Irreducible affine varieties over a free group. II: Systems in triangular quasi-quadratic form and description of residually free groups

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We shall prove the conjecture of Myasnikov and Remeslennikov [**?**] which states that a finitely generated group is fully residually free (every finite set of nontrivial elements has nontrivial images under some homomorphism into a free group) if and only if it is embeddable in the Lyndon's exponential group $F^{\mathbf{Z}[x]}$, which is the $\mathbf{Z}[x]$-completion of the free group. Here $\mathbf{Z}[x]$ is the ring of polynomials of one variable with integer coefficients. Historically, Lyndon's attempts to solve Tarski's famous problem concerning the elementary equivalence of free groups of different ranks led him to introduce $F^{\mathbf{Z}[x]}$.

An $\exists$-free group is a group G such that the class of $\exists$-formulas, true in G, is the same as the class of $\exists$-formulas, true in a nonabelian free group. A finitely generated group is $\exists$-free if and only if it is fully residually free [**?**]. Our result gives an algebraic description of $\exists$-free groups.

We shall give an algorithm to represent a solution set of arbitrary system of equations over F as a union of finite number of irreducible components in the Zariski topology on F^n. The solution set for every system is contained in the solution set of a finite number of systems in triangular form with quadratic words as leading terms. The possibility of such a decomposition for a solution set was conjectured by Razborov in [**?**] and also by Rips.

We shall give a description of systems of equations determining irreducible components using methods developed in [**?**] and [**?**]; it is possible to find some of these methods in [**?**].

We are thankful to E. Rips for attracting our attention to these techniques.

0. Introduction

All the necessary definitions can be found in [**?**]. Nevertheless, we repeat here most of them to make this paper selfcontained.

Let G be a group, $F(X)$ the free group with basis $X = \{x_1, x_2, \dots, x_n\}$, and $G[X] = G * F(X)$ the free product of G and $F(X)$.

An element s from $G[X]$ is called *an equation over the group* G. We write this as $s(x_1, \dots, x_n, g_1, \dots, g_m) = 1$ or, simply, as $s(\bar{x}, \bar{g}) = 1$. *A system of equations over group* G is an arbitrary set of equations $S = \{s_i = 1 \mid i \in I\}$ (in more succint notation: $S = 1$). *A solution* of a system $S(x_1, \dots, x_n, g_1, \dots, g_m) = 1$ over a group G is a tuple of elements $a_1, \dots, a_n \in G$ such that after replacement of each x_i by a_i in every equation $s(x, g) = 1$ one gets a trivial element in the group G. In other words, a solution of the system $S = 1$ over G can be described as a G-homomorphism (i.e. a homomorphism which is identical on G) $\pi_{\bar{X}} : G[X] \longrightarrow G$ such that $\phi(S) = 1$. If by $V(S)$ we denote the set of all solutions in G of the system $S = 1$, then $V(S)$ is called an *algebraic subset* or an (affine) variety in G^n.

For any $S \subseteq G[X]$ we have $V(S) = V(ncl(S))$, where $ncl(W)$ is the normal closure of W in $G[X]$.

A group G is called a CSA-group if every maximal abelian subgroup M of G is malnormal, i.e. $M^g \cap M = 1$ for any $g \notin M$.

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It was shown in [?] that for a nonabelian CSA-group G all algebraic sets in G^n define a topology on G^n in which they are exactly the closed sets. The topology defined by algebraic sets as closed subsets is said to be a Zariski topology.

Below G is always a nonabelian group.

Definition 1 *Let $Y \subseteq G^n$. Define a set*

$$I(Y) = \{s \in G[X] \mid s(g_1, \ldots, g_n) = 1 \quad \forall (g_1, \ldots, g_n) \in Y\}$$

The set $I(Y)$ has a description in terms of homomorphisms. Any tuple $g = (g_1, \ldots, g_n) \in Y$ defines a homomorphism $\pi_g : G[X] \longrightarrow G$ by the condition $x_i \longrightarrow g_i$. Then

$$I(Y) = \bigcap_{g \in Y} ker(\pi_g).$$

$I(Y)$ is a normal subgroup of $G[X]$, and if G is a torsion-free group then $I(Y)$ is an isolated normal subgroup of $G[X]$, in particular, $I(V(S))$ contains the intersection $\sqrt{S}$ of all normal isolated subgroups containing S.

Definition 2 *Let $V(S)$ be a variety defined by $S \subset G[X]$. Then $I(V(S))$ is called the radical of the system $S = 1$ and is denoted by $Rad(S)$. Denote $G[X]/Rad(S)$ by $G_{R(S)}$, and $G[X]/ncl(S)$ by G_S.*

A system $S = 1$ over G is called consistent if there is a G-homomorphism $\pi : G[X] \to H \geq G$ such that $S \in ker(\pi)$. Otherwise it is inconsistent over G. If a system $S = 1$ over G is consistent then the canonical homomorphism $G \to G_{R(S)}$ is monic. An inconsistent system $S = 1$ defines the empty variety over G. Therefore, for non-empty varieties $V(S)$ we will assume that G is a subgroup of $G_{R(S)}$.

Definition 3 *Let H be a group and $\mathcal{G}$ be a family of groups.*

1) A homomorphism of groups $\psi : H \longrightarrow G$ separates a nontrivial element $h \in H$ if $\psi(h) \neq 1$;

2) A family of homomorphisms $\Psi = \{\psi : H \longrightarrow G \mid G \in \mathcal{G}\}$ is called a separating (discriminating) family of homomorphisms if any nontrivial $h \in H$ (any finite number of nontrivial elements $h_1, \ldots, h_n \in H$) can be separated by some $\psi \in \Psi$. In this case H is called a residually $\mathcal{G}$ group (ω-residually $\mathcal{G}$ group or fully residually $\mathcal{G}$ group).

In the case when $\mathcal{G}$ consists of a single group G, which is also a subgroup of H and if the separating (discriminating) homomorphisms in Ψ are all G-homomorphisms, we say that H is separated (discriminated) by G-homomorphisms.

A group G is called *Equationally Noetherian* (EN) if for every system S of equations over G there is a finite subsystem S_0 such that $V(S) = V(S_0)$. A free group is EN group [?].

A closed set in a topological space is called *irreducible* if it is not a union of two proper closed subsets.

Lemma 1 *[?] Let G be a EN CSA-group. Then $V(S)$ is irreducible if and only if $G_{R(S)}$ is discriminated in G by G-homomorphisms.*

Definition 4 *An equation is said to be quadratic if every variable occurs in the equation not more than twice. An equation is said to be strictly quadratic if every variable occurs in the equation exactly twice. A system is said to be quadratic if every variable occurs in the equations of the system not more than twice (it may not occur at all). A system is said to be strictly quadratic if every variable occurs in the equations of the system exactly twice.*

Let the set X consist of three types of variables: x_i, y_i, z_i. We call a quadratic equation *standard* if it has one of the following forms:

$$\Pi_{i=1}^{n}[x_i, y_i] = 1 \quad (n > 0), \tag{1}$$

$$\Pi_{i=1}^{n}[x_i, y_i]\Pi_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \tag{2}$$

$$\Pi_{i=1}^{n} x_i^2 = 1 \quad (n > 0), \tag{3}$$

$$\Pi_{i=1}^{n} x_i^2 \Pi_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \tag{4}$$

where $d, c_i (i = 1, \dots m)$ are nontrivial elements in G.

Definition 5 *Let G be a group, C(u) the centralizer of the element u in G. Suppose $C(u)$ is abelian. Then the group $C(u,t) =< G, t | [v, t] = 1, v \in C(u) >$ is called a free extension of the centralizer of u.*

Let A be an arbitrary associative ring with identity and G a group. Fix an action of the ring A on G, i.e. a map $G \times A \to G$. The result of the action of $\alpha \in A$ on $g \in G$ is written as g^α. Consider the following axioms:

1. $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$;
2. $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;
3. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;
4. $[g, h] = 1 \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha$.

Definition 6 *Groups with A-actions satisfying axioms 1)–4) are called A*–groups.

In particular, an arbitrary group G is a **Z**-group. We now recall the definition of A-completion.

Definition 7 *Let G be a group . Then an A–group G^A together with a homomorphism $\lambda : G \to G^A$ is called a* tensor *A–completion of the group G if G^A satisfies the following universal property: for any A–group H and a homomorphism $\varphi : G \to H$ there exists a unique A–homomorphism $\psi : G^A \to H$ (a homomorphism that commutes with the action of A) such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & G^A \arrow[dl, "\psi"] \\ H & \end{tikzcd}$$

By $\mathbf{Z}[x]$ we denote as usual the ring of polynomials of one variable with integer coefficients.

Lemma 2 *[?] Every group obtained from a CSA group G by a sequence of free extensions of centralizers is embeddable into $G^{\mathbf{Z[x]}}$.*

Below $\bar{x}$ denotes several variables.

Definition 8 *Let G be a group, $\bar{c}$ a tuple of elements from G, $\bar{x}_1, \ldots, \bar{x}_n$ disjoint tuples of variables. A system $\bigcup_{i=1}^{m} S_i(\bar{c}, \bar{x}_i, \ldots, \bar{x}_m) = 1$ is said to be triangular quasi-quadratic if for every i the equation $S_i(\bar{c}, \bar{x}_i, \ldots, \bar{x}_m) = 1$ is quadratic in the variables from $\bar{x}_i$*

Such a system is said to be nondegenerate if for each i the equation $S_i = 1$ over $G[\bar{x}_{i+1}, \ldots \bar{x}_m]/R(\bigcup_{j=1}^{i-1} S_j)$ (with elements $\bar{x}_i$ considered as variables and elements from $\bar{c}, \bar{x}_{i+1} \ldots \bar{x}_m$ as coefficients) has a solution.

In [?] the following result was proved.

Theorem 1 *If S is a nondegenerate triangular quasi-quadratic system over a fully residually free group G, then $G_{R(S)}$ is isomorphic to a subgroup of a group obtained from G by a sequence of free extensions of centralizers and hence a subgroup of $G^{\mathbf{Z}[x]}$.*

If G is fully residually free, then every finitely generated subgroup of $G^{\mathbf{Z}[x]}$ is a subgroup of a group obtained from G by a finite series of free extensions of centralizers, and hence is discriminated by G-homomorphisms [?]. This and Lemma 1 implies

Corollary 1 *For a nondegenerate triangular quasi-quadratic system S over a fully residually free group G the solution set $V(S)$ is irreducible.*

Theorem 2 *For any finite system $S(\bar{x}) = 1$ over a free group F, one can find effectively a finite family of nondegenerate triangular quasi-quadratic systems $U_1, \ldots, U_k$ and word mappings $p_i : V_F(U_i) \to V_F(S)$ $(i = 1, \ldots, k)$ such that for every $b \in V_F(S)$ there exists i and $c \in V_F(U_i)$ for which $b = p_i(c)$, i.e.*

$$V_F(S) = p_1(V_F(U_1)) \cup \ldots \cup p_k(V_F(U_k))$$

and all sets $p_i(V_F(U_i))$ are irreducible; moreover, every irreducible component of $V_F(S)$ can be obtained as a closure of some $p_i(V_F(U_i))$ in the Zariski topology.

This theorem will be proved in Sections 1-10. A system S is said to be *irreducible* if the solution set $V(S)$ is irreducible. The main objective in this paper is to prove the following

Theorem 3 *For a system $S = 1$ over a free group, $V(S)$ is irreducible if and only if $F_{R(S)} \subseteq F_{R(S_1)}$ for a nondegenerate triangular quasi-quadratic system S_1.*

Sections 1-9 will be devoted to proving that for any irreducible system $S = 1$ over a free group F, $F_{R(S)} \subseteq F_{R(S_1)}$ for a nondegenerate triangular quasi-quadratic system S_1.

Notice that in [?] it was shown that $F^{\mathbf{Z[x]}}$ is fully residually free.

Theorem 3 implies

Theorem 4 *A finitely generated group is fully residually free if and only if it is isomorphic to a subgroup of $F^{\mathbf{Z[x]}}$.*

Proof Consider a finitely generated fully residually free group G given by generators $x_1, \ldots, x_n$ and relations $s_j(x_1, \ldots, x_n), j \in J$. Consider $S = \{s_j(x_1, \ldots, x_n), j \in J\}$ as a system of equations over F. Then $G * F = F_S$ and $ncl(S) = Rad(S)$, since F_S is fully residually free. Hence $G \leq F_{R(S)}$ is embeddable into $F^{\mathbf{Z}[\mathbf{x}]}$.

Now we can describe the algebraic structure of finitely generated subgroups of $F^{\mathbf{Z}[\mathbf{x}]}$ in terms of free constructions.

Let $H =< G, t | A^t = B >$ be an HNN extension of G with associated subgroups A and B. H is called a *separated HNN-extension* if for any $g \in G$, $A^g \cap B = 1$.

Corollary 2 *Every finitely generated residually free group G is a subgroup of a direct product of finitely many fully residually free groups; hence G is embeddable into $F^{\mathbf{Z}[x]} \times \ldots \times F^{\mathbf{Z}[x]}$*

Theorem 5 *Let V be an irreducible variety over F. Then there exists a finite system of equations $S = 1$ over F which defines the variety V and satisfies the Nullstellensatz.*

Theorem 6 *(joint with V. Remeslennikov) Let a group G be obtained from a free group F by a series of finitely many free extensions of centralizers. Then every finitely generated subgroup H of G is obtained from free abelian groups of finite rank by finitely many operations of the following type:*

1. *free products;*
2. *amalgamated products with abelian amalgamated subgroups at least one of which is maximal abelian;*
3. *free extensions of centralizers;*
4. *separated HNN-extensions with abelian associated subgroups at least one of which is maximal abelian.*

The following four corollaries will be proved in section 10.

Corollary 3 *Every finitely generated fully residually free group is finitely presented.*

Corollary 3 was also announced by Z. Sela.

Corollary 4 *All finitely generated subgroups of $F^{\mathbf{Z}[x]}$ in which all proper centralizers are cyclic, are hyperbolic.*

Corollary 5 *Every finitely generated group H which is $\forall\exists$-equivalent to a nonabelian free group is torsion-free hyperbolic; moreover, H can be obtained from infinite cyclic groups by finitely many operations of the following type:*

1. *free products;*
2. *amalgamated products with infinite cyclic amalgamated subgroups at least one of which is maximal abelian;*

3. *separated HNN-extensions with infinite cyclic associated subgroups at least one of which is maximal abelian.*

In [?] Remeslennikov proved that every finitely generated fully residually free group acts freely on some $\mathbf{Z}^n$-tree with some order for a suitable natural number n. In [?] he asks (Question A) if such group acts freely on some $\mathbf{Z}^n$-tree with lexicographic order. Corollary 6 gives a positive answer to his question.

Corollary 6 *Every finitely generated fully residually free group acts freely on some $\mathbf{Z}^n$-tree, where $\mathbf{Z}^n$ is a direct sum of n copies of $\mathbf{Z}$ with lexicographic order.*

Let $U = \{u_1, \ldots, u_n\}$ be a set of parametric words, i.e. a subset of $F^{\mathbf{Z}^k}$. By the definition we have fixed some pure cyclic subgroup $\mathbf{Z}$ in $\mathbf{Z}^k$ in such a way that the action of this subgroup $\mathbf{Z}$ coincides with the integer powers in F. Because $\mathbf{Z}$ is pure in $\mathbf{Z}^k$ we have $\mathbf{Z}^k = \mathbf{Z} \oplus B$, where B is a free abelian group with a free base $t_1, \ldots, t_n$. These generators t_i-s are called parameters in $F^{\mathbf{Z}^k}$. Any homomorphism $\xi : B \longrightarrow \mathbf{Z}$ gives rise to a F-homomorphism $\xi^\star : F^{\mathbf{Z}^k} \longrightarrow F$. In this case we say that the image $U^{\xi^\star}$ is obtained from U by specializing parameters by ξ. Let

$$U^\star = \bigcup \{U^{\xi^\star} \mid \xi \in Hom(\mathbf{Z}^k, \mathbf{Z})\}$$

be the union of all specializations of the set U.

We can slightly generalize the construction of a specialization. Instead of a set U we can consider a set of tuples of words from $F^{\mathbf{Z}^k}$ and specialize them coordinatewise. Then we will get the set $U^\star$ of tuples of elements from F.

Theorem 7 *Let $S(X) = 1$ be a system of equations over a free group F. Then there exists a finite set of n-tuples of parametric words $U = (u_1, \ldots, u_n) \in (F^{\mathbf{Z}^k})^n$ such that the set of all their specializations $U^\star$ is a dense subset of the variety $V_F(S)$ in the Zariski topology.*

Corollary 7 *Any system $S = 1$ over a free group F has a dense subset which can be parametrized by finitely many parametric words.*

Definition 9 *The existential theory of G is the set of all formulas of the form*

$$\Phi = \exists \bar{x} (\bigwedge_1^s u_i(\bar{x}, \bar{g}) = 1 \bigwedge_1^t v_j(\bar{x}, \bar{g}) \neq 1),$$

that are true on G.

It was proved in [?] that the existential theory of a free group is decidable; this implies that for a finite system $S = 1$ the group $F_{R(S)}$ has decidable word problem.

Definition 10 *A fundamental sequence of length k for a system of equations ϕ is a triple*

$$(\mathcal{M}, Hom, Aut),$$

where $\mathcal{M}$ consists of n systems of equations $\phi_1 = 1, \ldots \phi_k = 1$, $\phi = \phi_1$ and ϕ_k is an empty system. Hom is a collection of $k-1$ homomorphisms $\pi_1, \ldots, \pi_{k-1}$ where $\pi_i : F_{R(\phi_i)} \to F_{R(\phi_{i+1})}$, and π_i is a retract on F. Aut is a collection of k finitely generated automorphism groups $P_1, \ldots, P_k$ of the groups $F_{R(\phi_1)}, \ldots, F_{R(\phi_k)}$ respectively. A fundamental sequence $\Phi = (\mathcal{M}, Hom, Aut)$ is effectively given if the systems in $\mathcal{M}$, homomorphisms from Hom, and automorphisms from Aut are effectively given. To effectively define a homomorphism from $F_{R(\phi)} \to F_{R(\psi)}$ means to define the images of the generators of the group $F_{R(\phi)}$.

If Φ is some fundamental sequence of length k for the system $\phi = 1$, $\pi : F_{R(\phi_n)} \to F$ a homomorphism of free groups, and $\sigma_1, \sigma_2, \ldots \sigma_k$ are automorphisms from $P_1, P_2 \ldots, P_k$ respectively, then the composition

$$F_{R(\phi)} \to_{\sigma_1} F_{R(\phi)} \to_{\pi_1} F_{R(\phi_2)} \to_{\sigma_2} F_{R(\phi_2)} \to_{\pi_2} \ldots F_{R(\phi_k)} \to_{\sigma_k} F_{R(\phi_k)} \to_{\pi} F \qquad (5)$$

equals $\pi_{\bar{X}}$ for some solution $\bar{X}$ of the system ϕ. We say that Φ describes a solution $\bar{X}$ of the system ϕ if $\pi_{\bar{X}}$ can be represented in the form (5) for some choice of $\pi_1, \ldots, \pi_k, \sigma_1, \sigma_2, \ldots \sigma_k$.

Lemma 3 *([**?**], Lemma 1.1) In an infinite sequence*

$$G_1 \to_{\pi_1} G_2 \to_{\pi_2} \ldots \to_{\pi_{r-1}} G_r \to_{\pi_r} \ldots$$

of finitely generated residually free groups $G_1, \ldots, G_r, \ldots$ and surjective homomorphisms, almost all homomorphisms are isomorphisms.

Proof Let $g_1, \ldots, g_n$ be a finite family of generators of G_1. Consider system of equations

$$\{\phi(x_1, \ldots, x_n) = 1 \mid \exists r(\pi_r \ldots \pi_1(\phi(g_1, \ldots, g_n)) = 1)\}. \qquad (6)$$

By Guba's theorem [**?**] there exists a finite subsystem $\phi_1(\bar{x}) = 1, \phi_2(\bar{x}) = 1, \ldots, \phi_m(\bar{x}) = 1$ of system (6) which is equivalent to (6). Let r_0 be such number that $\pi_{r_0} \ldots \pi_1(\phi_i(g_1, \ldots, g_n)) = 1, 1 \leq i \leq m$. We claim that π_r is an isomorphism for $r \geq r_0$.

