = 1, 2,\ Y - \{e\} = Y_1 \cup Y_2,\ T_1 \cup T_2 \subseteq T$ is the maximal subtree of $Y - \{e\}$ and $\mathcal{G}_i$ is a restriction of $\mathcal{G}$ on $Y_i,\ i = 1, 2$. Hence, we say that H *splits over* e *as a free product with amalgamation.* We can assume that $G_{\sigma(e)} \leq H_1(e)$ and $G_{\tau(e)} \leq H_2(e)$

Now, combining Lemma 2 with Lemma 4 we obtain the following result.

Lemma 5 *Let $G = A *_U B$ be an extension of a centralizer U of a CSA-group A by an abelian group $B = U \times C$, where C is torsion free. Suppose a subgroup H of G is the fundamental group of the graph of groups $(\mathcal{G}, Y, T)$ and $e \in E(Y)$.*

1. *If H splits over e as an HNN-extension and G_e is not trivial then there exist maximal abelian subgroups $M_1,\ M_2$ of $H(e)$ such that*

$$M_1 = G_e \times D_1,\ M_2 = G_e^{\phi_e} \times D_2,$$

where $D_1,\ D_2$ are torsion free abelian and at least one of them is trivial.

2. *If H splits over e as a free product with amalgamation and G_e is not trivial then there exist maximal abelian subgroups $M_1 \leq H_1(e),\ M_2 \leq H_2(e)$ such that*

$$M_1 = G_e \times D_1,\ M_2 = G_e \times D_2,$$

where $D_1,\ D_2$ are torsion free abelian and at least one of them is trivial.

Proof. Let $e \in E(Y)$. Consider two cases.

1. H splits over e as an HNN-extension.

Then

$$\pi(\mathcal{G}, Y, T) = \langle H(e), t_e \mid t_e^{-1} G_e t_e = G_e^{\phi_e} \rangle,$$

where $H(e) = \pi(\mathcal{G}', Y', T'),\ Y' = Y - \{e\},\ T' \subseteq T$ is the maximal subtree of $Y',\ \mathcal{G}'$ is a restriction of $\mathcal{G}$ on Y' and $\phi_e = i_{\bar{e}} \circ \phi$, where ϕ is a canonical embedding of $G_{\tau(e)}$ into $H(e)$.

By Lemma 2 G_e is maximal in $G_{\sigma(e)}$, while $G_e^{\phi_e}$ is a direct summand of $G_{\tau(e)}$ which is maximal abelian in H (so in $H(e)$ too) since $G_{\tau(e)} = H \cap gBg^{-1}$ for some $g \in G$. By Lemma 4 there exists the unique maximal abelian subgroup M_1 of H (so of $H(e)$ too) such that $M_2 = G_e \times D_1$, where D_1 is torsion free abelian. On the other hand we set $M_2 = G_{\tau(e)}$ and then $M_2 = G_e^{\phi_e} \times D_2$, where D_1 is torsion free abelian. Finally, since H is CSA then by Proposition 3 [8] it follows that either D_1 or D_2 is trivial.

2. H splits over e as a free product with amalgamation.

The same argument as above.

□

The following theorem is a direct corollary of Lemma 5.

Theorem 2 *Let $A*_U B$ be an extension of a centralizer U of a non-abelian CSA-group A by an abelian group $B = U \times C$, where C is torsion-free. Then every finitely generated subgroup H of G can be obtained from finitely many subgroups of A and B by finitely many operations of the following types: free products, extensions of centralizers, free products with amalgamation along maximal abelian subgroups in both factors and separated HNN-extensions with maximal abelian associated subgroups.*

Proof. Let H be a subgroup of G. Then H is a fundamental group of the graph of groups $(\mathcal{G}, Y, T)$ described above. We prove by induction that the statement of the lemma holds for the fundamental group of $(\mathcal{G}', Y', T')$, where Y' is any connected subgraph of Y and T' is the corresponding subtree of T.

If $|E(Y)| = 0$ then Y contains only one vertex and either $H = H \cap A^g$ or $H = H \cap B^g$ for some $g \in G$. In both cases H is canonically isomorphic to a subgroup of either A or B.

