Irreducible affine varieties over a free group. I: Irreducibility of quadratic equations and Nullstellensatz

O. Kharlampovich, A. Myasnikov[1]

0. Introduction

The object of this paper (which consists of two parts) is to describe irreducible varieties over free groups and to characterize finitely generated fully residually free groups. We prove that any variety over a free group F can be defined by a finite number of systems of equations $S = 1$ in triangular form where quadratic words play the role of leading terms. Algebraically, irreducible varieties are exactly the varieties whose coordinate groups are fully residually F. The crucial point of the classification of fully residually free groups is to prove that the coordinate groups of irreducible varieties are embeddable into Lyndon's free $\mathbf{Z}[x]$-group $F^{\mathbf{Z}[x]}$. Since every finitely generated fully residually free group is a free factor of the coordinate group of an irreducible variety, and the group $F^{\mathbf{Z}[x]}$ is fully residually F, we obtain a characterization of finitely generated fully residually free groups as subgroups of $F^{\mathbf{Z}[x]}$.

The group $F^{\mathbf{Z}[x]}$ and its subgroups have been studied extensively during the last several years. In particular, every finitely generated subgroup of $F^{\mathbf{Z}[x]}$ (hence, every finitely generated fully residually free group) can be obtained from free abelian groups of finite rank by finitely many free products with amalgamation and HNN-extensions of the type, where amalgamated and associated subgroups are free abelian of finite rank. In particular, this implies that every finitely generated fully residually free group is finitely presented.

There are three parts to this paper: algebraic geometry over free groups, the theory of free exponential groups, and Makanin-Razborov's machinery to deal with equations over free groups [11],[14],[13].

The algebraic geometry approach has been shown to be very usefull in dealing with equations over groups. It provides necessary topological means and a method to transcribe geometric notions into pure group-theoretic language. Following Baumslag, Myasnikov, Remeslennikov [1] we use the standard algebraic geometry notions such as variety, Zariski topology, irreducibility of varieties, radicals and coordinate groups. Some of the ideas of the algebraic geometry approach go back to R. Lyndon [9], E. Rips, J. Stallings [16].

[1]The first author was supported by NSERC grant; the second author was supported by the NSF Grant DMS-9103098

The theory of exponential groups (i.e. groups admitting exponents in some ring A) starts with results of P. Hall, A. Malcev, G. Baumslag and R. Lyndon. It provides a technique to deal with noncommutative modules over the ring A. R. Lyndon gave an axiomatic description of the notion of an exponential group. He described and studied the group $F^{\mathbf{Z}[x]}$ (free exponential group over the ring of integral polynomials $\mathbf{Z}[x]$) and showed the crucial importance of this group in the study of equations over free groups. A modern treatment of exponential groups was given by Myasnikov and Remeslennikov in [12] . In particular, they showed that $F^{\mathbf{Z}[x]}$ can be described using HNN-extensions of a very special type, namely extensions of centralizers. Basically, to obtain $F^{\mathbf{Z}[x]}$ from F one needs just to extend all centralizers of F up to free $\mathbf{Z}[x]$-modules of rank 1. Namely, $F^{\mathbf{Z}[x]}$ is the union of groups:

$$F < G_1 < G_2 < \dots,$$

where $G_{i+1} =< G_i, t | u_i^t = u_i,\ u_i \in C(u_i) >$, where $C(u_i)$ is some proper centralizer in G_i.

The groups $F^{\mathbf{Z}[x]}$ happen to be fully residually F [8]. Basically there exists only one known method of proving that a group is fully residually free. This method is due to G. Baumslag who showed that the surface groups (except the non-orientable case of genus 1,2,3) are embeddable into an extension of a centralizer of a free group, and hence they are fully residually free. B. Baumslag gave other examples of fully residually free groups using the same method.

In 1992 Myasnikov and Remeslennikov while studying ultrapowers of free groups came to the following conjecture: every finitely generated fully residually free group is a subgroup of Lyndon's group $F^{\mathbf{Z}[x]}$.

In the paper [4], the conjecture was proved for 3-generated fully residually free groups; moreover it was proved that there are just three types of 3-generated fully residually free groups: free groups, free abelian groups, and extensions of centralizers $< x, y, t | u^t = u >$, where u is an arbitrary element in $F(x, y)$ that is not a proper power.

In the second part of this paper we shall prove the conjecture and will discuss the applications.

In the first part we concentrate on quadratic equations over a free group F. Quadratic equations have been widely studied, see for example [6],[5],[3],[2], [10].