Indeed, π_r is surjective by definition; so we only have to verify that it is injective. Let $g \in G_r; \pi_r(g) = 1$. Choose $g' \in G_1$ such that $\pi_{r-1} \ldots \pi_1(g') = g$ and consider $g' = \phi(g_1, \ldots, g_n)$. Then $\phi(x_1, \ldots, x_n) = 1$ is an equation of the system (6), and for any $X_1, \ldots, X_n \in F$ the following implication is true

$$\bigwedge_{i=1}^{m} \phi_i(X_1, \ldots, X_n) = 1 \to \phi(X_1, \ldots, X_n) = 1.$$

Suppose now that $g \neq 1$. Besause G_r is residually free, there exists a homomorphism $\pi : G_r \to F$ such that $\pi(g) \neq 1$. Let $X_j = \pi\pi_{r-1} \ldots \pi_1(g_j)(1 \leq j \leq n)$. Then for any $1 \leq i \leq m$ one has $\phi_i(\bar{X}) = \pi\pi_{r-1} \ldots \pi_1\phi_i(\bar{g}) = 1$ since $r \geq r_0$. But $\phi(\bar{X}) = \pi\pi_{r-1} \ldots \pi_1(g') = \pi(g) \neq 1$. This gives a contradiction with the implication above. $\square$

1. Reduction to a generalized equation.

Everywhere below $G = F$ will denote a free group $F(\bar{a})$, and $F(\bar{x})$ will denote a free group with generators $x_1, \ldots, x_n$. We will consider now a finite system of equations $S(\bar{x}, \bar{a}) = 1$.

A generalized equation is defined to be a collection consisting of the following:

1. An interval I, subdivided into ρ items $h_1, \ldots, h_\rho$ which play the role of the unknowns. The points of division are called "boundaries". This number ρ is called the number of unknowns. We have $\rho + 1$ boundary.

2. A system of $2n$ oriented subintervals, divided into pairs (a base and the dual base) and corresponding system of n basic equations. If λ is the number of a base, then $\Delta(\lambda) = n + \lambda$, if $\lambda \leq n$ and $\Delta(\lambda) = \lambda - n$, if $\lambda > n$ denotes the dual base; $\alpha(\lambda)$ and $\beta(\lambda)$ denote the initial and terminal boundary of λ.

The corresponding system of basic equations consists of the n equations

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \ldots h_{\beta(\lambda)-1}]^{\varepsilon(\lambda)} = [h_{\alpha(\Delta(\lambda))} h_{\alpha(\Delta(\lambda))+1} \ldots h_{\beta(\Delta(\lambda))-1}]^{\varepsilon(\Delta(\lambda))},$$

where $\varepsilon \in \{1, -1\}$.

3. A system of m coefficient equations $h_{i_l} = a_{j_l}^{\varepsilon_l}, (1 \leq l \leq m; t(l) = (i_l, j_l, \varepsilon_l))$.

4. A system of k boundary connections and corresponding system of k boundary equations. A boundary connection is a connection between boundary p on the base λ and boundary q on the base $\Delta(\lambda)$. A corresponding boundary equation is an equation

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \ldots h_{p-1}] = [h_{\alpha(\Delta(\lambda))} h_{\alpha(\Delta(\lambda))+1} \ldots h_{q-1}],$$

if $\varepsilon(\lambda) = \varepsilon(\Delta\lambda)$ and

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \ldots h_{p-1}] = [h_q h_{q+1} \ldots h_{\beta(\Delta(\lambda))-1}]^{-1},$$

if $\varepsilon(\lambda) = -\varepsilon(\Delta\lambda)$.

So there is a system of equations corresponding to the generalized equation. A solution of the generalized equation Ω is defined to be a collection $\bar{H}$ of nonempty words $H_1, \ldots, H_\rho$, which , when substituted into this system, turn it into graphical equalities, and the left and right sides of the basic equations are irreducible after this substitution.

The notation $(\Omega, \bar{H})$ means that $\bar{H}$ is a solution of the generalized equation Ω.

If $\bar{\phi}(\bar{h}) = \bar{\psi}(\bar{h})$ is an arbitrary list of equations, then the same list with asterisk (for example Ω^*) denotes the system of equations of the form $\bar{\phi}(\bar{h})(\bar{\psi}(\bar{h}))^{-1} = 1$ in the free group. Obviously, if $\bar{H}$ turns all the equations of Ω into a graphical equality, then $\bar{H}$ is a solution of the system Ω^*. The converse is false.

For a solution $\bar{H}$ of a generalized equation Ω we introduce the notation

$$X_\mu \equiv [H_{\alpha(\mu)} \ldots H_{\beta(\mu)-1}]^{\varepsilon(\mu)}.$$

In the cases when several solutions are being considered at the same time, superscripts on the words X_μ will indicate which solution they relate to.

The *length* of the word B will be denoted by $d(B)$. The *length of a solution* $\bar{H}$ of a generalized equation is defined to be

$$d(\bar{H}) = \sum_{i=1}^{\rho} d(H_i).$$

The *periodicity exponent* of a list of words in the maximal number m such that some of the words in the list contains a subword c^m for some c.

The *periodicity exponent* of a solution $\bar{H}$ is defined to be the periodicity exponent of the list of words $X_\mu, \mu \in \{1, \ldots, 2n\}$.

Lemma 4 *[?] For a given system of equations in a free group $S(\bar{x}, \bar{a}) = 1$ it is possible to construct effectively a finite list of generalized equations $\Omega_1, \ldots, \Omega_r$ and homomorphisms π_i $F_{R(S)} \to F_{R(\Omega_i^*)}$ such that for any solution $\bar{X}$ of the system $S = 1$ there exists $i \in \{1, \ldots, r\}$ and a solution $\bar{H}$ of Ω_i such that the following diagram commutes.*

$$\begin{tikzcd} F_{R(S)} \arrow[r, "\pi_i"] \arrow[d, "\pi_{\bar{X}}"'] & F_{R(\Omega_i^*)} \arrow[dl, "\pi_{\bar{H}}"] \\ F(\bar{a}) & \end{tikzcd}$$

Proof Every system S can be transformed by adding new variables into a system S_1 such that every equation in S_1 contains not more than 3 terms, and $F_{R(S)}$ is isomorphic to $F_{R(S_1)}$. Thus we can suppose that the system $S = 1$ has this property write it in the form

$$r_{11}r_{12}r_{13} = 1$$

$$r_{21}r_{22}r_{23} = 1,$$

$$\ldots$$

$$r_{m1}r_{m2}r_{m3} = 1,$$

where r_{ij} are letters in the alphabet $\bar{X}^{\pm 1} \cup \bar{a}^{\pm 1}$.

A partition table is defined to be a set of irreducible words $\{V_{ij}(z_1, \ldots, z_p)\}$ $(1 \leq i \leq m, 1 \leq j \leq 3)$ in the alphabet $\{z_1^{\pm 1}, \ldots, z_p^{\pm 1}\}$ such that the following condition are satisfied:

1. The equality $V_{i1}V_{i2}V_{i3} = 1, 1 \leq i \leq m$ holds in the free group with basis $\bar{z}$;
2. $d(V_{ij}) \leq 2$;
3. if $r_{ij} \in \bar{a}^{\pm 1}$, then $d(V_{ij}) = 1$.

The finite set of all partition tables can be effectively constructed for a system $S = 1$. An example of a partition table for equation $x_1x_2x_3 = 1$ is the following: $V_{11} = z_1z_2,\ V_{12} = z_2^{-1}z_3,\ V_{13} = z_3^{-1}z_1^{-1}$.

To each partition table $T = \{V_{ij}\}$ assign a generalized equation Ω_T in the following way. (Below we will use the notation $\doteq$ for graphical equality.) Let

$$V \doteq V_{11}V_{12}V_{13} \ldots V_{m1}V_{m2}V_{m3}.$$

Let $\rho = d(V)$. The equation Ω_T contains ρ variables $h_1, \dots, h_\rho$ corresponding to the letters of the word V. For any two distinct occurrences of $z_i^{\pm 1}$ introduce a basic equation $h_{j1}^{\varepsilon_1} = h_{j2}^{\varepsilon_2}$, where unknowns h_{j1}, h_{j2} correspond to the selected occurrences of $z_i^{\pm 1}$, and ε_1 and ε_2 are determined by the signs of these occurrences.

For all $1 \leq i_1, i_2 \leq m, 1 \leq j_1, j_2 \leq 3$ such that $r_{i_1 j_1}^{\pm 1} = r_{i_2 j_2}^{\pm 1} = x_k$ we introduce the basic equation

$$[h_{\alpha_1} \dots h_{\beta_1 - 1}]^{\varepsilon_1} = [h_{\alpha_2} \dots h_{\beta_2 - 1}]^{\varepsilon_2},$$

where the words $[h_{\alpha_1} \dots h_{\beta_1 - 1}]$ and $[h_{\alpha_2} \dots h_{\beta_2 - 1}]$ correspond to the occurrences of the words $V_{i_1 j_1}$ and $V_{i_2 j_2}$ in V.

For any $r_{ij} = a_k^{\pm 1}$ introduce the coefficient equation $h_\alpha = a_k^{\pm 1}$, where h_α corresponds to the occurrence of V_{ij} in V.

The list of boundary equations is empty.

For an arbitrary letter x_k in $\bar{x}$ we choose some occurrence $r_{i_k j_k}$ of the letter $x_k^{\varepsilon_k}$ in the system. Suppose that the word $h_{\alpha_k} h_{\alpha_k + 1} \dots h_{\beta_k - 1}$ corresponds to the occurrence of $V_{i_k j_k}$ in V. We define a homomorphism $\pi : F_{R(S)} \to F_{R(\Omega_T^*)}$ as follows $\pi(x_k) = (h_{\alpha_k} \dots h_{\beta_k - 1})^{\varepsilon_k}$. The value of $\pi(x_k)$ does not depend on the choice of the occurrence of $r_{i_k j_k}$. $\square$

A pair of dual bases $(\mu, \Delta(\mu))$ is said to be *matched* if $\alpha(\mu) = \alpha(\Delta(\mu))$.

We note some trivial properties satisfied by all generalized equations having at least one solution:

a) If $\varepsilon(\mu) = -\varepsilon(\Delta(\mu))$, then the bases μ and $\Delta(\mu)$ do not intersect.

b)If two boundary equations have respective parameters (p, λ, q) and (p_1, λ, q_1) with $p \leq p_1$, then $q \leq q_1$ in the case when $\varepsilon(\lambda)\varepsilon(\Delta(\lambda)) = 1$, and $q \geq q_1$ in the case $\varepsilon(\lambda)\varepsilon(\Delta(\lambda)) = -1$.

c) For a matched pair of bases $(\mu, \Delta(\mu))$ and a boundary connection (p, μ, q) we must have $p = q$.

d) A variable cannot occur in two distinct coefficient equations.

e) If h_i is a variable from some coefficient equation, and if $(i, \mu, q_1), (i+1, \mu, q_2)$ are boundary connections, then $|q_1 - q_2| = 1$.

Generalized equations satisfying these restrictions will be called *nondegenerate*.

2. Elementary transformations

We say that an item h_i belongs to the base μ if $\alpha(\mu) \leq i \leq \beta(\mu) - 1$. An item is said to be empty if it does not belong to any base. A boundary i cuts the base μ if $\alpha(\mu) < i < \beta(\mu)$. A boundary i touches the base μ if $i = \alpha(\mu)$ or $i = \beta(\mu)$. A boundary is said to be *open* if it cuts at least one base and is closed otherwise. A boundary is said to be *free* if it does not touch any base and is not connected by any boundary connection. A set of items $\{h_i, \dots, h_{i+j-1}\}$, denoted by $[i, i+j]$ is called a *section*. A section is said to be *closed* if the boundaries i and $i+j$ are closed and all the boundaries between them are open.

An elementary transformation of a nondegenerate generalized equation Ω gives a set of generalized equations $\Omega_1, \dots, \Omega_r$ and a collection of surjective homomorphisms $\theta_i : G_{R(\Omega^*)} \to G_{R(\Omega_i^*)}$ such that for every pair $(\Omega, \bar{H})$ there exists an unique pair $(\Omega_i, \bar{H}^{(i)})$ for which the following diagram commutes.

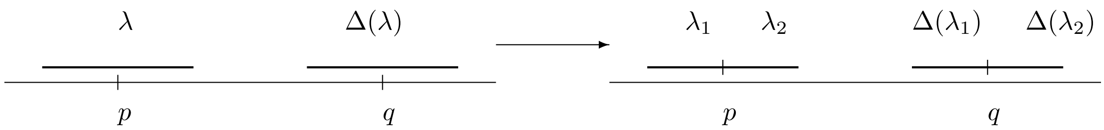
$$\begin{tikzcd}
F_{R(\Omega^*)} \arrow[r, "\theta_i"] \arrow[d, "\pi_{\bar H}"'] & F_{R(\Omega_i^*)} \arrow[dl, "\pi_{\bar H^{(i)}}"] \\
F(\bar a) &
\end{tikzcd}$$

Here $\bar H = (H_1, \ldots, H_n)$ and $\pi_{\bar H}(x_j) = H_j$.

We need 5 types of elementary transformations.

E1 (Cutting a base): (Fig. 1) Suppose there is a boundary connection $< p, \lambda, q >$. Then we cut the base λ into two bases λ_1 and λ_2 in the boundary p. We also cut $\Delta(\lambda)$ into $\Delta(\lambda_1)$ and $\Delta(\lambda_2)$ in the boundary q, replace the corresponding basic equation by the two equations and correct all the remaining boundary equations.

Fig. 1

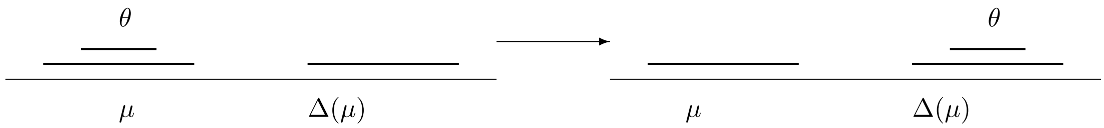


If Ω is a generalized equation, then by $\tilde\Omega$ we denote a generalized equation obtained from Ω by a consequent application of all possible E1 transformations. The groups $F_{R(\Omega^*)}$ and $F_{R(\tilde\Omega^*)}$ are isomorphic.

E2 (Transfer of a base): (Fig. 2) Suppose that the base θ is contained in the base μ ($\alpha(\mu) \leq \alpha(\theta) < \beta(\theta) \leq \beta(\mu)$). Suppose futher that there are boundary connections $< \alpha(\theta), \mu, \gamma_1 >$ and $< \beta(\theta), \mu, \gamma_2 >$ and that if there are some boundary connections for some boundaries cut by θ then these boundaries are connected through boundary connections to the corresponding boundaries on $\Delta(\mu)$.

Then we transfer θ from the situation on the base μ to the situation on the base $\Delta(\mu)$ and adjust all the basic and boundary equations.

Fig. 2



E3 (Removal of matched bases): Remove a pair of matched bases.

For the transformations E1-E3 the output consists of a single equation Ω_1; the list of unknowns remains the same; every solution $\bar{H}$ of Ω is a solution of Ω_1, and the systems Ω^* and Ω_1^* in the free groups are equivalent. The homomorphism π_1 is induced by the identity isomorphism on G and is itself an isomorphism.

E4 (Removal of a single base): Suppose the section $[h_{\alpha(\mu)} \ldots h_{\beta(\mu)-1}]^{\varepsilon}$ is covered by the a single base μ and that for all i $(1 \leq i \leq \beta(\mu) - \alpha(\mu) - 1)$ there exists a $w(i)$ such that the list of boundary connections contains $< \alpha(\mu) + i, \mu, w(i) >$.

The transformation E4 carries Ω into a unique generalized equation Ω_1 obtained from Ω by deleting $h_{\alpha(\mu)}, \ldots, h_{\beta(\mu)-1}$ from the list of unknowns. We define the homomorphism π_1 as follows:

$\pi_1(h_j) = h_j$ if $j < \alpha(\mu)$ or $j \geq \beta(\mu)$;

$$\pi_1(h_{\alpha(\mu)} + i - 1) = \{ \begin{array}{ll} h_{w(i-1)} \ldots h_{w(i)-1}, & if \varepsilon(\mu) = \varepsilon(\Delta\mu), \\ h_{w(i)} \ldots h_{w(i-1)-1}, & if \varepsilon(\mu) = -\varepsilon(\Delta\mu) \end{array}$$

for $1 \leq i \leq \beta(\mu) - \alpha(\mu)$. π_1 is obviously an isomorphism.

E5 (Introduction of a boundary): Suppose the list of boundary connections does not contain any connections with the first two parameters $< p, \mu, .. >$. Let q be a boundary on $\Delta(\mu)$. Then we perform one of the following two transformations:

1. Introduce the boundary connection $< p, \mu, q_i >$ if the new generalized equation is nondegenerate (the corresponding homomorphism from $G_{R(\Omega^*)}$ into $G_{R(\Omega_i^*)}$ will be induced by the identity isomorphism on $G[\bar{h}]$ and is not necessary an isomorphism).

2. Replace the unknown h_{q_i} by the two items h' and h'' and introduce the new connection, connecting boundary p with the boundary between h' and h'' (the corresponding homomorphism π_i from $G_{R(\Omega^*)}$ onto $G_{R(\Omega_i^*)}$ will be induced by the following homomorphism on $G[\bar{h}]$: $\hat{\pi}_i(h_k) = h_k$ if $k \neq q_i$, and $\hat{\pi}_i(h_{q_i}) = h'h''$. And π_i is an isomorphism).

From now on we consider solutions of generalized equations in the extended alphabet $\bar{a} \cup \bar{b}$. Let now $F = F(\bar{a}, \bar{b})$. Suppose we have a generalized equation Ω and a solution $\bar{H}$.

Let P be a group of automorphisms of $F_{R(\Omega^*)}$ and $\bar{H}^{(1)}$ and $\bar{H}^{(2)}$ be two solutions of the generalized equation Ω. We will write $\bar{H}^{(1)} <_P \bar{H}^{(2)}$ if there exists an endomorphism π of the group F which is an $\bar{a}$-homomorphism, and an automorphism $\sigma \in P$ such that $\pi_{\bar{H}^{(2)}} = \pi\pi_{\bar{H}^{(1)}}\sigma$ and $d(H_k^{(1)}) \leq d(H_k^{(2)})$ for all $1 \leq k \leq \rho$ and such that at least for one k, $d(H_k^{(1)}) < d(H_k^{(2)})$. A solution $\bar{H}$ of Ω is called *minimal with respect to the group of automorphisms* P if there is no solution $\bar{H}^+$ of the equation Ω such that $\bar{H}^+ < \bar{H}$. A solution $\bar{H}$ of Ω is called *minimal* if it is minimal with respect to the canonical group of automorphisms of $F_{R(\Omega^*)}$ (to be defined below).

3. Some results about irreducible systems

The following lemma was proved in [**?**] (Lemma 19).

Lemma 5 *Let H be a CSA-group and*

$$\Phi = \{\phi : H \longrightarrow H_\phi\}$$

a separating family of homomorphisms of H. Then for any finite partition $\Phi = \bigcup_{i=1}^n \Phi_i$ there exists an index $i, (1 \leq i \leq n)$, such that Φ_i is also a separating family of homomorphisms.

Every fully residually free group is CSA; hence for any irreducible system S we can apply this lemma to $H = F_{R(S)}$ and any separating family of homomorphisms.

4. Kernel of a generalized equation

Let Ω be a generalized equation and let γ_i denote the number of bases containing h_i.

Suppose first that Ω does not contain boundary connections. The base μ is called *eliminable* in the equation Ω if at least one of the following two conditions is satisfied:

a) There exists h_i such that $h_i \in \mu, \gamma_i = 1$ and h_i is not contained in the coefficient equations.

b) At least one of the boundaries $\alpha(\mu), \beta(\mu)$ is different from $1, \rho + 1$, and does not touch any other base and any coefficient equation.

Consider a sequence

$$\Omega = \Omega_0 \rightarrow \Omega_1 \rightarrow \ldots \rightarrow \Omega_l, \tag{7}$$

in which Ω_{i+1} is obtained from Ω_i by deleting some eliminable base μ_{i+1} together with $\Delta(\mu_{i+1})$. Suppose Ω_l does not contain eliminable bases.