We assume now that we have proved the required result for any connected subgraph of Y' with $|E(Y')| < |E(Y)|$.

Choose any edge $e \in E(Y)$ and consider $Y' = Y - \{e\}$.

1. Y' is connected.

Hence, H splits over e as an HNN-extension

$$\pi(\mathcal{G}, Y, T) = \langle H(e), t_e \mid t_e^{-1} G_e t_e = G_e^{\phi_e} \rangle,$$

where $H(e) = \pi(\mathcal{G}', Y', T')$ and by the induction hypothesis $H(e)$ can be obtained from finitely many subgroups of A and B by finitely many operations described.

If G_e is trivial then $H = H(e)$ and we are done. Let $G_e \neq 1$ then by Lemma 5 there exist maximal abelian subgroups $M_1,\ M_2$ of $H(e)$ such that

$$M_1 = G_e \times D_1,\ M_2 = G_e^{\phi_e} \times D_2,$$

where $D_1,\ D_2$ are torsion free abelian and at least one of them is trivial. Without loss of generality we can assume $D_2 = 1$. Below we use the operation which is called a *sliding of* $H(e)$ *along* M_1. That is, if we set

$$H^* = H(e) *_{G_e^{\phi_e} = t_e^{-1} G_e t_e} (t_e^{-1} M_1 t_e)$$

which can be viewed as a centralizer extension of $G_e^{\phi_e}$ then

$$H = \langle H^*, t_e \mid t_e M_1 t_e^{-1} = M_1^{\phi} \rangle,$$

where $M_1^{\phi} = M_1$ and ϕ is an identity map, that is, H is obtained from H^* by a separated HNN-extensions with maximal abelian associated subgroups.

2. Y' is disconnected.

Hence, $Y' = Y_1 \cup Y_2$ and H splits over e as a free product with amalgamation

$$\pi(\mathcal{G}, Y, T) = H_1(e) *_{G_e} H_2(e),$$

where $H_i(e) = \pi(\mathcal{G}_i, Y_i, T_i),\ i = 1, 2$ and by the induction hypothesis $H_1(e)$, $H_2(e)$ can be obtained from finitely many subgroups of A and B by finitely many operations described.

If G_e is trivial then H is a free product $H_1(e) * H_2(e)$ and we are done. Let $G_e \neq 1$ then by Lemma 5 there exist maximal abelian subgroups $M_1 \leq H_1(e),\ M_2 \leq H_2(e)$ such that

$$M_1 = G_e \times D_1,\ M_2 = G_e \times D_2,$$

where $D_1,\ D_2$ are torsion free abelian and at least one of them is trivial. Without loss of generality we can assume $D_1 = 1$. Below we use the operation which is called a *sliding of* H *along* M_2. That is, if we set $H^* = H_1(e) *_{G_e} M_2$ which can be viewed as a centralizer extension of G_e then $H = H^* *_{M_2} H_2(e)$, where M_2 is maximal in both $H_2(e)$ and H^*.

□

One can generalize the theorem above in the following way.

Theorem 3 *Let A be a non-abelian CSA-group and let a group G be obtained from A by finitely many successive free extensions of centralizers. Then every* *finitely generated subgroup H of G can be obtained from finitely many subgroups of A and B by finitely many operations of the following type: free products, extensions of centralizers, free products with amalgamation along maximal abelian subgroups in both factors and separated HNN-extensions with maximal abelian associated subgroups.*

Proof. If G is an extension of a single centralizer of A, then the result follows from Theorem 2. In the case of several extensions we proceed by induction in the following way. Let G be obtained from A by n consecutive centralizer extensions. Then we have the following chain of groups

$$A = G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

where G_{i+1} is obtained from G_i by a single centralizer extension. Without loss of generality we can assume n to be the minimal natural number such that $H \leq G_n$, otherwise the result follows by the induction hypothesis.