Lemma 6 *Equation Ω_l in the sequence (7) depends only on Ω but not on the choice of sequence (7).*

Proof Suppose there is another sequence

$$\Omega = \Omega_0 \rightarrow \Omega'_1 \rightarrow \ldots \rightarrow \Omega'_{l'}, \tag{8}$$

with the same properties and $\Omega_l \neq \Omega'_{l'}$. Without loss of generality we can suppose that the pair $(\mu, \Delta(\mu))$ belongs to Ω_l but not to $\Omega'_{l'}$. Suppose this pair is deleted with the transformation $\Omega'_k \rightarrow \Omega'_{k+1}$. Let k be minimal with these two properties. Then the set of bases of Ω_l is contained in the set of bases of Ω'_k. This implies that μ is eliminable in Ω_l. This contradicts the definition of sequence (7). □

Equation Ω_l in (7) will be called *the kernel* of the equation Ω and denoted $Ker(\Omega)$. We say that h_i belongs to the kernel, if h_i belongs to at least one base in the kernel or h_i occurs in some coefficient equation.

Let the generalized equation $\bar{\Omega}_l$ be obtained from Ω_l by deleting variables $h_i \notin Ker(\Omega)$.

Lemma 7 *[?] $F_{R(\Omega^*)}$ is isomorphic to $F(\bar{y}) * F_{R(\bar{\Omega}_l^*)}$, where $F(\bar{y})$ is a free group with a finite basis $\bar{y}$.*

5. Construction of $T(\Omega)$

Let Ω be a nondegenerate generalized equation. We describe the construction of the tree $T(\Omega)$, which is oriented from the root. To each vertex of $T(\Omega)$ we assign a generalized equation Ω_v, and an equation corresponding to the root v_0. For any edge $e : v \to v'$ we assign a surjective homomorphism $\pi(v, v') : F_{R(\Omega_v^*)} \to F_{R(\Omega_v'^*)}$. If $v \to v_1 \to \ldots \to v_s \to v'$ is a path in $T(\Omega)$, then $\pi(v, v')$ is a composition of $\pi(v_s, v'), \pi(v_{s-1}, v_s), \ldots, \pi(v, v_1)$. The set of all edges is subdivided into principal and auxiliary edges.

Closed sections of Ω_v are subdivided into working and constant sections. We will suppose that the union of working closed sections forms the section $[1, j_v]$ for some boundary j_v of the equation Ω_v, and the union of constant sections forms section $[j_v, \rho_v + 1]$. The edges are also subdivided into two classes: principal and auxiliary.

The construction begins with announcing all closed sections as working sections.

Denote by ρ' the number of variables in the working sections of some equation Ω, and by n' the number of bases on these sections, by ν' the number of open boundaries in the working sections, σ' the number of closed boundaries in the working sections. The number of closed working sections containing zero bases, one base, or more than one base are denoted by t', u', w' respectively. The *complexity* of the equation Ω is the number

$$\tau' = n' - u' - 2w' = \Sigma max\{0, n_i - 2\},$$

where n_i is the number of bases on the closed working section with number i, and the summation is taken over all closed working sections.

It is obvious that $\tau' \geq 0$, and equality holds if and only if each closed working section contains not more than two bases.

Suppose we are at the vertex v. The outgoing edges of this vertex depend on which of the cases described below takes place. If we have Case i, $(i < 15)$, then suppose that Cases $1 \ldots, i-1$ do not take place. Cases 14 and 15 can take place for the same vertex v.

In Cases 1 and 2 the vertex is said to be the end vertex.

Case 1. The homomorphism $\pi(v_0, v)$ is not an isomorphism.

Case 2. Ω_v does not contain working sections.

Case 3. Ω_v contains h_k, which belongs to the working sections and to some coefficient equation, and the section $[k, k+1]$ is not closed. Then we first perform a series of elementary E5 transformations, continuing the boundaries k and $k+1$ through all bases they intersect. Then perform a series of E1 transformations, cutting these bases on the introduced boundaries. In all the equations obtained this way $[k, k+1]$ is closed.

Case 4. The generalized equation contains h_k, which belongs to the closed section $[k, k+1]$ contained in some coefficient equation. The section $[k, k+1]$ becomes constant and the corresponding edge is auxiliary.

Case 5. Ω_v contains a fictitious unknown h_q belonging to the working section. The section $[q, q+1]$ is transferred into constant sections and the edge is auxiliary.

Case 6. Ω_v contains a pair of matched bases in a working section. Perform E3 and delete it.

Case 7. $\gamma_i = 1$ for some h_i belonging to a working section, such that both boundaries i and $i+1$ are closed. Apply $E4$ and delete the closed section $[i, i+1]$ together with unique base that is contained in this section.

Case 8. $\gamma_i = 1$ for some h_i belonging to a working section, and one of the boundaries $i, i+1$ is open and the other is closed. Without loss of generality we can consider i as a closed boundary. Perform E5 and continue $i+1$ through the only base μ it intersects; cut μ in $i+1$, and delete $[i, i+1]$ which is now closed.

Case 9. $\gamma_i = 1$ for some h_i belonging to a working section, and both $i, i+1$ are open. In addition, some closed section $[j_1, j_2]$ contains exactly two bases μ_1, μ_2, such that $\alpha(\mu_1) = \alpha(\mu_2)$, and $\beta(\mu_1) = \beta(\mu_2)$ and all the bases of $\tilde{\Omega}_v$, obtained from μ_1, μ_2 by cuttings, do not belong to the kernel of $\tilde{\Omega}_v$.

Using E5 continue through μ_1 all the boundaries that intersect it. Using E2 transfer μ_2 from the situation on μ_1 to the situation on $\Delta(\mu_1)$. Delete μ_1 together with the closed section $[j_1, j_2]$.

Case 10. The first assumption in case 9 holds and the second does not. Perform E5, continue i and $i+1$ through μ, perform twice E1, and then cut μ into 3 new bases. Finally, delete $[i, i+1]$ together with the unique base that is contained in it.

Case 11. Some boundary ℓ on the working part is free. Since we do not have case 5, ℓ intersects at least one base μ. Continue ℓ through μ using E5.

Before considering Case 12 let us proceed to the consideration of the *entire transformation* composed in a definite way from the elementary ones. We apply this transformation only to equations with $\gamma_i \geq 2$, for each i. We can perform the entire transformation on the union of some closed sections of the equation Ω_v. First suppose that these sections are all situated on the interval $[1, j+1]$. A base μ of the equation Ω is called a leading base, if $\alpha(\mu) = 1$. A leading base is said to be maximal if $\beta(\lambda) \leq \beta(\mu)$, for any other leading base λ. The base having largest index among the maximal bases is called the carrier base. A base λ is called a transfer base if $\beta(\lambda) \leq \beta(\mu)$ and $\lambda \neq \mu$, where μ is the carrier base. Let μ be the carrier base of an equation Ω. Take a transfer base λ and applying an $E5$ transformation, continue through μ all the boundaries on λ. Using E2 we transfer all the transfer bases from the situation at the base μ to the situation at the base $\Delta(\mu)$. Now, there exists some $w < \beta(\mu)$ such that $h_1, \dots, h_w$ belong to only one base μ, while the interval h_{w+1} belongs to at least two bases. Applying E1 we cut μ along the boundary $i+1$. An application of E4 annihilates the section $[1, w+1]$ which has become closed together with the unique base belonging to it. Notice that the entire transformation does not increase complexity.

Case 12. $\gamma_i \geq 2$ for each h_i belonging to working sections. In addition, for some base μ section $[\alpha(\mu), \beta(\mu)]$ is closed. Using E5 continue all the boundaries which intersect μ through μ. Using E3 transfer all the bases situated on μ to the situation on $\Delta(\mu)$. Using E2 delete $[\alpha(\mu), \beta(\mu)]$ together with the pair μ, $\Delta(\mu)$.

Case 13. $\gamma_i \geq 2$ for each h_i belonging to working sections. In addition some boundary ℓ, belonging to a working section and touching some base intersects some base μ and is not continued through μ by a boundary connection. Continue ℓ through μ using E5.

Case 14. $\gamma_i \geq 2$ for each h_i and $\gamma_i = 2$ for some h_i and in addition $F_{R(\Omega_v^*)}$ is not isomorphic to $F_{R(s_2^*)}$, where s_2 is a generalized equation as defined below.

Notice that the function γ_i is constant when h_i belongs to some closed section of $\tilde{\Omega}_v$.

Consider the following transformation of Ω_v. Applying E1 transformations to cut the bases containing h_i covered exactly twice, we finally get that the union of bases covered twice becomes a union of closed sections.

Renumbering h_i's we can suppose that the section $[1, j+1]$ is covered exactly twice. We say now that this is a quadratic section.

If μ and $\Delta\mu$ both belong to the quadratic section, then μ is called a *variable base.* If μ belongs and $\Delta\mu$ does not belong to the quadratic section, then μ is called a *constant base.*

Apply now the entire transformations to the quadratic section of $\tilde{\Omega}_v$. Each time we apply the entire transformation we do not increase complexity and do not increase the total number of items in the whole interval.

Every time we express some items of the quadratic section through the other items of the quadratic section and the rest of the items. The number of items on the quadratic section and the number of bases can not increase. We also delete pairs of matched bases. If the process continues for too long then the equation with the same quadratic part will occur twice, and the corresponding homomorphism is an automorphism invariant with respect to the items in the nonquadratic part ([**?**], Lemma 3.3, second part). Lemma 8 in [**?**] and [**?**] imply that this group of automorphisms is finitely generated and there is an effective procedure to obtain the generating set.

After we get a repetition of the equation, we have to introduce a new boundary equation without introducing a new boundary in the quadratic section. This operation decreases the number of items in the quadratic section. Finally, we find a solution of a quadratic equation expressed in terms of h's not belonging to the quadratic part.

There are several new h's and several new equations on the h's not belonging to the quadratic part obtained after the process stopped.

Let s_1 be a generalized equation consisting of bases such that one of the paired bases is either variable or a constant base with respect to the quadratic part. Let p_2 be a generalized equation on h's not belonging to the quadratic part before the process started. Let s_2 be a generalized equation on h's not belonging to the quadratic part which we get after we have finished the process with the quadratic part. We have $\Omega_v = s_1 \cup p_2$.

There are two possibilities.

1. The canonical homomorphism $F_{R(s_1 \cup p_2)} \to F_{R(s_2)}$ is an isomorphism. In this case we do not apply the transformation described above, instead we construct outgoing edges as described in Case 15 below.

2. The canonical homomorphism $F_{R(s_1 \cup p_2)} \to F_{R(s_2)}$ is not an isomorphism. Then we construct a path

$$v = v_1 \to v_2 \to \dots v_n \tag{9}$$

in $T(\Omega)$, such that each edge $v_i \to v_{i+1}$ corresponds to one entire transformation. We say that for each v_i in this path we have Case 14 and for v_n $\Omega_{v_n} = s_2$. To each v_i assign the group of automorphisms P_i of $F_{R((\Omega_{v_i})^*)}$ invariant with respect to the nonquadratic part of Ω_{v_i} and call it a canonical group of automorphisms for the equation Ω_{v_i} in Case 14. We have a piece of Razborov's fundamental sequence

$$F_{R(\Omega_v^*)} \to_{\sigma_1} F_{R(\Omega_v^*)} \to_{\pi(v_1,v_2)} F_{R(\Omega_{v_1}^*)} \to_{\sigma_2} F_{R(\Omega_{v_1}^*)} \to_{\pi(v_2,v_3)} \dots F_{R(\Omega_{v_n}^*)} \to_{\sigma_n} F_{R(\Omega_{v_n}^*)}, \tag{10}$$

where $\sigma_i \in P_i$, correspond to some epimorphisms $\pi : F_{R((s_1 \cup p_2)^*)} \to F_{R(s_2^*)}$.

Let P be the group of all such epimorphisms. In Case 15 we can also consider the group P which will be a group of automorphisms of $F_{R(\Omega_v^*)}$. We call it canonical for Case 15.

Let λ be a natural homomorphism $F_{R((s_1 \cup p_2)^*)} \to F_{R((s_1 \cup s_2)^*)}$. Then for any epimorphism $\pi \in P$ there is a natural epimorphism $\phi : F_{R((s_1 \cup s_2)^*)} \to F_{R(s_2^*)}$ such that the following diagram commutes.

$$\begin{tikzcd}
F_{R((s_1 \cup p_2)^*)} \arrow[r, "\lambda"] \arrow[d, "\pi"'] & F_{R((s_1 \cup s_2)^*)} \arrow[dl, "\phi"] \\
F_{R(s_2^*)} &
\end{tikzcd}$$

In the situation where the canonical homomorphism $F_{R((s_1 \cup p_2)^*)} \to F_{R(s_2^*)}$ is an isomorphism , λ is an embedding of $F_{R((s_1 \cup p_2)^*)}$ into $F_{R((s_1 \cup s_2)^*)}$, because all the epimorphisms π are isomorphisms.

Lemma 8 *The natural homomorphism $\psi : F_{R((s_2)^*)} \to F_{R((s_1 \cup s_2)^*)}$ is a monomorphism.*

Proof Let H be the subgroup $\psi(F_{R(s_2^*)})$ in $F_{R((s_1 \cup s_2)^*)}$. The epimorphism $\phi : F_{R((s_1 \cup s_2)^*)} \to F_{R(s_2^*)}$ defined above is identical on H and determines a solution of the system s_1^* over $F_{R(s_2^*)}$. Hence $\phi \circ \psi$ is an identity on $F_{R(s_2^*)}$. $\square$

Let $\tilde{F}_{R((s_1 \cup p_2)^*)}$ be the factor-group of $F_{R((s_1 \cup p_2)^*)}$ over the intersection of the kernels of all epimorphisms in P.

Lemma 9 *The homomorphism λ induces an embedding of $\tilde{F}_{R(s_1 \cup p_2)}$ into $F_{R((s_1 \cup s_2)^*)}$.*

Proof Take an element $g \in F_{R((s_1 \cup p_2)^*)}$ which does not belong to the intersection of all these kernels. Then there is an epimorphism $\pi : F_{R((s_1 \cup p_2)^*)} \to F_{R(s_2^*)}$ such that $\pi(g) \neq 1$. This implies that $\lambda(g) \neq 1$ by the commutativity of the diagram above.

Case 15. $\gamma_i \geq 2$ for each h_i belonging to working sections (and the application of Case 14 would give an isomorphism $F_{R((s_1 \cup p_2)^*)} \to F_{R(s_2^*)}$, so we do not apply the transformation of Case 14). In this case it must be some h_i with $\gamma_i > 2$. Apply the entire transformation. Continue all boundaries that touch at least one base through all the bases they intersect.

In Case 15 it is also possible that there are some auxiliary edges coming out of the vertex v, this is described below in Case 15.1.

Case 15.1. All the assumptions of Case 15 hold. In addition the carrier base μ of the equation Ω_v intersects with $\Delta(\mu)$. First construct some equation $\Omega_{v'}$ in the following way. Introduce the new closed section $[\rho_v + 1, \rho_v + 2]$, and announce this section as a constant section. Introduce a new pair of bases $(\lambda, \Delta(\lambda))$, such that $\alpha(\lambda) = 1, \beta(\lambda) = \beta(\Delta(\mu)), \alpha(\Delta(\lambda)) = \rho_v + 1, \beta(\Delta(\lambda)) = \rho_v + 2$. In other words we introduce the new basic equation $h' = h[1, \beta(\Delta(\mu))]$, where h' is a new variable. Let $\pi(v, v')$ be a natural isomorphism. Notice that Ω_v can be obtained from $\Omega_{v'}$ with the use of E4 by deleting $\delta(\lambda)$ together with the closed section $[\rho_v + 1, \rho_v + 2]$. For the equation $\Omega_{v'}$ we have Case 15, but λ is a carrier base. Applying to $\Omega_{v'}$ transformations described for Case 15, we obtain the list of all auxiliary edges coming out of the vertex v.

The tree $T(\Omega)$ is described. In Case 14 we can not say that every solution of one of the equations $\Omega_{v'}$ is a solution of Ω_v, but we can say that every solution of one of the equations $\Omega^*_{v'}$ is a solution of Ω^*_v. We can also say that every solution of Ω_v is a solution of one of the equations $\Omega_{v'}$ and every solution of Ω^*_v is a solution of one of the equations $\Omega^*_{v'}$.

Notice that our first 11 cases coincide with 11 cases in Razborov's thesis. Our Cases 12 and 13 correspond to his Cases 13 and 14 respectively. Our Case 14 is different; our Case 15 is a partial case of his Case 15.

If Case i ($1 \leq i \leq 13$) takes place for a vertex v, we say that v has type i and write $tp(v) = i$. In Case 14 (resp. 15) we say that $tp(v) = 14$ (resp. $tp(v) = 15$) depending on whether we apply to v the transformation of Case 14 or 15.

Lemma 10 *(Lemma 3.1, [?]) If $v_1 \to v_2$, is the principal edge of the tree $T(\Omega)$, then*

1. $n'_2 \leq n'_1$, *if* $tp(v_1) \neq 3, 10$. *This inequality is proper if* $tp(v_1) = 6, 7, 9, 12$.
2. *If* $tp(v_1) = 10$, *then* $n'_2 \leq n'_1 + 2$.
3. $\nu'_2 \leq \nu'_1$ *if* $tp(v_1) \leq 12$ *and* $tp(v_1) \neq 3, 11$.
4. $\tau'_2 \leq \tau'_1$, *if* $tp(v_1) \neq 3, 14$.

All these assertions can be verified directly.

Lemma 11 *Let $v_1 \to v_2 \to \ldots \to v_r \ldots$ be an infinite path in the tree $T(\Omega)$. Then there exists N such that all the edges of this path starting with N are principal edges, and one of the following holds:*

1. $7 \leq tp(v_n) \leq 10$ *for all* $n \geq N$,
2. $tp(v_n) = 15$ *for all* $n \geq N$.

Proof Notice that if some generalized equation contains a coefficient equation $h_i = a_j^{\pm 1}$, such that h_i belongs to the working part, then we apply transformations of Cases 3,4, decreasing the number of such equations. So in generalized equations of $2t$'s level (where t is the number of coefficient equations in initial Ω) unknowns on the working part will not belong to coefficient equations, and without loss of generality we can think that Ω already has this property. Then we do not use Cases 3,4 in the construction of the tree. Case 14 can only occur finitely many times , because the transformation 14 gives a proper homomorphism $F_{R(\Omega^*_v)} \to F_{R(s^*_2)}$. So we can suppose that we do not have it. So all our transformations do not increase complexity. We can suppose that $tp(v_i) \geq 5$ for all i.

We show that the number of vertices for which $tp(v_i) = 5$ is not more than ρ. Indeed, if we denote by Ω' the generalized equation obtained from Ω by deleting all the coefficient equations, then the tree $T(\Omega')$ can be obtained from $T(\Omega)$ by replacing all generalized equations Ω_v by $\Omega_{v'}$; hence for any vertex there is a surjective homomorphism from $F_{R(\Omega'^*)}$ to $F_{R(\Omega'^*_v)}$. This implies that $F_{R(\Omega'^*_v)}$ can be generated by $\rho + \omega$ elements, where $\omega = card(\bar{a})$. If the path from the root v_0 to v contains at least $\rho + 1$ vertex of type 5, then Ω'_v would have at least $\rho + 1$ fictitious variables (on the constant sections). Sending all the other variables into identity we would have a homomorphism from $F_{R(\Omega'^*_v)}$ onto a free group of rank $\rho + \omega + 1$, which gives a contradiction (see Proposition 1.2.7 [?]).

So we can suppose that $tp(v_i) \neq 5$.

The value of complexity must be stabilized for the infinite path. If we have an auxiliary edge then it only can be constructed by using the Case 15.1. But by applying the transformation of Case 15 to the equation Ω'_v, constructed for the Case 15.1, both bases μ and $\Delta(\mu)$ will be transferred from the base λ to the constant part, so the complexity will be decreased by 2. But $\tau_{v'} = \tau_i + 1$, hence $\tau'_{i+1} < \tau'_i$. Hence the number of auxiliary edges in the path must be finite. So we can suppose that τ'_i is a constant and all the edges of the path are principal edges.

If now $tp(v_i) = 6$, then the closed section, containing matched bases $\mu, \Delta(\mu)$, can not contain any other bases, because then complexity would be decreased. But if this section does not contain any other bases, then $tp(v_{i+1}) = 5$ which is also impossible.

So we can suppose that $tp(v_i) \geq 7$. If the equation Ω_1 does not contain free boundaries and Ω_2 is obtained from it by an elementary transformation other than E3, then Ω_2 does not contain free boundaries. Hence $tp(v_i) \neq 6$ implies that $tp(v_i) \neq 11$.

If $12 \leq tp(v_i) \leq 13$, or $tp(v_i) = 15$, then $tp(v_{i+1}) \in \{6, 13, 14, 15, 12\}$. Since $tp(v_i) \neq 6, 14$, this implies that for all vertices $v_j (j \geq i)$ we also have $12 \leq tp(v_i) \leq 13$ or $tp(v_i) = 15$. In this case the sequence n' stabilizes by lemma 10. In addition, if $tp(v_j) = 12$, then $n'_{j+1} < n'_j$. Hence $tp(v_j) \neq 12$ for all j. There can not be more than $8(n')^2$ vertices of type 13 in a row; hence there exists $j \geq i$ such that $tp(v_j) = 15$. The series of transformations E5 in Case 15 guarantees the inequality $tp(v_{j+1}) \neq 13$; hence $tp(v_{j+1}) = 15$, and we have assertion 2 of the lemma.

So we can suppose $tp(v_i) \leq 10$ for all the vertices of our path. Then we have assertion 1 of the lemma. $\square$

6. Periodized equations

This section is basically a translation of the corresponding section from [**?**].

Let us assume at first that the equation Ω contains no boundary connections and is nondegenerate. The *periodic structure* of the equation Ω is a pair $\langle \mathcal{P}, R \rangle$, where $\mathcal{P}$ is a set of unknowns, bases, and closed sections of the equation Ω; R is an equivalence relation on a certain set of boundaries (which will be defined below – see item e)), and where the pair $\langle \mathcal{P}, R \rangle$ satisfies the following six properties:

a) if $h_i \in \mathcal{P}$ and $h_i \in \mu$, then $\mu \in \mathcal{P}$; moreover, this holds $\forall h_i \in \mathcal{P}$ $(\gamma_i \geq 1)$;

b) if $\mu \in \mathcal{P}$, then $\Delta(\mu) \in \mathcal{P}$;

c) if $\mu \in \mathcal{P}$ and $\mu \in [i, j]$, then $[i, j] \in \mathcal{P}$;

d) there exists a function $\mathcal{X}$ mapping the set of closed sections from $\mathcal{P}$ into $\{-1, +1\}$ such that for every $\mu, [i_1, j_1], [i_2, j_2] \in \mathcal{P}$, the condition that $\mu \in [i_1, j_1]$ and $\Delta(\mu) \in [i_2, j_2]$ implies
$\varepsilon(\mu) \cdot \varepsilon(\Delta(\mu)) = \mathcal{X}([i_1, j_1]) \cdot \mathcal{X}([i_2, j_2])$;

e) R is an equivalence relation on the set of those boundaries l for which there exists $[i, j] \in \mathcal{P}$ such that $i \leq l \leq j$. Furthermore, if a boundary l is closed, and both closed sections $[j, l]$ and $[l, j]$ belong to $\mathcal{P}$, then we consider two copies of the boundary l, not related to each other, one of which is associated with $[i, l]$ and the other with $[l, j]$;

f) if $\mu \in \mathcal{P}$, then $R(\alpha(\mu), \alpha(\Delta(\mu)))$, $R(\beta(\mu), \beta(\Delta(\mu)))$ in the case where $\varepsilon(\mu) = \varepsilon(\Delta(\mu))$ and $R(\alpha(\mu)), \beta(\Delta(\mu))), R(\beta(\mu), \alpha(\Delta(\mu)))$ in the case where $\varepsilon(\mu) = -\varepsilon(\Delta(\mu))$. Here the boundaries $\alpha(\mu)$, $\beta(\mu)$ are associated with the closed section on which the base μ lies.

A solution $\bar{H}$ of a generalized equation Ω is called *periodic with respect to a period* P (P is a primitive cyclically irreducible word), if for every closed section $[i,j]$ containing at least one base either $d(H[i,j]) = 1$ or the word $H[i,j]$ can be represented in the form

$$H[i,j] = A^r A_1 \tag{11}$$

$$(r \geq 1, A = A_1 A_2,\ A \text{ is a primitive word, } d(A) \leq d(P)),$$

where for at least one such section the word A in presentation (11) is a cyclic shift of the word $P^{\pm 1}$, and $r \geq 2$.

Now we will show how one associates to each solution $\bar{H}$ of a generalized equation Ω a periodic structure $\langle \mathcal{P}, R\rangle$, which will be denoted by $\mathcal{P}(\bar{H}, P)$. A closed section $[i,j]$ is included in the list $\mathcal{P}$ if and only if it contains at least one base and has a presentation (11) in which A is a cyclic shift of the word $P^{\pm 1}$ and $r \geq 2$. An unknown h_i is included in the list $\mathcal{P}$ if and only if h_i belongs to a closed section from $\mathcal{P}$ and $d(H_i) \geq 2d(P)$. A base μ is included in $\mathcal{P}$ if and only if either μ or $\Delta(\mu)$ contains an unknown from $\mathcal{P}$.

For a set $\mathcal{P}$ defined in this way, items a) and b) from the definition of a periodic structure can be trivially verified.

Let $\mu \in \mathcal{P}$ and $\mu \in [i,j]$. There exists an unknown $h_k \in \mathcal{P}$ such that $h_k \in \mu$ or $h_k \in \Delta(\mu)$. If $h_k \in \mu$, then, obviously, $[i,j] \in \mathcal{P}$. If $h_k \in \Delta(\mu)$ and $\Delta(\mu) \in [i',j']$, then $[i',j'] \in \mathcal{P}$, and hence, the word $H[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]$ can be written in the form $Q^{r'} Q_1$, where $Q = Q_1 Q_2$; Q is a cyclic shift of the word $P^{\pm 1}$ and $r' \geq 2$. Now let (11) be a presentation for the section $[i,j]$. Then $H[\alpha(\mu), \beta(\mu)] = B^s B_1$, where B is a cyclic shift of the word $A^{\pm 1}$, $d(B) \leq d(P)$, $B = B_1 B_2$, and $s \geq 0$. From the equality $H[\alpha(\mu), \beta(\mu)]^{\varepsilon(\mu)} = H[\alpha(\Delta(\mu)), \beta(\Delta(\mu)))]^{\varepsilon(\Delta(\mu))}$ and Lemma 1.2.9 [1] it follows that B is a cyclic shift of the word $Q^{\pm 1}$. Consequently, A is a cyclic shift of the word $P^{\pm 1}$ and $r \geq 2$ in (11), since $d(H[i,j]) \geq d(H[\alpha(\mu), \beta(\mu)]) \geq 2d(P)$. Therefore, $[i,j] \in \mathcal{P}$; i.e, part c) of the definition of a periodic structure holds.

Put $\mathcal{X}([i,j]) = \pm 1$ depending on whether in (11) the word A is conjugate to P or to P^{-1}. If $\mu \in [i_1, j_1]$, $\Delta(\mu) \in [i_2, j_2]$, and $\mu \in \mathcal{P}$, then the equality $\varepsilon(\mu) \cdot \varepsilon(\Delta(\mu)) = \mathcal{X}([i_1, j_1]) \cdot \mathcal{X}([i_2, j_2])$ follows from the fact that given $A^r A_1 = B^s B_1$ and $r, s \geq 2$, the word A cannot be a cyclic shift of the word B^{-1}. Hence, part d) also holds.

Now let $[i,j] \in \mathcal{P}$ and $i \leq l \leq j$. Then there exists a subdivision $P = P_1 P_2$ such that if $\mathcal{X}([i,j]) = 1$, then the word $H[i,l]$ is the end of the word $(P^{\infty}) P_1$ and $H[l,j]$ is the beginning of the word $P_2(P_{\infty})$, and if $\mathcal{X}([i,j]) = -1$, then the word $H[i,l]$ is the end of the word $(P^{-1})^{\infty} P_2^{-1}$ and $H[l,j]$ is the beginning of $P_1^{-1}(P^{-1})^{\infty}$. Again, Lemma 1.2.9 [**?**] implies that the subdivision $P = P_1 P_2$ with the indicated properties is unique; denote it by $\delta(l)$. Let us define a relation R in the following way: $R(l_1, l_2) \rightleftharpoons \delta(l_1) = \delta(l_2)$. Item e) of the definition of a periodic structure obviously holds.

Item f) follows from the graphic equality $H[\alpha(\mu), \beta(\mu)]^{\varepsilon(\mu)} = H[\alpha(\Delta(\mu)),\ \beta(\Delta(\mu))]^{\varepsilon(\Delta(\mu))}$ and Lemma 1.2.9 [**?**].

Now let us fix a nonempty periodic structure $\langle \mathcal{P}, R\rangle$. Item d) allows us to assume (after replacing the variables $h_i, \ldots, h_{j-1}$ by $h_{j-1}^{-1}, \ldots, h_i^{-1}$ on those sections $[i,j] \in \mathcal{P}$ for which $\mathcal{X}([i,j]) = -1$) that $\varepsilon(\mu) = 1$ for all $\mu \in \mathcal{P}$. For a boundary k, we will denote by (k) the equivalence class of the relation R to which it belongs.

Let us construct an oriented graph Γ whose set of vertices is the set of R–equivalence classes. For each unknown h_k lying on a certain closed section from $\mathcal{P}$, we introduce an oriented edge e leading from (k) to $(k+1)$ and an inverse edge e^{-1} leading from $(k+1)$ to (k). This edge e is assigned the label $h(e) \rightleftharpoons h_k$ (respectively, $h(e^{-1}) \rightleftharpoons h_k^{-1}$.) For every path $r = e_1^{\pm 1} \dots e_s^{\pm 1}$ in the graph Γ denote by $h(r)$ its label $h(e_1^{\pm 1}) \dots h(e_j^{\pm 1})$. The periodic structure $\langle \mathcal{P}, R \rangle$ is called *connected*, if the graph Γ is connected. Suppose first that $\langle \mathcal{P}, R \rangle$ is connected.

Lemma 12 *Let $\bar{H}$ be a solution of a generalized equation Ω periodic with respect to a period P, $\langle \mathcal{P}, R \rangle = \mathcal{P}(\bar{H}, P)$; c a cycle in the graph Γ at the vertex (l); $\delta(l) = P_1P_2$. Then there exists $n \in \mathbf{Z}$ such that $H(c) = (P_2P_1)^n$.*

Proof If e is an edge in the graph Γ with initial vertex V' and terminal vertex V'' and $P = P_1'P_2'$, $P = P_1''P_2''$ are two subdivisions corresponding to the boundaries from V', V'' respectively, then, obviously, $H(e) = P_2'P^{n_k}P_1''$ ($n_k \in \mathbf{Z}$). The claim is easily proven by multiplying together the values $H(E)$ for all the edges e taking part in the cycle c.

A generalized equation Ω is called *periodized* with respect to the periodic structure $\langle \mathcal{P}, R \rangle$ of this equation, if for every two cycles c_1 and c_2 in the graph Γ having the same initial vertex, the following equality holds in the group $F_{R(\Omega^*)}$:

$$[h(c_1), h(c_2)] = 1. \tag{12}$$

Let Γ_0 be the subgraph of the graph Γ having the same set of vertices and consisting of the edges e whose labels do not belong to $\mathcal{P}$. Choose a maximal subforest T_0 in the graph Γ_0 and extend it to a maximal subforest T of the graph Γ. Since $\langle \mathcal{P}, R \rangle$ is connected by assumption, it follows that T is a tree. Let V_0 be an arbitrary vertex of the graph Γ and $r(V_0, V)$ the (unique) path from V_0 to V all of whose vertices belong to T. For every edge $e : V \to V'$ not lying in T, introduce a cycle $c_e = r(V_0, V)e(r(V_0, V'))^{-1}$. Then (see the proof of Proposition 3.2.1 [**?**]) the fundamental group $\pi_1(\Gamma, V_0)$ is generated by the cycles c_e. This and the decidability of the universal theory of a free group imply that the property of a generalized equation "to be periodized with respect to a given periodic structure" is algorithmically decidable.

Furthermore, the set of elements

$$\{h(e) \mid e \in T\} \cup \{h(c_e) \mid e \notin T\} \tag{13}$$

forms a basis of the free group with the set of generators $\{h_k \mid h_k$ is an unknown lying on a closed section from $\mathcal{P}\}$. If $\mu \in \mathcal{P}$, then $(\beta(\mu)) = (\beta(\Delta(\mu)))$, $(\alpha(\mu)) = (\alpha(\Delta(\mu)))$ by part f) from the definition of a periodic structure and, consequently, the word $h[\alpha(\mu), \beta(\mu)]h[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]^{-1}$ is the label of a cycle $c'(\mu)$ from $\pi_1(\Gamma, (\alpha(\mu)))$. Let $c(\mu) \rightleftharpoons r(V_0, (\alpha(\mu)))c'(\mu)r(V_0, (\alpha(\mu)))^{-1}$. Then

$$h(c(\mu)) = uh[\alpha(\mu), \beta(\mu)]h[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]^{-1}u^{-1}, \tag{14}$$

where u is a certain word. Since $c(\mu) \in \pi_1(\Gamma, V_0)$, it follows that $c(\mu) = b_\mu(\{c_e \mid e \notin T\})$, where b_μ is a certain word in the indicated generators which can be effectively constructed on the basis of the proof of Proposition 3.2.1 [**?**].

Let $\tilde{b}_\mu$ denote the image of the word b_μ in the factor group of $\pi_1(\Gamma, V_0)$ over the derived subgroup. Denote by $\widetilde{Z}$ the free abelian group consisting of formal linear combinations $\sum_{e \notin T} n_e \tilde{c}_e$ ($n_e \in \mathbf{Z}$), and by $\widetilde{B}$ its subgroup generated by the elements $\tilde{b}_\mu$ ($\mu \in \mathcal{P}$) and the elements $\tilde{c}_e$ ($e \notin T,\ h(e) \notin \mathcal{P}$). Let

$\widetilde{A} = \widetilde{Z}/\widetilde{B}$, $T(\widetilde{A})$ the torsion subgroups of the group $\widetilde{A}$, and $\widetilde{Z}_1$ the preimage of $T(\widetilde{A})$ in $\widetilde{Z}$. The group $\widetilde{Z}/\widetilde{Z}_1$ is free; therefore, there exists a decomposition of the form

$$\widetilde{Z} = \widetilde{Z}_1 \oplus \widetilde{Z}_2, \ B \subseteq \widetilde{Z}_1, \ (\widetilde{Z}_1 : \widetilde{B}) < \infty. \tag{15}$$

Note that it is possible to express effectively a certain basis $\widetilde{\bar{c}}^{(1)}$, $\widetilde{\bar{c}}^{(2)}$ of the group $\widetilde{Z}$ in terms of the generators $\widetilde{c}_e$ so that for the subgroups $\widetilde{Z}_1$, $\widetilde{Z}_2$ generated by the sets $\widetilde{\bar{c}}^{(1)}$, $\widetilde{\bar{c}}^{(2)}$ respectively, relation (15) holds. For this it suffices, for instance, to look through the bases one by one, using the fact that under the condition $\widetilde{Z} = \widetilde{Z}_1 \oplus \widetilde{Z}_2$ the relations $\widetilde{B} \subseteq \widetilde{Z}_1$, $(\widetilde{Z}_1 : \widetilde{B}) < \infty$ hold if and only if the generators of the groups $\widetilde{B}$ and $\widetilde{Z}_1$ generate the same linear subspace over $\mathbf{Q}$, and the latter is easily verified algorithmically (a more economical algorithm can be constructed by analyzing the proof of the classification theorem for finitely generated abelian groups). By Proposition 1.4.4 [7], one can effectively construct a basis $\bar{c}^{(1)}$, $\bar{c}^{(2)}$ of the free (nonabelian) group $\pi_1(\Gamma, V_0)$ so that $\widetilde{\bar{c}}^{(1)}$, $\widetilde{\bar{c}}^{(2)}$ are the natural images of the elements $\bar{c}^{(1)}$, $\bar{c}^{(2)}$ in $\widetilde{Z}$.

A generalized equation Ω is called *singular* with respect to a connected periodic structure $\langle \mathcal{P}, R \rangle$, if at least one of the following three conditions holds:

a) Ω is not periodized with respect to $\langle \mathcal{P}, R \rangle$;

b) $\mathrm{rk}(A \otimes \mathbf{Q}) \geq 2$;

c) $\mathrm{rk}(A \otimes \mathbf{Q}) = 1$ and there exists $e \notin T$ such that $h(e) \notin \mathcal{P}$ and $h(c_e) \neq 1$ in the group $F_{R(\Omega^*)}$.

Otherwise, the equation Ω is called *regular*. Thus , Ω is regular with respect to $\langle \mathcal{P}, R \rangle$ if and only if Ω is periodized, $\mathrm{rk}(A \otimes \mathbf{Q}) \leq 1$, and in the case $\mathrm{rk}(A \otimes \mathbf{Q}) = 1$ for all $e \notin T$ such that $h(e) \notin \mathcal{P}$, we have $h(c_e) = 1$ in the group $F_{R(\Omega^*)}$. The definitions of singularity and regularity formally depend on the tree T, therefore we assume that T is fixed once and for all in an arbitrary way.

Now assume that $\langle \mathcal{P}, R \rangle$ is an arbitrary periodic structure of a generalized equation Ω, not necessarily connected. Let $\Gamma_1, \ldots, \Gamma_r$ be the connected components of the graph Γ constructed above. The labels of edges of the component Γ_i form in the equation Ω a union of closed sections from $\mathcal{P}$; moreover, if a base $\mu \in \mathcal{P}$ belongs to such a section, then its dual $\Delta(\mu)$, by item f) of the definition of a periodic structure, also possesses this property. Therefore, by taking for $\mathcal{P}_i$ the set of labels of edges from Γ_i belonging to $\mathcal{P}$, sections to which these labels belong, and bases $\mu \in \mathcal{P}$ belonging to these sections, and restricting in the corresponding way the relation R, we obtain a periodic connected structure $\langle \mathcal{P}_i, R_i \rangle$ with the graph Γ_i. A generalized equation Ω is called *singular* with respect to $\langle \mathcal{P}, R \rangle$ if it is singular with respect to at least one structure $\langle \mathcal{P}_i, R_i \rangle$ $(1 \leq i \leq r)$ and *regular* otherwise.

The notation $\langle \mathcal{P}', R' \rangle \subseteq \langle \mathcal{P}, R \rangle$ means that $\mathcal{P}' \subseteq \mathcal{P}$, and the relation R' is a restriction of the relation R. In particular, $\langle \mathcal{P}_i, R_i \rangle \subseteq \langle \mathcal{P}, R \rangle$ in the situation described above.

Lemma 13 *Let Ω be a nondegenerate generalized equation with no boundary connections, singular with respect to the periodic structure $< \mathcal{P}, \mathcal{R} >$. Then $F_{R(\Omega^*)}$ is isomorphic to $F_{R(S)}$ where S is such that the list of variables $\bar{x}$ of Ω is subdivided into two parts $\bar{y}$ and $\bar{z}$, and the list of equations in S is subdivided into two parts θ and ψ, such that $\bar{y}$ does not occur in the equations ψ, and θ has the following form.*

a) If Ω is singular of type a, then θ is empty.

b) If Ω is singular of type b, then

$$[y_1, y_2] = [y_1, U_i(\bar{z}, \bar{a})] = [y_2, U_i(\bar{z}, \bar{a})] = 1,\ i = 1, \dots k. \tag{16}$$

c) If Ω is singular of type c, then

$$[y_1, U_i(\bar{z}, \bar{a})] = 1, \quad i = 1, \dots k. \tag{17}$$

The group $F_{R(S)}$ is isomorphic to $F_{(\theta, R(\psi))}$. There is a finite family of cycles $c_1, \dots, c_r$ in the graph Γ such that $h(c_i) \neq 1$ $(1 \leq i \leq r)$ in the group $F_{R(\Omega^*)}$ and for any solution $\bar{H}$ of the equation Ω periodic with respect to some period P, such that $< \mathcal{P}, R >= \mathcal{P}(\bar{H}, P)$, there is an automorphic image $\bar{H}^+$ of $\bar{H}$ with respect to the group of automorphisms P_0 of $F_{R(S)}$ invariant on elements from $\bar{a}, \bar{z}$ $(\pi_{\bar{H}^+} = \pi_{\bar{H}}\sigma,\ \sigma \in P_0)$ such that there exists i $(1 \leq i \leq r)$ such that $\bar{H}^+(c_i) = 1$. In case a) $r = 1$. In case b) $r = 1$ and $y_1 = h(c_1)$.