Since G is obtained from G_{n-1} by a centralizer extension and G_{n-1} is a CSA-group then by Theorem 2 it follows that H is isomorphic to the fundamental group of a graph of groups $(\mathcal{G}, Y, T)$, in which every vertex group is either a subgroup of G_{n-1} or a free abelian group of a finite rank and every edge represents either a free product, or a free product with amalgamation along a maximal abelian subgroup, or a separated HNN-extension with an association along a maximal abelian subgroup or an extension of a centralizer. Since H is finitely generated, then all vertex groups of $(\mathcal{G}, Y, T)$ are finitely generated and the induction hypothesis holds for them, which completes the proof.

□

Theorem 3 can be reformulated for G-constructible groups as follows.

Theorem 4 *Let G be a non-abelian CSA-group. Then any G-constructible group H can be obtained from $\mathbb{Z}$ and finitely many subgroups of G by finitely many operations of the following type: free products, extensions of centralizers, free products with amalgamation along maximal abelian subgroups in both factors and separated HNN-extensions with maximal abelian associated subgroups.*

Observe that if G is a CSA-group with cyclic centralizers (for example, a torsion-free hyperbolic group) then free products with amalgamation and HNN-extensions in Theorem 3 are taken along cyclic subgroups. Namely, the following corollary holds.

Corollary 2 *Let G be a non-abelian CSA-group with cyclic centralizers. Then any G-constructible group H can be obtained from $\mathbb{Z}$ and finitely many subgroups of G by finitely many operations of the following type: free products, extensions of centralizers, free products with amalgamation along maximal cyclic subgroups in both factors and separated HNN-extensions with maximal cyclic associated subgroups.*

Notice, that if Γ is a finite graph of groups and $G = \pi(\Gamma)$ then G can be presented by a finite directed graph $T(G)$ (see Section 1) which corresponds to a sequence of collapses of all edges of Γ (in some particular order). In this event, $\chi(T(G))$ is equal to the number of edges in Γ.

This observation makes it possible, given a splitting of G as a graph of groups, to find a construction tree for G effectively. Moreover, in the case of fully residually free groups such a splitting can be found effectively and one can talk about effectiveness of the overall procedure. Indeed, in algebraic geometry over a free group fully residually free groups arise as coordinate groups of irreducible algebraic sets, hence the initial object from which one obtains G is a system of equations S over a free group F, which can be assumed to have a solution and can be given effectively. In [11] Kharlampovich and Myasnikov give an algorithm which effectively finds finitely many irreducible systems $S_1, \ldots, S_k$ (their union is equivalent to S), computes radicals of these systems and G arises as a coordinate group of an algebraic set of S_{i_0} for some $i_0 \in [1, k]$. Finally, the algorithm computes a corresponding finite presentation of G and a $\mathbb{Z}$-splitting of G as a graph of groups. Hence the following result follows immediately.

Corollary 3 *Let G be a finitely generated fully residually free group. Then one can effectively find a construction tree $T(G)$ of G.*

5 Homological and cohomological dimensions of fully residually free groups

Let G be a group and R a commutative ring with unit element $1 \neq 0$. Define

$$hd_R(G) = \inf\{n \mid R \text{ as an } RG\text{−module admits a flat resolution of length } n\},$$

$$cd_R(G) = \inf\{n \mid R \text{ as an } RG\text{−module admits a projective resolution of length } n\}.$$

$hd_R(G)$ ($cd_R(G)$) is called the *homology (cohomology) dimension of G over R*. Observe that $hd_R(G)$ ($cd_R(G)$) can be equal ∞.

Our main tool for computing homological and cohomological dimensions of a group G, denoted $hd(G)$ and $cd(G)$ respectively is the following result.

Proposition 1 *(Proposition 6.1 and Proposition 6.12 [3])*

1) *Let $G = G_1 *_S G_2$ be a free product with amalgamated subgroup S and let $n = \max\{cd_R(G_1), cd_R(G_2)\}$ and $m = \max\{hd_R(G_1), hd_R(G_2)\}$. Then*

$$n \leq cd_R(G) \leq n + 1, \;\; m \leq hd_R(G) \leq m + 1.$$

Moreover, $cd_R(G) = n + 1$ implies $cd_R(G_1) = cd_R(G_2) = cd_R(S) = n$ and $hd_R(G) = m + 1$ implies $hd_R(G_1) = hd_R(G_2) = hd_R(S) = m$.