In all cases every solution of the system ψ can be extended to a solution of the system $\theta \cup \psi$.

In other words, every solution $\bar{H}$ of Ω can be obtained as a composition of a solution of θ over a factor-group of $F_{R(\Omega^*)}$ over the normal subgroup generated by one of the $h(c_i)$ and a canonical homomorphism from this factor-group into F corresponding to a solution $\bar{H}^+$.

Proof We can restrict ourselves to the case of a connected graph Γ.

Consider 3 types of equations singular with respect to the periodic structure $< \mathcal{P}, R >$.

In case a) the list $\{c_i\}$ consists of some cycle $[c_{e_1}, c_{e_2}],\ e_1, e_2 \notin T$ which is not equal to the identity in $F_{R(\Omega^*)}$, and we put $\bar{H}^+ = \bar{H}$.

In case b): $\Omega_v = \Omega$ is periodized and $rk(A \bigotimes \mathbf{Q}) \geq 2$. Adding to the system Ω^* equations (12) for all pairs of cycles c_{e_1}, c_{e_2} $(e_1, e_2 \notin T)$, we have an equivalent system. Consider in the free group $F(\Omega^*)$ a new basis $\bar{a}, \bar{x}$ consisting of $\bar{a}$, variables not belonging to the closed sections from $\mathcal{P}$, variables $\{h(e) | e \in T\}$ and words $h(\bar{c}^{(1)}), h(\bar{c}^{(2)})$. Notice that $|\bar{c}^{(2)}| = rk(A \bigotimes \mathbf{Q}) \geq 2$. Let $y_1 = h(c_1^{(2)}),\ y_2 = h(c_2^{(2)})$, and the rest of the variables from the list $\bar{x}$ will be considered as variables from $\bar{z}$. All the equations of the system Ω^* can be rewritten modulo (12) in the variables $\bar{z}$ as a system $\bar{\psi}^{(0)}(\bar{z}, \bar{a}) = 1$.

The relations in (12) can be rewritten in the form

$$[h(c), h(c')] = [y_1, y_2] = [y_1, h(c)] = [y_2, h(c)] = 1,\ (c, c' \in \bar{c}^{(1)}, c_3^{(2)}, c_4^{(2)}, \dots, c_m^{(2)}). \tag{18}$$

The system $\bar{\phi}$, obtained as a union of equations (18) and equations from $\bar{\psi}^{(0)}$ is equivalent to Ω^* so there is a natural isomorphism between $F_{R(\Omega^*)}$ and $F_{R(\bar{\phi})}$.

If in these relations some $h(c)$ is a proper power, we can replace it by the corresponding root, get a new system $\bar{\phi}_1$, and by lemma 12 $R(\phi) = R(\bar{\phi}_1)$.

We assign equations from $\bar{\psi}^{(0)}$ and $[h(c), h(c')] = 1$ to the list $\bar{\psi}$ and the rest of the equations (18) to the list $\bar{\theta}$. We have a splitting of equations and can consider canonical group of automorphisms connected to this splitting.

The list $c_1, \dots, c_r$ consists of the one cycle $c_1^{(2)}$.

Every solution $\bar{H}$ of Ω can be obtained as a composition of a solution of θ over a factor-group $F_{R(\Omega^*)}$ over the normal subgroup generated by $h(c_1)$, and a canonical homomorphism from this factor-group into F corresponding to the solution $\bar{H}^+$.

If the equation Ω has at least one solution, then ψ also has a solution $\bar{Z}$. Take as Y_1 and Y_2 an arbitrary nontrivial word that commutes with components $\bar{C}^{(1)}, C_3^{(2)}, \ldots, C_m^{(2)}$ of the solution $\bar{Z}$ we will have a solution of system $\bar{\phi}$; this implies that $h(c_1^{(2)}) \neq 1$ in the group $F_{R(\Omega^*)}$.

Let solution $\bar{H}$ of generalized equation Ω be periodized with respect to the period P, and $< \mathcal{P}, R > \subseteq \mathcal{P}(\bar{H}, P)$. By lemma 12, $H(c_1^{(2)}) = Q^{n_1}, H(c_2^{(2)}) = Q^{n_2}$.

Applying the automorphism from the canonical group of automorphisms we can make $y_1 = 1$. This means that sending $y_1 = c_1^{(2)}$ into a trivial element we have a proper homomorphism from $F_{R(\Omega^*)}$ into the subgroup generated by the rest of the generators including y_2, and $F_{R(\Omega^*)}$ is the extension of a centralizer, since the subgroup generated by $h_(c), c \in \bar{c}^{(1)}, c_3^{(2)}, c_4^{(2)}, \ldots, c_m^{(2)}$ is maximal abelian in the group generated by $\bar{z}$.

Consider now case c). The system of equations is equivalent to some list $\bar{\psi}^{(0)}$ which does not contain the variable $w = h(c_1^{(2)})$ and has commutativity relations:

$$[w, h(c)] = 1 \ (c \in \bar{c}^{(1)}), \tag{19}$$

$$[h(c), h(c')] = 1 \ (c, c' \in \bar{c}^{(1)}). \tag{20}$$

These relations can be also rewritten in the form

$$u = h(c_{e_0}); w^{-1}uw = h(c_{e_0}); [u, h(c)] = 1 \ (c \in \bar{c}^{(1)}). \tag{21}$$

The group $F_{R(\Omega^*)}$ is isomorphic to the extension of a centralizer of a maximal abelian subgroup of the group generated by all the generators $\bar{a}, \bar{x}$ except w.

The epimorphism from $F_{R(\Omega^*)}$ to the subgroup generated by all the generators except w is proper.

As a list of cycles $c_1, \ldots c_r$ we can take $c_{e_0}^i (c_1^{(2)})^j$, where i, j run through the set of pairs of integers not simultaneously equal to zero and $|i|, |j| \leq 2\rho$ (ρ is the number of items in Ω).

Verify that if $h(c_{e_0})^i h(c_1^{(2)})^j = 1$ in $F_{R(\Omega^*)}$, then $i = j = 0$. Suppose $h(c_{e_0})^i h(c_1^{(2)})^j = 1$. Let σ_0 be a generator of the group of automorphisms P_0 such that $\sigma_0(h(c_{e_0})) = (h(c_{e_0}))$ and $\sigma_0(h(c_1^{(2)})) = (h(c_{e_0}))(h(c_1^{(2)}))$. Hence $(h(c_{e_0}))^{i+j} h(c_1^{(2)})^j = 1$ in $F_{R(\Omega^*)}$ and $(h(c_{e_0}))^j = 1$. This implies $(h(c_{e_0})) = 1$ (because $F_{R(\Omega^*)}$ is torsion-free) unless $j = 0$. But $(h(c_{e_0})) \neq 1$; hence $j = 0$. In the same way we get $i = 0$.

Let a solution $\bar{H}$ of the generalized equation Ω be periodic with respect to the period P and $\langle \mathcal{P}, R \rangle \subseteq \mathcal{P}(\bar{H}, P)$. Observe that $e_0 = r_1 c_{e_0} r_2$, where r_1 and r_2 are paths in the tree T. Since $e_0 \in \Gamma_0$, it follows that the initial vertex and the terminal vertex of the edge e_0 lie in the same connected component of the graph Γ_0 and, consequently, are connected by a path s in the forest T_0. Furthermore, r_1 and sr_2^{-1} are paths in the tree T connecting the same vertices; therefore, $r_1 = sr_2^{-1}$. Hence, $c_{e_0} = r_2 c'_{e_0} r_2^{-1}$, where c'_{e_0} is a certain cycle in the graph Γ_0.

From the equality $H(c_{e_0}) = H(r_2) H(c'_{e_0}) H(r_2)^{-1}$ it follows that the cyclically irreducible words $H(c_{e_0})$ and $H(c'_{e_0})$ are conjugate, and hence $d(H(c_{e_0})) = d(H(c'_{e_0})) \leq 2\rho d(P)$, since the cycle c'_{e_0} is primitive and for every unknown $h_k \notin \mathcal{P}$ the inequality $d(H_k) < 2d(P)$ is true by the definition of the structure $\mathcal{P}(\bar{H}, P)$.

Without loss of generality we may assume that $\delta(V_0) = \Lambda P$, where Λ is the empty word, so by Lemma 12, $H(c_{e_0}) = P^{n_0}$, and $H(C_1^{(2)}) = W = P^n$ ($|n_0| \leq 2\rho$).

If $n_0 = 0$, we can take $\sigma = 1$, $\bar{H}^+ = \bar{H}$ and the set of cycles $\{c_{e_0}\}$.

Let $n_0 \neq 0$, $n = tn_0 + n'$, and $|n'| \leq 2\rho$. Take as σ the power σ_0^t of the generator σ_0 and define the vector $\bar{H}^+$ by the formula $\pi_{\bar{H}^+} = \pi_{\bar{H}}\sigma$. If we take the cycle $c = (c_{e_0})^{-n'}(c_1^{(2)})^{n_0}$, then $H^+(c_{e_0}) = P^{n_0}$, $H^+(c_1^{(2)}) = P^{n'}$ and $H^+(c) = 1$. The proof of Lemma 13 is complete.

Lemma 14 *(lemma 2.11 from [**?**]) Let Ω be a consistent generalized equation without boundary connections, regular with respect to a periodic structure $\langle \mathcal{P}, R\rangle$. Then it is possible to effectively construct a group of automorphisms P_0 of the group $F_{R(\Omega^*)}$, which is a direct product of a finite number of canonical groups of automorphisms, so that the following condition is satisfied.*

Let $\bar{H}$ be a solution of the generalized equation Ω, periodic with respect to a period P, and $\langle \mathcal{P}, R\rangle = \mathcal{P}(\bar{H}, P)$. If the solution $\bar{H}$ is minimal with respect to the group of automorphisms P_0, then for every $h_k \in \mathcal{P}$ the inequality $d(H_k) \leq f_2(\Omega, \mathcal{P}, R) \cdot d(P)$ holds, where f_2 is a certain computable function.

Proof Let Γ be the graph corresponding to the periodic structure $\langle \mathcal{P}, R\rangle$, and $\Gamma_1, \ldots, \Gamma_r$ its connected components. Let $\langle \mathcal{P}_i, R_i\rangle$ be the corresponding connected periodic structures. If we were able to prove Lemma 14 for each of the structures $\langle \mathcal{P}_i, R_i\rangle$ and construct the required groups of automorphisms $P_1, \ldots, P_r$, then every solution minimal with respect to $P_0 =< P_1, \ldots, P_r >= P_1 \times \ldots \times P_r$ would also be minimal with respect to all the P_i, which would imply Lemma 14 for $\langle \mathcal{P}, R\rangle$. Therefore, it suffices to restrict ourselves to the case of a connected periodic structure.

Let $e_1, \ldots, e_m$ be all the edges of the graph Γ from $T \setminus T_0$. Since T_0 is the spanning forest of the graph Γ_0, it follows that $h(e_1), \ldots, h(e_m) \in \mathcal{P}$. Let us choose a basis $\bar{x}, \bar{a}$ in the same way as in the proof of the previous lemma and study in more detail how the unknowns $h(e_i)$ $(1 \leq i \leq m)$ can participate in the equations from Ω^* rewritten in this basis.

If h_k does not lie on a closed section from $\mathcal{P}$, or $h_k \in \mathcal{P}$, but $e \notin T$ (where $h(e) = h_k$), then h_k belongs to the basis $\bar{x}, \bar{a}$ and is distinct from each of $h(e_1), \ldots, h(e_m)$. Now let $h(e) = h_k$, $h_k \notin \mathcal{P}$ and $e \notin T$. Since $e \in \Gamma_0$, the vertices (k) and $(k+1)$ lie in the same connected component of the graph Γ_0, and hence are connected by a path s in the forest T_0. Furthermore, r_1 and sr_2^{-1} are paths in the tree T connecting the vertices (k) and V_0; consequently, $r_1 = sr_2^{-1}$. Thus, $e = sr_2^{-1}c_e r_2$ and $h_k = h(s)h(r_2)^{-1}h(c_e)h(r_2)$. The unknown $h(e_i)$ $(1 \leq i \leq m)$ can occur in the right-hand side of the expression obtained (written in the basis $\bar{x}, \bar{a}$) only in $h(r_2)$ and at most once. Moreover, the sign of this occurrence (if it exists) depends only on the orientation of the edge e_i with respect to the root V_0 of the tree T. If $r_2 = r_2' e_i^{\pm 1} r_2''$, then all the occurrences of the unknown $h(e_i)$ in the words h_k written in the basis $\bar{x}, \bar{a}$, with $h_k \notin \mathcal{P}$, are contained in the occurrences of words of the form $h(e_i)^{\mp 1} h((r_2')^{-1} c_e r_2') h(e_i)^{\pm 1}$, i.e., in occurrences of the form $h(e_i)^{\mp 1} h(c) h(e_i)^{\pm 1}$, where c is a certain cycle of the graph Γ starting at the initial vertex of the edge $e_i^{\pm 1}$. The system Ω^* is equivalent to the following system: we introduce new variables $\bar{u}^{(i)} = \{u_{ie} | e \notin T\}$, $\bar{z}^{(i)} = \{z_{ie} | 1 \leq i \leq m, e \notin T\}$ and add to Ω^* equations

$$u_{ie} = h(r(V_0, V_i)^{-1} c_e r(V_0, V_i)), \tag{22}$$

$$h(e_i)^{-1} u_{ie} h(e_i) = z_{ie}, \tag{23}$$

$$[u_{ie_1}, u_{ie_2}] = 1, \tag{24}$$

where e runs over the list of edges not belonging to T and i is fixed. Because $h(e_i)$ does not belong to the right part of (22), we can rewrite Ω^* in the form $\bar{\psi}^{(1)}(\bar{x}, \bar{z}^{(i)}, \bar{a}) = 1$, such that $h(e_i)$ does not occur in $\bar{\psi}^{(1)}$. Include now all the variables except $h(e_i)$ into the list $\bar{z}$ and also all the variables $\bar{u}^{(i)}$ except some fixed u_{ie_0}. Let $\bar{\psi}$ consist of the equations $\bar{\psi}^{(1)}$ and those equations (22), (24) which do not contain this u_{ie_0}. θ consists of (23) and the rest of (22), and (24).

We write $u = u_{ie_0}, w = h(e)$, $U_1 = h(r(V_0, V_i)^{-1} c_e r(V_0, V_i)), U_0 = z_{ie_0}$, and let pairs $< U, V >$ be pairs $< u_{ie}, z_{ie} >$ $(e \neq e_0)$.

Then we have a presentation

$$w^{-1}uw = U_0(\bar{z}, \bar{a}), u = U_1, \tag{25}$$

and several pairs

$$w^{-1}Uw = V, [u, U] = 1. \tag{26}$$

Canonical automorphisms have the form $u \to u, w \to u^r w$.

Now let $\bar{H}$ be a solution of the generalized equation Ω periodic with respect to some period P, $\langle \mathcal{P}, R \rangle$ a connected component of the structure $\mathcal{P}(\bar{H}, P)$, and let the solution $\bar{H}$ be minimal with respect to the group of automorphisms P_0. Without loss of generality, we can assume that $\delta(V_0) = \Lambda P$. Then, by Lemma 12, there is a homomorphism $\gamma : \widetilde{Z} \to \mathbf{Z}$ such that for every cycle $c \in \pi_1(\Gamma, V_0)$ the condition $H(c) = P^{\gamma(\tilde{c})}$ holds. Let us first verify that if for some variable $h_k \in \mathcal{P}$

$$d(H_k) \geq 2\rho^2 d(P), \tag{27}$$

then $\gamma(\tilde{\mathbf{Z}})$ contains a certain n such that $1 \leq n \leq 2\rho$ (ρ is the number of unknowns in the equation Ω).

To verify this, let us construct a chain

$$(\Omega, \bar{H}) = (\Omega_0, \bar{H}^{(0)}) \to (\Omega_1, \bar{H}^{(1)}) \to \cdots \to (\Omega_t, \bar{H}^{(t)}), \tag{28}$$

in which every term is obtained from the previous one by extending a certain boundary through a certain base $\mu \in \mathcal{P}$ with the help of the E.5 transformation. The construction of the chain (28) terminates when all boundaries intersecting bases from $\mathcal{P}$ turn out to be extended through these bases. Let Ω_i' be the equation obtained from Ω_i by deleting all boundary connections. It is obvious that the solution $\bar{H}^{(i)}$ of the equation Ω_i' is periodic with respect to the period P. Denote by $\langle \mathcal{P}_i, R_i \rangle$ the periodic structure $\mathcal{P}(\bar{H}^{(i)}, P)$ of the equation Ω_i' restricted to the closed sections of $\mathcal{P}$, and by $\Gamma^{(i)}$, $\widetilde{Z}^{(i)}$, γ_i the corresponding graph, abelian group of cycles and homomorphism $\widetilde{Z}^{(i)} \to \mathbf{Z}$, respectively.

If $\langle P, \mu, q \rangle$ $(\mu \in \mathcal{P})$ is a boundary connection of the equation Ω_i $(1 \leq i \leq t)$, then $\delta(p) = \delta(q)$; therefore, all the graphs $\Gamma^{(0)}, \Gamma^{(1)}, \ldots, \Gamma^{(t)}$ have the same set of vertices, whose cardinality does not exceed ρ. The solution $\bar{H}^{(t)}$ of the equation Ω_t is minimal with respect to the trivial group of automorphisms. Suppose that for some unknown h_l lying on a closed section from $\mathcal{P}$ the inequality $d(H_l^{(t)}) > 2d(P)$ holds. In the vector $\bar{H}^{(t)}$, replace all the components that are graphically equal to $(\bar{H}_l^{(t)})^{\pm 1}$ and correspond to the unknowns lying on the closed sections from $\mathcal{P}$, by a letter $u^{\pm 1}$ of the alphabet Σ_2 not participating in the solution $\bar{H}^{(t)}$. The resulting vector obviously satisfies the conditions of nonemptiness and irreducibility. It satisfies all basic equations of the generalized equation Ω_t with numbers $\mu \in \mathcal{P}$ and all the corresponding boundary equations, since in the equation Ω_t all the boundaries from $\mathcal{P}$ are extended through all possible boundaries. If, on the other hand, $\mu \notin \mathcal{P}$, then for every unknown $h_k \in \mu$ of the equation Ω lying on a closed section from $\mathcal{P}$, we have $h_k \notin \mathcal{P}$ and, consequently, $d(H_k) \leq 2d(P)$. In particular, this inequality holds for the unknowns $h_k \in \mu$ of the equation Ω_t; therefore, such unknowns have not been replaced in the vector $\bar{H}^{(t)}$. Consequently, the vector constructed is a solution to the equation Ω_t, which contradicts the minimality of the solution $\bar{H}^{(t)}$.

Thus we have established the fact that $d(H_l^{(t)}) \leq 2d(P)$, if h_l lies on a closed section from $\mathcal{P}$. In particular, the unknown h_k of the equation Ω for which inequality (27) holds was divided during the transition to the equation Ω_t into at least ρ distinct unknowns. Since the graph $\Gamma^{(t)}$ contains at most ρ vertices, in the equation Ω_t' we can choose boundaries l and l' such that $l < l'$, $(l) = (l')$, and $l' - l \leq \rho$. The word $h[l, l']$ is a label of a cycle c_t of the graph $\Gamma^{(t)}$ for which $0 < d(H(c_t)) \leq 2\rho d(P)$, i.e., $\gamma_t(\widetilde{Z}^{(t)})$ contains a number n with the property $1 \leq n \leq 2\rho$. By π_{ij} $(0 \leq i < j \leq t)$ we denote from now on the homomorphism $G(\Omega_i^*) \to G(\Omega_j^*)$ defined by the sequence (28). It remains to prove the existence of a cycle c_0 of the graph Γ for which $\pi_{0t}(h(c_0)) = h(c_t)$.

To do this, it suffices to show that for every path $r_{i+1} : V \to V'$ in the graph $\Gamma^{(i+1)}$ there exists a path $r_i : V \to V'$ in the graph $\Gamma^{(i)}$ such that $\pi_{i,i+1}(h(r_i)) = h(r_{i+1})$. In turn, it suffices to verify the latter statement for the case where r_{i+1} is the edge e. If the unknown $h(e)$ of the equation Ω_{i+1} is also an unknown of the equation Ω_i, then this is obvious. Otherwise one should use the formulas

$$\pi_{i,i+1}^{-1}(h') = h[\alpha(\Delta(\mu)), q]^{-1} h[\alpha(\mu), p],$$
$$\pi_{i,i+1}^{-1}(h'') = h[\alpha(\mu), p]^{-1} h[\alpha(\Delta(\mu)), q+1],$$

defining the inverse isomorphism to $\pi_{i,i+1}$, and notice that the right-hand sides of these formulas are labels of paths in $\Gamma^{(i)}$, since $(\alpha(\mu)) = (\alpha(\Delta(\mu)))$.

Thus, we have deduced from (27) that $\gamma(\widetilde{Z})$ is a nonzero subgroup in $\mathbf{Z}$ whose generator n_0 satisfies the inequality $|n_0| \leq 2\rho$. Assume first that, let $\mathrm{rk}(A \otimes \mathbf{Q}) = 1$. Then the regularity of the equation Ω implies that for all $e \notin T$ with $h(e) \notin \mathcal{P}$ we have $H(c_e) = 1$, i.e., $\gamma(\tilde{c}_e) = 0$. Since $\bar{H}$ is a solution, it follows that $\gamma(\tilde{b}_\mu) = 0$ $(\mu \in \mathcal{P})$. Therefore, $\gamma(\widetilde{B}) = 0$. By (2.54) we obtain that $\gamma(\widetilde{Z}_1) = 0$ and, consequently, $\gamma(\widetilde{Z})$ is generated by the single element $\gamma(\tilde{c}_1^{(2)})$. Therefore, $|\gamma(\tilde{c}_1^{(2)})| \leq 2\rho$. By the representation (15), one can effectively construct an expression $\tilde{c}_e = n_e \tilde{c}_1^{(2)} + \tilde{z}_e^{(1)}$ $(\tilde{z}_e^{(1)} \in \widetilde{Z}_1)$ of the elements $\tilde{c}_e$ $(e \notin T)$ in terms of the basis elements. Hence $|\gamma(\tilde{c}_e)| = |n_e \gamma(\tilde{c}_1^{(2)})| \leq 2\rho n_e$, and we finally obtain

$$|\gamma(\tilde{c}_e)| \leq g_1(\Omega, \mathcal{P}, R), \tag{29}$$

where g_1 is a computable function.

Now let us analyze the case $\mathrm{rk}(A \otimes \mathbf{Q}) = 0$, i.e., $\widetilde{Z} = \widetilde{Z}_1$. As we have already seen in the proof of Lemma 13, the cycle c_e $(e \notin T,\ h(e) \notin \mathcal{P})$ is conjugate to a certain cycle of the graph Γ_0, and $d(H(c_e)) \leq 2\rho d(P)$. Hence, $|\gamma(\tilde{c}_e)| \leq 2\rho$ for $h(e) \notin \mathcal{P}$. Because $(\tilde{Z} : \tilde{B}) < \infty$, for every $e_0 \notin T$ one can effectively construct a valid equality of the form $n_{e_0} \tilde{c}_{e_0} = \sum_{h(e) \notin \mathcal{P}} n_e \tilde{c}_e + \sum_{\mu \in P} n_\mu \tilde{b}_\mu$, which implies

$$|\gamma(\tilde{c}_{e_0})| \leq |\gamma(n_{e_0} \tilde{c}_{e_0})| \leq \sum_{h(e) \notin \mathcal{P}} |n_e \gamma(\tilde{c}_e)| \leq 2\rho \cdot \sum_{h(e) \notin \mathcal{P}} |n_e|.$$

Thus, in this case we have also demonstrated the estimate (29) for a certain computable function g_3.

Let $\delta((k)) = P_1^{(k)} P_2^{(k)}$. Denote by $t(c, h_k)$ the number of occurrences of the edge with label h_k in the cycle c, calculated taking into account the orientation. Finally, let

$$H_k = P_2^{(k)} P^{n_k} P_1^{(k+1)} \tag{30}$$

(h_k lies on a closed section from $\mathcal{P}$), where the equality in (30) is graphic whenever $h_k \in \mathcal{P}$. Direct calculations show that

$$H(c) = P^{\sum_k t(c,h_k)(n_k+1)}. \tag{31}$$

Since $\gamma(\widetilde{Z}) \neq 0$, $e_0 \notin T$ can be chosen in such a way that $\gamma(\tilde{c}_{e_0}) \neq 0$. Let $n_k = |\gamma(\tilde{c}_{e_0})| m_k + r_k$, where $0 < r_k \leq |\gamma(\tilde{c}_{e_0})|$. Equation (31) implies that the vector $\{m_k | h_k \in \mathcal{P}\}$ is a solution to the following system of Diophantine equations in variables $\{z_k | h_k \in \mathcal{P}\}$:

$$\sum_{h_k \in \mathcal{P}} t(c_e, h_k)(|\gamma(\tilde{c}_{e_0})| z_k + r_k + 1) + \sum_{h_k \notin \mathcal{P}} t(c_e, h_k)(n_k + 1) = \gamma(\tilde{c}_e) \ (e \notin T). \tag{32}$$

Note that the number of unknowns and coefficients of the system (32) are bounded from above (this follows from (29), the simplicity of the cycles c_e and the inequality $|n_k| \leq 2\rho$ ($h_k \notin \mathcal{P}$)) by a certain computable function of Ω, $\mathcal{P}$, and R.

A solution $\{m_k\}$ of a system of linear Diophantine equations is called *minimal* [**?**], if $m_k \geq 0$ and there is no other solution $\{m_k^+\}$ such that $0 \leq m_k^+ \leq m_k$ for all k, and that at least one of the inequalities $m_k^+ \leq m_k$ is strict. Let us verify that the solution $\{m_k | h_k \in \mathcal{P}\}$ of the system (32) is minimal.

Indeed, let $\{m_k^+\}$ be another solution to the system (32) such that $0 \leq m_k^+ \leq m_k$ for all k, and at least for one k the inequality is strict. Let $n_k^+ = |\gamma(\tilde{c}_{e_0})| m_k^+ + r_k$. Form a vector $\bar{H}^+$, putting $H_k^+ = H_k$ if $h_k \notin \mathcal{P}$, and $H_k^+ = P_2^{(k)} P^{n_k^+} P_1^{(k+1)}$ if $h_k \in \mathcal{P}$. Since the words H_k^+ and H_k start (and end) with the same letter, it follows that

$$T(\bar{H}^+) = T(\bar{H}). \tag{33}$$

Obviously, the vector $\bar{H}^+$ satisfies all the coefficient equations and the basic equations with numbers $\mu \notin \mathcal{P}$. Since $\{m_k^+\}$ is a solution of the system (32), $H^+(c_e) = P^{\gamma(\tilde{c}_e)} = H(c_e)$. Therefore, for every cycle c we have $H^+(c) = H(c)$ and, in particular, $H^+(b_\mu) = H(b_\mu) = 1$. Thus the vector $\bar{H}^+$ is a solution of the system Ω^*.

The vector $\bar{H}^+$ satisfies the condition of nonemptiness, and by (33) it also satisfies the condition of irreducibility. Since for every μ it is true that

$$H^+[\alpha(\mu), \beta(\mu)] H^+[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]^{-1} = 1,$$

and the words $H^+[\alpha(\mu), \beta(\mu)]$, $H^+ \ [\alpha(\Delta(\mu)), \beta(\Delta(\mu))]$ are irreducible; it follows that $H^+[\alpha(\mu), \beta(\mu)] = H^+[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]$. Thus, $\bar{H}^+$ is a solution to the generalized equation Ω.

Denote by δ_{ie_0} the generator of the group of automorphisms P_{ie_0} constructed above. In the basis $\bar{x}, \bar{a}$ the map δ_{ie_0} acts in the following way: $\delta_{ie_0} : h(e_i) \mapsto h(r(V_0, V_i)^{-1} c_{e_0} r(V_0, V_i)) h(e_i)$ (the other unknowns remain unchanged). Therefore, if $\pi_{\bar{H}'} = \pi_{\bar{H}} \delta_{ie_0}$ and $h(e_i) = h_k \in \mathcal{P}$, then $H'_k = P_2^{(k)} P^{n_k + \gamma(\tilde{e}_0)} P_1^{(k+1)}$, and all the other components of $\bar{H}'$ (in the basis $\bar{x}, \bar{a}$) are the same as in $\bar{H}$. Denote $\delta = \prod_{i=1}^{m} \delta_{ie_0}^{\Delta_i}$, where $h(e_i) = h_{k_i}$, $\Delta_i = (m_{k_i}^+ - m_{k_i}) \cdot \mathrm{sgn}(\gamma(\tilde{e}_0))$. Let us verify the equality

$$\pi_{\bar{H}^+} = \pi_{\bar{H}} \delta. \tag{34}$$

Let $\pi_{\bar{H}} \delta = \pi_{\bar{H}^{(1)}}$. Then, by construction, $H_k^{(1)} = P_2^{(k)} P^{m_k^+} P_1^{(k+1)} = H_k^+$ for all h_k that are labels of edges from $T \setminus T_0$. If the edge with label h_k lies in T_0, or h_k does not lie on a closed section from

$\mathcal{P}$, then $h_k \notin \mathcal{P}$ and $H_k^{(1)} = H_k = H_k^+$. Finally, note that for every $e \notin T$ $H^{(1)}(c_e) = H(c_e) = H^+(c_e)$. Since $cx_e = r_1 e r_2$, where r_1, r_2 are paths in the tree T, and for every unknown h_k which is a label of an edge from T, the equality $H_k^{(1)} = H_k^+$ has already been established, it follows that $H^{(1)}(e) = H^+(e)$. This proves (34).

From (33) and (34) it follows that $\bar{H}^+ \leq_{P_0} \bar{H}$, which contradicts the minimality of the solution $\bar{H}$ with respect to the group P_0. Consequently, the solution $\{m_k | h_k \in \mathcal{P}\}$ of the system of linear Diophantine equations (32) is minimal.

Lemma 1.1 from [?] states that the components of the minimal solution $\{m_k | h_k \in \mathcal{P}\}$ can be bounded from above by a recursive function depending on the parameters of the system. Since the parameters of the system (32), as was mentioned earlier, are bounded from above by a computable function depending on Ω, $\mathcal{P}$ and R, we have the estimate $m_k \leq g_2(\Omega, \mathcal{P}, R)$. The conclusion of Lemma 14 holds if we put

$$f_2(\Omega, \mathcal{P}, R) \rightleftharpoons g_2(\Omega, \mathcal{P}, R)(2\rho + 1).$$

7. Construction of $T_0(\Omega)$

We assign to some vertices v of the tree $T(\Omega)$ the groups of automorphisms of groups $F_{R(\Omega_v^*)}$. We also assign for some paths $v \to w$ homomorphisms from $F_{R(\Omega_v^*)}$ into $F^*_{R(s_1 \cup s_2)}$, where s_1 is some system of equations over $F_{R(\Omega_w^*)}$ with a solution in $F_{R(\Omega_w^*)}$ and $s_2 = \Omega_w$.

For each vertex v such that $tp(v) = 14$, s_1 and s_2 are defined as in Case 14.

For each vertex v such that $7 \leq tp(v) \leq 10$ we assign the group of automorphisms invariant with respect to the kernel; in this case s_1 is an empty system over $F_{R(\Omega_w^*)}$.

For each vertex v such that $tp(v) = 15$ and the transformation of case 14 is not applicable (because it gives an isomorphism $\pi : F_{R(\Omega_v^*)} \to F_{R(s_2^*)}$) systems s_1 and s_2 are those that are defined in the description of case 14. Take the group P described in case 14 as the group of automorphisms of $F_{R(\Omega_v^*)}$ assigned to v.

Let $tp(v) = 2$. Equation Ω_v will be called nontrivial, if it has a closed section containing at least one base and not containing variables from the coefficient equations. From the construction it follows that Ω_v is nontrivial if and only if the path from v_0 to v contains an auxiliary edge, corresponding to the case 15.1 If all the auxiliary edges correspond to the cases 4,5, then the equation Ω_v is trivial.

For each vertex v such that $tp(v) = 2$, assign a group generated by the groups of automorphisms constructed in Lemma 14 that applied to Ω_v and all possible periodic structures of this equation with respect to which $\tilde{\Omega}_v$ is regular. For each periodic structure $< \mathcal{P}_i, R_i >$ there is a natural embedding of the group $F_{R(\Omega_v^*)}$ into a free extension of a centralizer of the element $U_1 \in F_{R(\Omega_v^*)}$ from equation (25) by a letter t, sending w into tw_0, where w_0 is a minimal solution of Ω_v with respect to the automorphism group P_i of $F_{R(\Omega_v^*)}$ from lemma 14.

Lemma 15 *[Lemma 3.3 from [?]] Let $v_1 \to v_2 \to \ldots \to v_k \to \ldots$ be an infinite path in the tree $T(\Omega)$, and $7 \leq tp(v_k) \leq 10$ for all k. Then among $\{\Omega_k\}$ some generalized equation occurs infinitely many times. If $\Omega_{v_k} = \Omega_{v_l}$, then $\pi(v_k, v_l)$ is an isomorphism invariant with respect to the kernel. .*

Proof By Lemma 10, $\tau'_k \leq \tau'_1$ and $\nu'_k \leq \nu'_1$ for all k. Hence, we can suppose $\tau'_k = \tau'_1$ and $\nu'_k = \nu'_1$ for all k. These equalities imply that all the transformations $E5$ introduce a new boundary.

For all k, $Ker(\tilde{\Omega}_{v_k})$ have the same bases. Indeed, consider equations $\tilde{\Omega}_{v_k}$ and $\tilde{\Omega}_{v_{k+1}}$. Because we do not have cases 3,4, the working part of $\tilde{\Omega}_{v_k}$ does not contain coefficient equations.

If $tp(v_k) = 7, 8, 10$, then $\tilde{\Omega}_{v_{k+1}}$ can be obtained from $\tilde{\Omega}_{v_k}$ by cutting some μ eliminable in $\tilde{\Omega}_{v_k}$ and then by deletion of one of the new bases (which is also eliminable by item a) in the definition of the eliminable base. But the rest of the base μ will also be eliminable by item b). So the set of bases from the kernel does not change.

Let $tp(v_k) = 9$. By similar reasoning one can show that all the bases of $\tilde{\Omega}_{v_{k+1}}$ obtained from μ_2 by cutting do not belong to the kernel. If we cut bases μ_1 and μ_2 in all boundaries that are continued in the equation Ω_{v_k} through both these bases, then we can suppose that the section $[j_1, j_2]$ does not contain closed boundaries of the equation $\tilde{\Omega}_{v_k}$; hence is closed in this equation. Construct some sequence (7) for the equation $\tilde{\Omega}_{v_k}$ and take the first equation Ω_i, such that one of the bases obtained by cutting from $\mu_1, \mu_2, \Delta\mu_1, \Delta\mu_2$ is eliminated in this equation. Denote it by ν. This base ν cannot be obtained from μ_1, μ_2. In addition, if ν is eliminable in the equation Ω_i using item b), then either $\alpha(\nu) \in \{\alpha(\Delta(\mu_1)), \alpha(\Delta(\mu_2))\}$ or $\beta(\nu) \in \{\beta(\Delta(\mu_1)), \beta(\Delta(\mu_2))\}$. We can start the construction of the sequence (7) for $\tilde{\Omega}_{v_{k+1}}$ by deletion of the same first i bases as was done for $\tilde{\Omega}_{v_k}$. Then some base ν' obtained by cutting from μ_2,$\Delta(\mu_2)$ of the equation $\Omega_{v_{k+1}}$ will become eliminable. But after deletion of ν' one can subsequently delete all the other bases obtained from μ_2 by cutting, using item b) of the definition since all the boundaries, touching these bases (except $\alpha(\mu_2), \beta(\mu 2)$) were not continued through $\Delta(\mu_1)$ in the equation Ω_{v_k}, and hence do not touch any other bases of $\tilde{\Omega}_{v_{k+1}}$. So, all the bases of equation $\tilde{\Omega}_{v_{k+1}}$, obtained from μ_2 do not belong to the kernel.

We have shown, that the number of bases is the same in all the equations $Ker(\tilde{\Omega}_{v_k})$. We denote this number by n''. We will now prove the inequality

$$n'_k \leq 3\tau' + 6n'' + 1. \tag{35}$$

Indeed, if k is the first number for which it fails, then

$$n'_{k-1} \leq 3\tau' + 6n'' + 1, n'_k > 3\tau' + 6n'' + 1. \tag{36}$$

By Lemma 10, $tp(v_{k-1}) = 10$. Hence we cannot apply transformations of Cases 5-9 to the equation $\Omega_{v_{k-1}}$. Hence every working section of $\Omega_{v_{k-1}}$ either contains at least three bases or contains some base of the equation $Ker(\tilde{\Omega}_{v_{k-1}})$. Hence $u'_{k-1} + w'_{k-1} \leq \frac{1}{3}n'_{k-1} + n''$ and by (10) $\tau' = n'_{k-1} - 2w'_{k-1} - u'_{k-1} \geq \frac{1}{3}n'_{k-1} - 2n''$, which contradicts (36). Now (10) and (35) imply that $u'_k + w'_k \leq n'_k \leq 3\tau' + 6n' + 1$ and $\rho'_k \leq \nu'_k + u'_k + w'_k + 1 \leq 3\tau' + 6n'' + \nu' + 2$. Hence the set $\{\Omega_{v_k} | k \in \mathbf{N}\}$ is finite and some generalized equation occurs in this set infinitely many times.

Let now $\Omega_{v_k} = \Omega_{v_l}$. $Ker(\tilde{\Omega}_{v_{i+1}})$ is obtained from $Ker(\tilde{\Omega}_{v_i})$ by cutting some variables and deletion of some variables not belonging to $Ker(\tilde{\Omega}_{v_i})$. So the number of variables belonging to the bases and coefficient equations of $Ker(\tilde{\Omega})$ can only increase, but $\Omega_{v_k} = \Omega_{v_l}$; hence this number is the same for all the vertices $v_k, v_{k+1}, \ldots, v_l$. Thus $\pi(v_k, v_l)(h_i) = h_i$ for any such variable. $\Box$

Let the tree $T_1(\Omega)$ be obtained from $T(\Omega)$ by replacing the infinite path in $T(\Omega)$ corresponding to the case $7 \leq tp(v_k) \leq 10$ by a finite initial subpath r such that every generalized equation with ρ variables in the set $\{\Omega_{v_k}\}$ occurs in r not more than once. For each vertex v in r assign an extra edge $v \to w$, where $\Omega_w = (\Omega_v)_l$ is the kernel of Ω_v (see Lemma 7). Then for w we have Case 1.

Introduce the new parameter

$$\tau''_v = \tau'_v + \rho - \rho''_v,$$

where ρ is the number of variables in the initial equation Ω, ρ''_v the number of variables belonging to the constant sections of the equation Ω_v. We have $\rho''_v \leq \rho$, hence $\tau''_v \geq 0$. In addition if $v_1 \to v_2$ is an auxiliary edge, then $\tau''_2 < \tau''_1$.

Define by the joint induction on τ''_v a finite subtree $T_0(\Omega_v)$ and a natural number $s(\Omega_v)$. The tree $T_0(\Omega_v)$ will have v as a root and consist of some vertices and edges of $T_1(\Omega)$ that lie higher than v. Let τ''_v=0; then in $T_1(\Omega)$ there can not be auxiliary edges and vertices of type 15 higher than v. Hence a subtree $T_0(\Omega_v)$ consisting of vertices of $T_1(\Omega_v)$ that are higher than v is finite.

Let now

$$s(\Omega_v) = max_w max_{<\mathcal{P},R>} \rho_w f_2(\Omega_w, \mathcal{P}, R), \tag{37}$$

where w runs through all the vertices of $T_0(v)$ for which $tp(w) = 2$ and Ω_w is nontrivial, $< \mathcal{P}, R >$ is the set of periodic structures of the equation $\tilde{\Omega}_w$, such that $\tilde{\Omega}_w$ is regular with respect to $< \mathcal{P}, R >$ and f_2 is a function appearing in Lemma 14.

Suppose now that $\tau''_v > 0$ and that for all v_1 with $\tau''_{v_1} < \tau''_v$ the tree $T_0(\Omega_{v_1})$ and the number $s(\Omega_{v_1})$ are already defined. We begin with the consideration of the paths

$$r = v_1 \to v_2 \to \ldots \to v_m, \tag{38}$$

where $tp(v_i) = 15$ $(1 \leq i \leq m)$. We have $\tau''_{v_i} = \tau''_v$.