2) *Let $G = G^* *_{S,\sigma}$ be an HNN-extension of G^* with associated cyclic subgroups S and T, and stable letter p. If $n = cd_R(G^*)$ and $m = hd_R(G^*)$ then*

$$n \leq cd_R(G) \leq n + 1, \;\; m \leq hd_R(G) \leq m + 1.$$

Moreover, $cd_R(G) = n + 1$ *implies* $cd_R(G^*) = cd_R(S) = n$ *and* $hd_R(G) = m + 1$ *implies* $hd_R(G^*) = hd_R(S) = m$.

Here are some general results about homological and cohomological dimensions of a group.

Proposition 2 *[3] Let G be a group and R a commutative ring with unit element $1 \neq 0$. Then*

1. $hd_R(G) \leq cd_R(G)$;
2. *if* $H \leq G$ *then* $hd_R(H) \leq hd_R(G), \quad cd_R(H) \leq cd_R(G)$.

Now we restrict ourselves to some special class of groups known as groups *of type FP*. The following definitions can be found in the book [3].

Let G be a group, R a commutative ring with unit element $1 \neq 0$ and A an RG-module.

A projective resolution $\underline{P} \twoheadrightarrow A$ is said to be *finitely generated* if the RG-modules P_i are finitely generated in each dimension $i \geq 0$. A is said to be *of type FP_n* if there is a projective resolution $\underline{P} \twoheadrightarrow A$ with P_i finitely generated for all $i \leq n$. If the modules P_i are finitely generated for all i then we say that A is of type FP_∞.

G is said to be *of type FP_n over* R, $n = \infty$ or an integer ≥ 0, if the trivial G-module R is of type FP_n as an RG-module. If G is of type FP_n over $\mathbb{Z}$ then we merely say that G is *of type FP_n*

A group G is *of type FP* if $\mathbb{Z}$ admits a finite projective resolution over $\mathbb{Z}G$.

From now on we assume $R = \mathbb{Z}G$ and respectively use the notation

$$hd_{\mathbb{Z}G}(G) = hd(G), \; cd_{\mathbb{Z}G}(G) = cd(G).$$

Proposition 3 *[4] A group G is of the type FP if and only if*

1. $cd(G) < \infty$;
2. *G is of type FP_∞.*

It turns out that groups of type FP possess many nice properties which make it easier to study them.

Proposition 4 *[3] If G is of type FP then $cd(G) = hd(G)$.*

From Propositions 1 and 4 one can obtain the following result.

Corollary 4 *1) If $G = G_1 *_S G_2$, where S is an infinite cyclic, G_1, G_2 are of type FP and $\max_{i=1,2}\{cd(G_i)\} \geq 2$ then*

$$cd(G) = hd(G) = \max_{1=1,2}\{cd(G_i)\}.$$

2) *If* $G = G^* *_{S,\sigma}$, *where* S *is an infinite cyclic,* G^* *is of type* FP *and* $cd(G^*) \geq 2$ *then*

$$hd(G) = cd(G) = cd(G^*).$$

The following lemma makes it possible to use all the results above for fully residually free groups.

Lemma 6 *If* $\mathcal{K}$ *consists of CSA-groups of type* FP *then any* $\mathcal{K}$*-constructible group is a CSA-group of type* FP*.*

Proof. If G is $\mathcal{K}$-constructible and $\mathcal{K}$ consists of CSA-groups then G is CSA because elementary operations preserve this property. Finally, the fact that G is of type FP follows from Proposition 2.13 [3].

□

Corollary 5 *If* G *is a fully residually free group then* $cd(G) = hd(G)$*.*

Let us denote $rank_C(G) = \max\{Spec(G)\}$ (see Section 2).