Denote by μ_i the carrier base of the equation Ω_{v_i}. The path (38) will be called μ-reducing if $\mu_1 = \mu$ and either there are no auxiliary edges from the vertex v_2 and μ occurs in the sequence $\mu_1, \ldots, \mu_{m-1}$ at least twice, or there are auxiliary edges $v_2 \to w_1, v_2 \to w_2 \ldots, v_2 \to w_k$ from v_2 and μ occurs in the sequence $\mu_1, \ldots, \mu_{m-1}$ at least $max_{1 \leq i \leq k} s(\Omega_{w_i})$ times.

The path (38) will be called prohibited, if it can be represented in the form

$$r = r_1 s_1 \ldots r_l s_l r', \tag{39}$$

such that for some sequence of bases $\eta_1, \ldots, \eta_l$ the following three properties hold:

1. every base occuring at least once in the sequence $\mu_1, \ldots, \mu_{m-1}$ occurs at least $40n^3 + 20n + 1$ times in the sequence $\eta_1, \ldots, \eta_l$, where n is the number of pairs of bases in Ω_{v_i};

2. the path r_i is η_i-reducing;

3. every transfer base of some equation of path r is a transfer base of some equation of path r'.

The property of path (38) of being prohibited is algorithmically decidable. Every infinite path (38) contains a prohibited subpath. Indeed, let ω be the set of all bases occuring in the sequence $\mu_1, \ldots, \mu_m, \ldots$ infinitely many times, and $\tilde{\omega}$ the set of all bases, that are carrier bases of infinitely many equations Ω_{v_i}. If one cuts out some finite part in the beginning of this infinite path, one can suppose that all the bases in the sequence $\mu_1, \ldots, \mu_m, \ldots$ belong to ω and each base that is a carrier base of at least one equation, belongs to $\tilde{\mu}$. Such infinite path for any $\mu \in \omega$ contains infinitely many non-intersecting μ-reducing finite subpaths. Hence it is possible to construct a subpath (39) of this path, satisfying the first two conditions in the definition of a prohibited subpath. Making r' longer one obtains a prohibited subpath.

Let $T_2(\Omega_v)$ be a subtree of $T_1(\Omega_v)$ consisting of the vertices v_1 for which the path from v to v_1 in $T(\Omega)$ contains neither prohibited subpaths nor vertices v_2 with $\tau''_{v_2} < \tau''_v$, except perhaps v_1. So the terminal vertices of $T_2(\Omega_v)$ are either vertices v_1 such that $\tau''_{v_1} < \tau''_v$, or terminal vertices of $T_1(\Omega_v)$. A subtree $T_2(\Omega_v)$ can be effectively constructed. $T_0(\Omega_v)$ is obtained by attaching of $T_0(\Omega_{v_1})$ (already constructed by the induction hypothesis) to those terminal vertices v_1 of $T_2(\Omega_v)$ for which $\tau''_{v_1} < \tau''_v$. The function $s(\Omega_v)$ is defined by (37). Let now $T_0(\Omega) = T_0(\Omega_{v_0})$.

Notice, that if $tp(v) \geq 6$ and $v \to w_1, \ldots, v \to w_m$ is the list of principal outgoing edges from v, then the generalized equations $\Omega_{w_1}, \ldots, \Omega_{w_m}$ are obtained from Ω_v by the application of several elementary transformations. Denote by e a function that assigns a pair $(\Omega_{w_i}, \bar{H}^{(i)})$ to the pair $(\Omega_v, \bar{H})$. For $tp(v) = 4, 5$ this function is identical.

If $tp(v) = 15$ and there are auxiliary edges from the vertex v, then the carrier base μ of the equation Ω_v intersects $\Delta(\mu)$. For any solution $\bar{H}$ of the equation Ω_v one can construct a solution $\bar{H}'$ of the equation $\Omega_{v'}$ by $H'_{\rho_v+1} = H[1, \beta(\Delta(\mu))]$. Let $e'(\Omega_v, \bar{H}) = e(\Omega_{v'}, \bar{H}')$.

Let $\bar{H}$ be a solution of the equation Ω with quadratic part $[1, j+1]$. Define the following numbers.

$$d_1(\bar{H}) = \Sigma_{i=1}^{j} d(H_i), \tag{40}$$

$$d_2(\bar{H}) = \Sigma_\mu d(H[\alpha(\mu), \beta(\mu)]), \tag{41}$$

where μ is a constant base.

Lemma 16 *If in case 14 $\pi : F_{R(\Omega_v^*)} \to F_{R(s_2^*)}$ is an isomorphism, then for any solution $\bar{H}$ of Ω_v there is another solution $\bar{H}^+$, which is an automorphic image of $\bar{H}$ with respect to the canonical group of automorphisms defined in the beginning of this section, such that*

$$d_1(\bar{H}^+) \leq n' d_2(\bar{H}^+).$$

Proof If π is an isomorphism, then every base (except one constant base) in the quadratic part can be transfered to the nonquadratic working part with the use of some constant base as a carrier base. Thismeans that the length of the transfered base is equal to the length of the part of the constant carrier base, which will then be deleted. □

In the beginning of this section we assigned to some vertices of $T(\Omega)$ the groups of automorphisms P_v. Denote by P the group of automorphisms of $F_{R(\Omega^*)}$, generated by all groups $\pi(v_0, v)^{-1} P_v \pi(v_0, v)$, $v \in T_0(\Omega)$. (Here $\pi(v_0, v)$ is an isomorphism, because $tp(v) \neq 1$.) In the proof of the following lemma a minimal solution of the equation Ω_v means a solution minimal with respect to the group $\pi(v_0, v) P \pi(v_0, v)^{-1}$.

Lemma 17 *For any solution $\bar{H}$ of a generalized equation Ω there exists a path $v_0 \to v_1 \to \ldots v_n = w$ into a terminal vertex w of the tree $T_0(\Omega)$ having type 1 or 2, and a solution $\bar{H}^{(w)}$ of a generalized equation Ω_w such that*

1. *$\pi_{\bar{H}} = \pi \pi_{\bar{H}^{(w)}} \sigma_n \ldots \sigma_1 \pi(v_0, v_1) \sigma_0$, where π is an endomorphism of a free group and σ_i is an automorphism in the canonical group of automorphisms of $F_{R(\Omega_{v_i}^*)}$.*

2. *if $tp(w) = 2$ and the equation Ω_w is nontrivial, then there exists a primitive cyclically reduced word P such that $\bar{H}^w$ is periodic with respect to P and the equation Ω_w is singular with respect to the periodic structure $P(\bar{H}^w, P)$.*

3. *Let s_{1i} be a linear or quadratic equation corresponding to the edge $v_i \to v_{i+1}$ described in the beginning of section 6, and $F_{R(\Omega^*_{v_{i+1}})} \simeq F_{R(s_{2i})}$, then there is a natural homomorphism $\gamma_w : F_{R(\Omega^*)} \to F_{R(s_{10} \cup s_{11} \cup \ldots s_{1(n-1)} \cup s_{2(n-1)})^*}$. If F_w is the factor-group of $F_{R(\Omega^*)}$ over the intersection of the kernels of all the homomorphisms from $F_{R(\Omega^*)}$ into $F_{R(\Omega^*_w)}$ corresponding to the path in $T_0(\Omega)$ from v_0 to w, then the induced homomorphism $\bar{\gamma} : F_w \to F_{R(s_{10} \cup s_{11} \cup \ldots s_{1(n-1)} \cup s_{2(n-1)})^*}$ is a monomorphism.*

Proof

Construct a sequence

$$(\Omega, \bar{H}) = (\Omega_{v_0}, \bar{H}^{(0)}) \to (\Omega_{v_1}, \bar{H}^{(1)}) \to \ldots \to (\Omega_{v_u}, \bar{H}^{(u)}) \to \ldots \tag{42}$$

in which the v_i are the vertices of the tree $T(\Omega)$ in the following way. Let $v_1 = v_0$ and let $\bar{H}^{(1)}$ be some minimal solution of the equation Ω with the property $\bar{H} \geq \bar{H}^{(1)}$. If $tp(v_i) = 15$ and there are auxiliary edges from vertex v_i: $v_i \to w_1, \ldots, v_i \to w_m$ (the carrier base μ intersects with its double $\Delta(\mu)$) and there exists a primitive word P such that

$$H^{(i)}[1, \beta(\Delta(\mu))] \equiv P^r P_1, P \equiv P_1 P_2, r \geq max_{1 \leq j \leq m} s(\Omega_{w_j}) \tag{43}$$

(note that in such a case $\tilde{\Omega}_{v_i}$ is not regular with respect to a periodic structure $< \mathcal{P}, R >= \mathcal{P}(\bar{H}^{(i)}, P)$), then we set $(\Omega_{v_{i+1}}, \bar{H}^{(i+1)}) = e'(\Omega_{v_i}, \bar{H}^{(i)})$. In all of the other cases we set $(\Omega_{v_{i+1}}, \bar{H}^{(i+1)}) = e(\Omega_{v_i}, \bar{H}^{(i)})$ and $\bar{H}^{(i+1)}$ is a minimal solution of $\Omega_{v_{i+1}}$ with respect to the canonical group of automorphisms assigned to v_{i+1} . The sequence (42) ends if $tp(v_i) \leq 2$.

We will show that in the sequence (42) $v_i \in T_0(\Omega)$. It can be proved by induction on $q - p$ that for $p < q$ solutions $\bar{H}^{(p)}$ and $\bar{H}^{(q)}$ in the sequence (42) are connected by the equation

$$\pi_{\bar{H}^{(p)}} = \pi \pi_{\bar{H}^{(q)}} \sigma_q \pi(v_{q-1}, v_q) \sigma_{q-2} \ldots \pi(v_p, v_{p+1}) \sigma_p. \tag{44}$$

Suppose $v_i \notin T_0(\Omega)$, and let i_0 be the first of such numbers. It follows from the construction of $T_0(\Omega)$ that there exists $i_1 < i_0$ such that the path from v_{i_1} into v_{i_0} contains a subpath prohibited in the construction of $T_2(\Omega_{v_{i_1}})$. From the minimality of i_0 it follows that this subpath goes from v_{i_2} $(i_1 \leq i_2 < i_0)$ to v_{i_0}. So $tp(v_i) = 15 (i_2 \leq i \leq i_0)$.

Suppose we have a subpath (38) corresponding to the fragment

$$(\Omega_{v_1}, \bar{H}^{(1)}) \to (\Omega_{v_2}, \bar{H}^{(2)}) \to \ldots \to (\Omega_{v_m}, \bar{H}^{(m)}) \to \ldots \tag{45}$$

of the sequence (42). Here $v_1, v_2, \ldots, v_{m-1}$ are vertices of the tree $T_0(\Omega)$, and for all vertices v_i having outcoming auxiliary edges condition (43) does not hold.

As before, let μ_i denote the carrier base of Ω_{v_i}, and $\omega = \{\mu_1, \ldots, \mu_{m-1}\}, and \tilde{\omega}$ denote the set of such bases which are transfer bases for at least one equation in (45). By ω_1 denote the set of such bases μ for which either μ or $\Delta(\mu)$ belongs to $\omega \cup \tilde{\omega}$; by ω_2 denote the set of all the other bases. Let

$$\alpha(\omega) = min(min_{\mu \in \omega_2} \alpha(\mu), j),$$

where j is the boundary between working and constant sections. Let $X_\mu \doteq H[\alpha(\mu), \beta(\mu)]$. If $(\Omega, \bar{H})$ is a member of sequence (45), then denote

$$d_\omega(\bar{H}) = \Sigma_{i=1}^{\alpha(\omega)-1} d(H_i), \tag{46}$$

$$\psi_\omega(\bar{H}) = \Sigma_{\mu\in\omega_1} d(X_\mu) - 2d_\omega(\bar{H}). \tag{47}$$

Every item h_i of the section $[1, \alpha(\omega)]$ belongs to at least two bases, and both bases are in ω_1, hence $\psi_\omega(\bar{H}) \geq 0$.

Consider the quadratic part of $\tilde{\Omega}_{v_1}$ which is situated to the left of $\alpha(\omega)$. If we apply the transformation of Case 14 to this part, we will get an isomorphism at the end. The solution $\bar{H}^{(1)}$ is minimal with respect to the canonical group of automorphisms corresponding to this vertex. By Lemma 16 we have

$$d_1(\bar{H}^{(1)}) \leq n' d_2(\bar{H}^{(1)}). \tag{48}$$

Using this inequality estimate $d_\omega(\bar{H}^{(1)})$ from above.

Denote by $\gamma_i(\omega)$ the number of bases $\mu \in \omega_1$ containing h_i. Then

$$\Sigma_{\mu\in\omega_1} d(X_\mu^{(1)}) = \Sigma_{i=1}^{\rho} d(H_i^{(1)})\gamma_i(\omega). \tag{49}$$

Let $I = \{i | 1 \leq i \leq \alpha(\omega) - 1 \& \gamma_i = 2\}$ and $J = \{i | 1 \leq i \leq \alpha(\omega) - 1 \& \gamma_i > 2\}$. By (46)

$$d_\omega(\bar{H}^{(1)}) = \Sigma_{i\in I} d(H_i^{(1)}) + \Sigma_{i\in J} d(H_i^{(1)}) = d_1(\bar{H}^{(1)}) + \Sigma_{i\in J} d(H_i^{(1)}). \tag{50}$$

Let $(\lambda, \Delta(\lambda))$ be a pair of constant bases of the equation $\tilde{\Omega}_{v_1}$, where λ belongs to the nonquadratic part. This pair can appear only from the bases $\mu \in \omega_1$. There are two types of constant bases.

Type 1. λ is situated to the left of the boundary $\alpha(\omega)$. Then λ is formed by items $\{h_i | i \in J\}$ and hence $d(X_\lambda) \leq \Sigma_{i\in J} d(H_i^{(1)})$. Thus the sum of the lengths $d(X_\lambda) + d(X_{\Delta(\lambda)})$ for constant bases of this type is not more than $2n'\Sigma_{i\in J} d(H_i^{(1)})$.

Type 2. λ is situated to the right of the boundary $\alpha(\omega)$. The sum of length of the constant bases of the second type is not more than $2\Sigma_{i=\alpha(\omega)}^{\rho} d(H_i^{(1)})\gamma_i(\omega)$.

We have

$$d_2(\bar{H}^{(1)}) \leq 2n'\Sigma_{i\in J} d(H_i^{(1)}) + 2\Sigma_{i=\alpha(\omega)}^{\rho} d(H_i^{(1)})\gamma_i(\omega). \tag{51}$$

Now (47) and (49) imply

$$\psi_\omega(\bar{H}_i^{(1)}) \geq \Sigma_{i\in J} d(H_i^{(1)}) + \Sigma_{i=\alpha(\omega)}^{\rho} d(H_i^{(1)})\gamma_i(\omega). \tag{52}$$

Inequalities (48), (50),(51),(52) imply

$$d_\omega(\bar{H}^{(1)}) \leq \psi_\omega(\bar{H}^{(1)})(2n'^2 + 1). \tag{53}$$

From the definition of Case 15 it follows that all the words $H^{(i)}[1, \rho_i + 1]$ are the ends of the word $H^{(1)}[1, \rho_1 + 1]$, that is

$$H^{(1)}[1, \rho_1 + 1] \doteq U_i H^{(i)}[1, \rho_i + 1]. \tag{54}$$

On the other hand bases $\mu \in \omega_2$ participate in these transformations neither as carrier bases nor as transfer bases; hence $H^{(1)}[\alpha(\omega), \rho_1 + 1]$ is the end of the word $H^{(i)}[1, \rho_i + 1]$, that is

$$H^{(i)}[1, \rho_i + 1] \doteq V_i H^{(1)}[\alpha(\omega), \rho_1 + 1]. \tag{55}$$

So we have

$$d_\omega(\bar{H}^{(i)}) - d_\omega(\bar{H}^{(i+1)}) = d(V_i) - d(V_{i+1}) = d(U_{i+1}) - d(U_i) = d(X_{\mu_i}^{(i)}) - d(X_{\mu_i}^{(i+1)}). \tag{56}$$

In particular (47),(56) imply that $\psi_\omega(\bar{H}^{(1)}) = \psi_\omega(\bar{H}^{(2)}) = \dots \psi_\omega(\bar{H}^{(m)}) = \psi_\omega$. Denote the number (56) by δ_i.

Let the path (38) be μ-reducing, that is either $\mu_1 = \mu$ and v_2 does not have auxiliary edges and μ occurs in the sequence $\mu_1, \dots, \mu_{m-1}$ at least twice, or v_2 does have auxiliary edges $v_2 \to w_1, \dots v_2 \to w_k$ and the base μ occurs in the sequence $\mu_1, \dots, \mu_{m-1}$ at least $max_{1 \le i \le k} s(\Omega_{w_i})$ times. Estimate $d(U_m) = \Sigma_{i=1}^{m-1} \delta_i$ from below. First notice that if $\mu_{i_1} = \mu_{i_2} = \mu (i_1 < i_2)$ and $\mu_i \ne \mu$ for $i_1 < i < i_2$ then

$$\Sigma_{i=i_1}^{i_2-1} \delta_i \ge d(H^{i_1+1}[1, \alpha(\Delta(\mu_{i_1+1}))]). \tag{57}$$

Indeed, if $i_2 = i_1 + 1$, then $\delta_{i_1} = d(H^{(i_1)}[1, \alpha(\Delta(\mu))] = d(H^{(i_1+1)}[1, \alpha(\Delta(\mu))]$. If $i_2 > i_1 + 1$, then $\mu_{i_1+1} \ne \mu$ and μ is a transfer base in the equation $\Omega_{v_{i_1+1}}$. Hence $\delta_{i_1+1} + d(H^{(i_1+2)}[1, \alpha(\mu)]) = d(H^{(i_1+1)}[1, \alpha(\mu_{i_1+1})])$. Now (57) follows from

$$\Sigma_{i=i_1+2}^{i_2-1} \delta_i \ge d(H^{(i_1+2)}[1, \alpha(\mu)]).$$

So if v_2 does not have outgoing auxiliary edges, that is the bases μ_2 and $\Delta(\mu_2)$ do not intersect in the equation Ω_{v_2}; then (57) implies that

$$\Sigma_{i=1}^{m-1} \delta_i \ge d(H^{(2)}[1, \alpha(\Delta\mu_2)]) \ge d(X_{\mu_2}^{(2)}) \ge d(X_\mu^{(2)}) = d(X_\mu^{(1)}) - \delta_1,$$

which implies that

$$\Sigma_{i=1}^{m-1} \delta_i \ge \frac{1}{2} d(X_\mu^{(1)}). \tag{58}$$

Suppose now there are outgoing auxiliary edges from the vertex v_2: $v_2 \to w_1, \dots, v_2 \to w_k$. The equation Ω_{v_1} has some solution. Let $H^{(2)}[1, \alpha(\Delta(\mu_2))] \doteq Q$, and P a primitive word (in the final h's) such that $Q \doteq P^d$, then $X_{\mu_2}^{(2)}$ and $X_\mu^{(2)}$ are beginnings of the word $H^{(2)}[1, \beta(\Delta(\mu_2))]$, which is a beginning of P^∞. By the construction of (42), relation (43) does not hold for v_2; hence

$$X_\mu^{(2)} \doteq P^r P_1, P \doteq P_1 P_2, r < max_{1 \le j \le k} s(\Omega_{w_j}). \tag{59}$$

Let $\mu_{i_1} = \mu_{i_2} = \mu; i_1 < i_2; \mu_i \ne \mu$ for $i_1 < i < i_2$. If

$$d(X_{\mu_{i_1+1}}^{(i_1+1)}) \ge 2d(P) \tag{60}$$

and $H^{(i_1+1)}[1, \rho_{i_1+1}+1]$ begins with a cyclic permutation of P^3, then $d(H^{(i_1+1)}[1, \alpha(\Delta(\mu_{i_1+1}))]) \ge d(P)$. Together with (57) this gives $\Sigma_{i=i_1}^{i_2-1} \delta_i \ge d(P)$. The base μ occurs in the sequence $\mu_1, \dots, \mu_{m-1}$ at least $max_{1 \le i \le k} s(\Omega_{w_i})$ times, so either (60) fails for some $i_1 \le m-1$ or $\Sigma_{i=1}^{m-1} \delta_i \ge (r-3)d(P)$.