Theorem 5 *Let* G *be a fully residually free group. Then*

1) *if* $rank_C(G) \geq 2$ *then* $hd(G) = cd(G) = rank_C(G)$*;*

2) *if* $rank_C(G) = 1$ *then* $hd(G) = cd(G) \leq 2$ *and* $hd(G) = cd(G) = 2$ *if and only if* G *is not free.*

Proof. $hd(G) = cd(G)$ follows from Corollary 5 since G is fully residually free.

Since a free group is CSA with cyclic centralizers then by Corollary 2 there exists a construction tree $T(G)$ for G such that the leaves groups of $T(G)$ are finitely generated free groups and G is built up using free products with amalgamation and HNN-extensions taken along cyclic subgroups. We prove by the induction on the height of $T(G)$.

If $\chi(T(G)) = 1$ then G is free and everything is proved. Suppose $\chi(T(G)) \geq 2$.

a) $G = G_1 *_S G_2$, where S is infinite cyclic.

Observe that $rank_C(G) = \max_{i=1,2}\{rank_C(G_i)\}$ and the induction hypothesis holds for G_1 and G_2.

If $\max_{i=1,2}\{rank_C(G_i)\} \geq 2$ then we can assume $rank_C(G_1) \geq rank_C(G_2)$ and $rank_C(G_1) \geq 2$. Then by the induction hypothesis we have $cd(G_1) = rank_C(G_1) \geq 2$. Hence, $\max_{i=1,2}\{cd(G_i)\} \geq 2$ and by Corollary 4

$$cd(G) = \max_{i=1,2}\{cd(G_i)\} \geq 2.$$

Now, if $rank_C(G_2) = 1$ then by induction $cd(G_2) \leq 2$ and we have

$$cd(G) = \max_{i=1,2}\{cd(G_i)\} = \max_{i=1,2}\{rank_C(G_i)\}.$$

Suppose $\max_{i=1,2}\{rank_C(G_i)\} = 1$. Then both G_1 and G_2 are free and G is a one relator group without torsion. It follows that either $hd(G) = cd(G) = 2$ (see [13]) or G is free so $hd(G) = cd(G) = 1$ (see [21]).

b) $G = G^* *_{S,\sigma}$ is an HNN-extension of G^* with associated cyclic subgroups S and T, and stable letter p.

The argument is similar to a).

□

It turns out that cohomological dimension of a fully residually free group G can be computed effectively. From Theorem 5 it follows that for this purpose it is enough to be able to compute effectively $rank_C(G)$ and decide if G is free. The following results are crucial.

Theorem 6 *[12] For any finitely generated fully residually free group G one can find the set $Spec(G)$ effectively.*

Theorem 7 *[11] There exists an algorithm which for every finitely generated fully residually free group G determines whether G is a free group or not.*

Combining the above results with Theorem 5 one obtains the following result.

Theorem 8 *There exists an algorithm which for every finitely generated fully residually free group G computes $cd(G)$.*

Proof. By Theorem 6 one can effectively find $Spec(G)$, hence the number $rank_C(G) = \max\{Spec(G)\}$. If $rank_C(G) \geq 2$ then $cd(G) = rank_C(G)$. If $rank_C(G) = 1$ then by Theorem 5 to compute $cd(G)$ it suffices to check whether the group G is free or not. Now the result follows from Theorem 7.

□

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Olga G. Kharlampovich,
McGill University, Department of Mathematics and Statistics, 805 Sherbrooke W., Montreal QC H3A 2K6, Canada
olga@math.mcgill.ca
http://www.math.mcgill.ca/olga/

Alexei G. Myasnikov,
McGill University, Department of Mathematics and Statistics, 805 Sherbrooke W., Montreal QC H3A 2K6, Canada
alexeim@att.net
http://home.att.net/˜ alexeim/index.htm

Vladimir N. Remeslennikov,
Omsk Branch of the Mathematical Institute SB RAS, 13 Pevtsova Street, 644099 Omsk, Russia
remesl@iitam.omsk.net.ru

Denis E. Serbin,
McGill University, Department of Mathematics and Statistics, 805 Sherbrooke W., Montreal QC H3A 2K6, Canada
zloidyadya@yahoo.com