If (60) fails, then the inequality $d(X_{\mu_i}^{(i+1)}) \le d(X_{\mu_{i+1}}^{(i+1)})$, the definition (56) and (59) imply that

$$\Sigma_{i=1}^{i_1} \delta_i \ge d(X_\mu^{(1)}) - d(X_{\mu_{i_1+1}}^{(i_1+1)}) \ge (r-2)d(P);$$

so everything is reduced to the second case.

Let

$$\Sigma_{i=1}^{m-1}\delta_i \geq (r-3)d(P).$$

Notice that (57) implies for $i_1 = 1$ $\Sigma_{i=1}^{m-1}\delta_i \geq d(Q) \geq d(P)$; so $\Sigma_{i=1}^{m-1}\delta_i \geq d(P)max\{1, r-3\}$. Together with (59) this implies $\Sigma_{i=1}^{m-1}\delta_i \geq \frac{1}{5}d(X_\mu^{(2)}) = \frac{1}{5}(d(X_\mu^{(1)}) - \delta_1)$. Finally,

$$\Sigma_{i=1}^{m-1}\delta_i \geq \frac{1}{10}d(X_\mu^{(1)}). \tag{61}$$

Comparing (58) and (61) we can see that for the μ-reducing path (38) inequality (61) always holds.

Suppose now that the path (38) is prohibited; hence it can be represented in the form (39). From definition (47) we have $\Sigma_{\mu\in\omega_1}d(X_\mu^{(m)}) \geq \psi_\omega$; so at least for one base $\mu \in \omega_1$ the inequality $d(X_\mu^{(m)}) \geq \frac{1}{2n}\psi_\omega$ holds. Because $X_\mu^{(m)} \doteq (X_{\Delta(\mu)}^{(m)})^{\pm 1}$, we can suppose that $\mu \in \omega \cup \tilde{\omega}$. Let m_1 be the length of the path $r_1 s_1 \dots r_l s_l$ in (39). If $\mu \in \tilde{\omega}$ then by the third part of the definition of a prohibited path there exists $m_1 \leq i \leq m$ such that μ is a transfer base of Ω_{v_i}. Hence, $d(X_{\mu_i}^{(m_1)}) \geq d(X_{\mu_i}^{(i)}) \geq d(X_\mu^{(i)}) \geq d(X_\mu^{(m)}) \geq \frac{1}{2n}\psi_\omega$. If $\mu \in \omega$, then take μ instead of μ_i. We proved the existence of a base $\mu \in \omega$ such that

$$d(X_\mu^{(m_1)}) \geq \frac{1}{2n}\psi_\omega. \tag{62}$$

By the definition of a prohibited path, the inequality $d(X_\mu^{(i)}) \geq d(X_\mu^{(m_1)})(1 \leq i \leq m_1)$, (61), and (62) we obtain

$$\Sigma_{i=1}^{m_1-1}\delta_i \geq max\{\frac{1}{20n}\psi_\omega, 1\}(40n^3 + 20n + 1). \tag{63}$$

By (56) the sum in the left part of the inequality (63) equals $d_\omega(\bar{H}^{(1)}) - d_\omega(\bar{H}^{(m_1)})$; hence

$$d_\omega(\bar{H}^{(1)}) \geq max\{\frac{1}{20n}\psi_\omega, 1\}(40n^3 + 20n + 1),$$

which contradicts (53).

This contradiction was obtained from the supposition that there are prohibited paths (45) in the sequence (42). Hence (42) does not contain prohibited paths. This implies that $v_i \in T_0(\Omega)$ for all v_i in (42). For all i $v_i \to v_{i+1}$ is an edge of a finite tree. Hence the sequence (42) is finite. Let $(\Omega_w, \bar{H}^w)$ is final term of this sequence. We show that $(\Omega_w, \bar{H}^w)$ satisfies all the properties formulated in the lemma.

The first property follows from (44).

Let $tp(w) = 2$ and let Ω_w be nontrivial. It follows from the construction of (42) that if $[j,k]$ is a constant section for Ω_i then $H^{(i)}[j,k] \doteq H^{(i+1)}[j,k] \doteq \dots H^{(w)}[j,k]$. Hence (43) and the definition of $s(\Omega_v)$ imply that the word $h_1 \dots h_{\rho_w}$ can be subdivided into subwords $h[i_1, i_2], \dots, h[i_{k-1}, i_k]$, such that for any a either $H^{(w)}$ has length 1, or $h[i_a, i_{a+1}]$ does not participate in basic and coefficient equations, or $H^{(w)}[i_a, i_{a+1}]$ can be written as

$$H^{(w)}[i_a, i_{a+1}] \doteq P_a^r P_a';\ \ P_a \doteq P_a' P_a''; r \geq max_{<\mathcal{P},\mathcal{R}>}\rho_w f_2(\Omega_w, P, R), \tag{64}$$

where P_a is a primitive word, and $< \mathcal{P}, \mathcal{R} >$ runs through all the periodic structures of $\tilde{\Omega}_w$ for which $\tilde{\Omega}_w$ is regular. Then for a maximal such P_a, $\tilde{\Omega}_w$ is singular, because if it were regular we would have h_k such that $d(H_k^{(w)}) \geq f_2(\Omega_w, P, R)$. This contradicts the minimality of $\bar{H}^{(w)}$.

The third assertion of the lemma follows by induction from the first and second assertions, by Lemma 9 and the fact that automorphisms corresponding to the equation (26) have the form $u \to u$, $w \to u^r w$. Thus Lemma 17 is proved.□

8. Tree $T_4(\Omega)$

Let w be the terminal vertex of $T_0(\Omega)$, such that $tp(w) = 2$ and Ω_w is nontrivial. Let $< \mathcal{P}, R >$ be a periodic structure such that $\tilde{\Omega_w}$ is singular, and let c be a cycle in Γ. There is a homomorphism ϕ from the group $F_{R(\Omega_w)^*}$ in the free extension of a centralizer of the group $F_{R(\Omega_w^* \cup h(c)=1)}$ by the element y, where $h(c) = h_{i_1} \dots h_{i_k}$ is the label of the cycle from the set $\{c_1, \dots, c_k\}$ from Lemma 13. Denote by Ω_T a generalized equation for the subequation consisting of bases in $\mathcal{P}$ and equation $h_{i_1} \dots h_{i_k} = 1$. Denote the new variables by z. Let the generalized equation $\bar{\Omega}$ obtained from $\tilde{\Omega}$ by deleting all the bases and variables in $\mathcal{P}$. Consider these two generalized equations Ω_T and $\bar{\Omega}$ together on the disjoint sets of variables. And add the following basic equations: $h_k = \tilde{\pi}^T(h_k)(h_k \notin \mathcal{P})$, where the h_k in the left side is considered as a variable in $\bar{\Omega}_v$, and in the right side as some section of the generalized equation Ω_T. Denote it by $\Omega_w(\mathcal{P}, R, c, T)$. The homomorphism ϕ induces a homomorphism ψ from the group $F_{R(\Omega_w)}$ in the free extension of a centralizer of the group $F_{R(\Omega_w(\mathcal{P},R,c,T))^*}$. Denote by $\tilde{F}_{R(\Omega_w)}$ the factor-group of $F_{R(\Omega_w)^*}$ over the intersection of the kernels of all the homomorphisms from $F_{R(\Omega_w)^*}$ into $F_{R(\Omega_w(\mathcal{P},R,c,T))^*}$ which can be obtained as a composition of $\sigma \in P_0$ and $\pi_{\bar{H}^+}$.(Lemma 13). Then ψ is monic on $\tilde{F}_{R(\Omega_w)}$.

Add the corresponding edge to the tree $T_0(\Omega)$ and denote by $T_3(\Omega)$ the tree obtained by using this procedure on each final vertex w of $T_0(\Omega)$, such that $tp(w) = 2$ and Ω_w is nontrivial. So if w' (corresponding to the edge $w \to w'$) is a final vertex of $T_3(\Omega)$, which is not the vertex of $T_0(\Omega)$ then $\pi(v_0, w)$ is not an isomorphism . Finally glue $T_3(\Omega_w)$ to those final vertices w of $T_3(\Omega)$, for which Ω_w is nontrivial, and iterate this process. Finally we get $T_4(\Omega)$. By Lemma 3 it does not contain infinite branches; so it is finite. The construction of $T_4(\Omega)$ is effective.

9. The proof of Theorems 2 and 3

We shall first prove Theorem 3. Consider an irreducible system $S = 1$. By Lemma 5, $F_{R(S)}$ can be approximated by the homomorphisms in only one Razborov's fundamental sequence, corresponding to some path in $T_4(\Omega)$ $v_0 \to v_{11} \to v_{12} \to \dots v_{1,n_1} = w_1 \to v_{21} \to v_{22} \to \dots v_{2,n_2} = w_2 \dots w_{m-1} \to v_{m1} \to v_{m2} \to \dots v_{m,n_m}$, where w_{i+1} is the terminal vertex of type 2 for the tree $T_3(w_i)$.

Let S_{ij} be a quadratic equation from the beginning of section 7 corresponding to the vertex v_{ij} (in case $j = n_i$ S_{in_i} corresponds to the extension of a centralizer). Denote by $F_{v_0,\dots,v_{m,n_m}}$ the factor-group of $F_{R((\Omega_{v_0})^*)}$ over the intersection of the kernels of all the homomorphisms from $F_{R((\Omega_{v_0})^*)}$ into $F_{R((\Omega_{v_{m,n_m}})^*)}$ corresponding to this sequence. The group $F_{R((\Omega_{v_{m,n_m}})^*)}$ is free. It follows from Lemma 17 that $F_{v_0,\dots,v_{m,n_m}}$ is embedded into $F_{R(S_0 \cup \dots \cup S_{mn_{m-1}})}$. But $F_{R(S)}$ is embedded into $F_{v_0,\dots,v_{m,n_m}}$; hence $F_{R(S)}$ is embedded into $F_{R(S_0 \cup \dots \cup S_{mn_{m-1}})}$.

The system $S_0 \cup \dots \cup S_{mn_{m-1}}$ is triangular quasi-quadratic. □

To prove Theorem 2 we have to follow the process described for the irreducible system in the proof of Theorem 3. Instead of one branch of the tree $T_4(\Omega)$ we will have several branches. The construction of T_4 is effective, hence this process is effective. □

10. The proof of Theorems 6,5 and Corollaries 2–5, 6

Proof of Corollary 2

Let G be a finitely generated residually free group, and $< X, S >$ be a finitely generated presentation for G. Let $F = F(A)$ be a nonabelian free group with some basis A disjoint with X. We can think of S as a system of equations $S = 1$ over F. The group $F[X]/S =< F * F(X)|S(X) = 1 >= F * G$ is approximated in F by F-homomorphisms; hence $R(S) = ncl(S)$ and $F * G = F[X]/R(S)$. Thus G is a free factor of the affine coordinate group $F_{R(S)}$. The variety $V(S)$ is a finite union of its irreducible components $V(S) = V(S_1)\cup \ldots \cup V(S_n)$. This implies that $F_{R(S)}$ is embedded into $F_{R(S_1)} \times \ldots \times F_{R(S_n)}$, and each group $F_{R(S_i)}$ is fully residually free. By the theorem G is embedded into $F^{\mathbf{Z}[x]} \times \ldots \times F^{\mathbf{Z}[x]}$. $\square$

Proof of Theorem 5

Let $F = F(A)$ be a free group, and $S(X)$ be a system of equations over F which determines an irreducible variety over F. Then $F_{R(S)} = F(A \cup X)/R(S)$ is a fully residually free group; hence it is finitely presented. So there are finitely many relations $r_i(A \cup X),\ i = 1, \ldots n$, such that $R(S) = ncl(r_1, \ldots r_n)$. The system $S' = \{r_1, \ldots, r_n\}$ is equivalent to S and satisfies the Nullstellensatz.

Remark. There exists a variety V (reducible) which cannot be defined by a finite system satisfying Nullstellensatz.

Indeed, this follows from the existence of finitely generated residually free and not finitely presented groups.

Proof of Theorem 6

Let $\mathcal{G}(X)$ be a graph of groups:

1. X is a connected graph;
2. For every vertex v of X and every edge e groups G_v and G_e are defined such that $G_e = G_{\bar{e}}$ (here $\bar{e}$ is the inverse edge for e);
3. For every edge $e \in X$, $G_e \leq G_{e\sigma}$ and there exists a monomorphism $\tau : G_e \to G_{e\tau}$ (here $e\sigma$ and $e\tau$ are initial and terminal vertices of e).

The fundamental droup $\pi_1(\mathcal{G}(X))$ of a graph of groups $\mathcal{G}(X)$ is defined as follows. Let T be a maximal subtree of X. Then

$$\pi_1(\mathcal{G}(X)) =< (*_{v\in V(X)}G_v), t_e(e \in E(X))|t_e = 1(e \in T), t_e^{-1}gt_e = g_\tau(g \in G_e), t_e t_{\bar{e}} = 1 > .$$

It is known that $\pi_1(\mathcal{G}(X))$ is independent (up to isomorphism) of T. The group $\pi_1(\mathcal{G}(X))$ can be obtained from the vertex groups by a tree product with amalgamation and then by HNN-extensions. Subgroups of $\pi_1(\mathcal{G}(X))$ are again fundamental groups of some special graphs of groups related to $\mathcal{G}(X)$.

Theorem 8 *[?] Let $\mathcal{G}(X)$ be a graph of groups, and let $H \leq \pi_1(\mathcal{G}(X))$. Then $H = \pi_1(\mathcal{G}(Y))$ where the vertex groups of $\mathcal{G}(Y))$ are $H \cap gG_vg^{-1}$ for all vertices $v \in X$, and g runs over a suitable set of (H, G_v) of double coset representatives, and the edge groups are $H \cap gG_eg^{-1}$ for all edges $e \in X$, where g runs over a suitable set of (H, G_e) double coset representatives.*

Let G be obtained as a union of the finite chain:

$$F < G_1 < \ldots < G_n = G,$$

where G_{i+1} is a free extension of a centralizer of G_i. We prove the theorem by induction on n. If $n = 0$ it is obvious, because all finitely generated subgroups of F are free of finite rank. By induction all finitely generated subgroups of G_{n-1} satisfy the conclusion of the theorem. The group G is a free product with amalgamation: $G = G_{n-1} *_{C=\bar{C}} (\bar{C}\times < t >)$, where $C = C_{G_{n-1}}(u)$ is a centralizer of some element $u \in G_{n-1}$, and $\bar{C}$ is an isomorphic copy of C. In particular, G is a fundamental group of the graph of groups with vertex groups G_{n-1} and $\bar{C}\times < t >$, and edge group C. By Theorem 8 a finitely generated subgroup H of G is a fundamental group of some graph of groups $\mathcal{H}(Y)$, where the vertex groups of Y are of the form $H \cap g^{-1}G_{n-1}g$ or $H \cap g^{-1}(\bar{C}\times < t >)g$ and edge groups are of the form $H \cap g^{-1}Cg$.

From general properties of amalgamated products one can deduce (see [**?**] for details) that centralizers in G are free abelian groups of finite rank ($\leq n$). Therefore all edge groups in the graph of groups $\mathcal{H}(Y)$ are finitely generated abelian groups. Since $H = \pi_1(\mathcal{H}(Y))$ and H is finitely generated, H is an HNN-extension with finitely many stable letters of a free product with amalgamation of finitely many vertex groups.

Notice, that if amalgamated subgroups are finitely generated and at least one of the free factors is not finitely generated, then the whole amalgamated product is not finitely generated (this follows from normal forms of elements in amalgamated products). Similarly, if the base group is not finitely generated, and associated subgroups are finitely generated, then an HNN-extension is not finitely generated. This implies, that the vertex groups $H \cap g^{-1}G_{n-1}g$ are finitely generated. Therefore, by induction, the vertex groups can be obtained from free abelian groups of finite rank by finitely many operations of the type 1–4.

The group G as well as all the subgroups of G are CSA-groups. It was shown in [**?**] that if a free product with abelian amalgamation results in a CSA-group, then at least one of the amalgamated subgroups is maximal abelian. Similarly, an HNN-extension with abelian associated subgroups is a CSA-group if and only if this HNN-extension is of type 3 or 4 [**?**]. □

Proof of Corollary 4

According to the theorem 6, if all proper centralizers in a finitely generated subgroup H of $F^{\mathbf{Z}[x]}$ are cyclic, then H is obtained from cyclic groups by operations 1, 2, 4, which preserve hyperbolicity (see [**?**], [**?**], [**?**]). □

Proof of Corollary 5

Consider the following formula

$$\forall x \forall y \forall z \exists u([x, y] = [x, z] = [y, z] = 1 \rightarrow (xy = u^2 \vee xz = u^2 \vee yz = u^2)).$$

This formula holds in all subgroups of $F^{\mathbf{Z}[x]}$ in which all centralizers are cyclic, and does not hold in any other subgroup. Hence every finitely generated group which is $\forall\exists$-equivalent to a free group is a subgroup of $F^{\mathbf{Z}[x]}$ with all centralizers cyclic.

Proof of Corollary 6 The assertion of the corollary follows from Theorem 4 and the results of ([**?**], part 1), where a length function on $F^{\mathbf{Z}[x]}$ with many useful properties has been constructed. The corollary can be also deduced from Theoren 4 and the results of Bass [**?**]

11. The proof of Theorem 6

It is enough to prove the theorem in the case when the variety $V_F(S)$ is irreducible. By the theorem we have an embedding

$$\mu : F[X]/Rad(S) \longrightarrow F^{\mathbf{Z}^k}$$

for some suitable number k. Let u_i be the image of the generator $x_i \in X$ under μ and $U = (u_1, \ldots, u_n)$ be the corresponding tuple of parametric words. Due to [**?**] the family of specializations

$$\Xi^\star = \{\xi^\star \mid \xi \in Hom(\mathbf{Z}^k, \mathbf{Z})\}$$

is a discriminating family of F-homomorphism s. In particular, $U^\star$ is a subset of $V_F(S)$. Let us prove that this subset is dense in $V_F(S)$ in the Zariski topology on F^n. Choose an arbitrary point $v \in V_F(S)$ and an open basic neighborhood $O_f = \{w \in F \mid f(w) \neq 1\}$ (here $f \in F[X]$) of v. Thus $f(v) \neq 1$ and hence f is not in $Rad(S)$. Therefore, f defines a nontrivial element in the affine coordinate group $F[X]/Rad(S)$. Now there exists a homomorphism $\phi \in \Xi^\star$ such that $f^\phi \neq 1$. But this means that the solution $U^\phi \in U^\star$ belongs to the same neighborhood O_f. This shows that $U^\star$ is dense in $V_G(S)$ in Zariski topology. □

12. An embedding theorem for affine groups

Theorem 9 *Suppose we have a generalized equation $w(\bar{h})$ such that w^* is irreducible, and a system $v(\bar{y}, \bar{h}) = 1, w^*(\bar{h}) = 1$. Then the following assertion is true: if for any solution $\bar{g} \in F$ of the system w^* there exists a solution $\bar{y}$ in F of the system $v(\bar{y}, \bar{h}) = 1, w^* = 1$, then there is an embedding of $F_{R(w^*)}$ into $F_{R(v(\bar{y},\bar{h}),w(\bar{h}))^*}$.*

Proof Suppose first that for any solution $\bar{g} \in F$ of the system w^* there exists a solution $\bar{y}$ in F of the system $v(\bar{y}, \bar{h}) = 1$.

Let H be a subgroup generated in $F_{R(w^*,v)}$ by the elements $\bar{h}$. Then for any homomorphism $\alpha : F_{R(w^*)} = F_{w^*} \to F$, this α can be extended to a homomorphism $\alpha' : H \to F$ such that the following diagram is commutative.

$$\begin{tikzcd}
F_{R(u)} \arrow[r, "\lambda"'] \arrow[dr, "\alpha"'] & H \arrow[r, "\mu"'] \arrow[d, "\alpha'"] & F_{R(w^*,v)} \arrow[dl, "\alpha''"] \\
& F &
\end{tikzcd}$$

Here λ is a canonical homomorphism $\lambda(h) = h$ (the h's in $F_{R(v,w^*)}$ satisfy $w(\bar{h}) = 1$), μ is an inclusion, and $v(\bar{h}^\alpha, \bar{y}^{\alpha''}) = 1$ in F.

$F_{R(w^*)}$ is residually F. If some nontrivial element $r \in F_{R(u^*)}$ belongs to the kernel of λ, then there exists some α such that $\alpha(r) \neq 1$ in F, but $\alpha' \circ \lambda(r) = 1$. This implies that λ is an isomorphism. So we have proved the existence of an embedding.