We shall prove that the coordinate group $F_{R(S)}$ of the quadratic equation $S = 1$ is embeddable into $F^{\mathbf{Z}[x]}$. This implies that the variety $V(S)$ is irreducible in the Zariski topology over F^n. Moreover, we completely describe the radical of the quadratic system $S = 1$. It turns out that the radical $Rad(S)$ coincides (with a few exceptions) with the normal closure of S in the group $F * F(X)$. In particular, this implies the Nullstellensatz for the system $S = 1$. The group theoretic formulation of the Nullstellensatz was first given by E. Rips. He actually announced in New York in the fall of 1995, the joint result with Z. Sela about the Nullstellensatz for quadratic equations over a free group.

We should mention that G. Baumslag and V. Remeslennikov took part in different stages of the discussions leading to the proof of Theorem 1 in the first paper, and in a sense they could be considered as coauthors of that theorem. We will follow the terminology given in [1].

Let G be a group, $F = F(X)$ the free group with basis $X = \{x_1, x_2, \dots x_n\}$, $G[X] = G * F$ the free product of G and F.

An element s from $G[X]$ is called *an equation over the group* G. We write this as $s = 1$. As an element of the free product, s can be written as a product of some elements $x_1, \dots, x_n$ from $X \cup X^{-1}$ (which are called *variables*) and elements $g_1, \dots, g_m$ from G (*constants*). We will write, sometimes, $s(x_1, \dots, x_n, g_1, \dots, g_m) = 1$ or, simply, $s(x, g) = 1$. *A system of equations over a group* G is an arbitrary set of equations $S = \{s_i = 1 \mid i \in I\}$ (we shall denote this as $S = 1$). *A solution* of a system $S(x_1, \dots, x_n, g_1, \dots, g_m) = 1$ over a group G is a tuple of elements $a_1, \dots, a_n \in G$ such that after replacement of each x_i by a_i in every equation $s(x, g) = 1$ from S one gets the trivial element in the group G. On the other hand, a solution of the system $S = 1$ over G can be described as a G-homomorphism (i.e. a homomorphism which is identical on G) $\phi : G[X] \longrightarrow G$ such that $\phi(S) = 1$. These definitions are equivalent. By $V(S)$ we denote the set of all solutions in G of the system $S = 1$.

Let S be a subset of $G[X]$. Then $V(S)$ is called an *algebraic subset* or an (affine) variety in G^n. Two systems $S = 1$ and $T = 1$ are equivalent over G if $V(S) = V(T)$. For any $S \subseteq G[X]$ we have $V(S) = V(ncl(S))$, where $ncl(S)$ is the normal closure of S in $G[X]$.

A group G is called CSA-group if every maximal abelian subgroup M of G is malnormal, i.e. $M^g \cap M = 1$ for any $g \notin M$.

It was shown in [1] that for a nonabelian CSA-group G all algebraic sets in G^n define a topology on G^n in which they are exactly the closed sets. The topology defined by algebraic sets as closed subsets is said to be a Zariski topology.

Below G is always a nonabelian CSA-group.

Definition 1 *Let $Y \subseteq G^n$. Define a set*

$$I(Y) = \{s \in G[X] \mid s(g_1, \dots, g_n) = 1 \forall (g_1, \dots, g_n) \in Y\}$$

The set $I(Y)$ has a nice description in terms of homomorphisms. Any tuple $g = (g_1, \dots, g_n) \in Y$ defines a G-homomorphism $f_g : G[X] \longrightarrow G$ by the condition $x_i \longrightarrow g_i$.

Then

$$I(Y) = \bigcap_{g \in Y} ker(f_g)$$

Let us recall that a subgroup N is *an isolated subgroup* in a group H if for any $x \in H$ and any nonzero integer n inclusion $x^n \in N$ implies that $x \in N$. For any set $S \subset H$ the intersection of all *normal isolated* subgroups containing S is denoted by $\sqrt{S}$.

Lemma 1 *[1]*

1) $I(Y)$ is a normal subgroup of $G[X]$;

2) If G is a torsion-free group then $I(Y)$ is an isolated normal subgroup of $G[X]$, in particular $I(V(S))$ contains $\sqrt{S}$.

Definition 2 *Let $V(S)$ be a variety defined by $S \subset G[X]$. Then $I(V(S))$ is called the radical of the system $S = 1$ and is denoted by $Rad(S)$. The quotient group $G_{R(S)} = G[X]/Rad(S)$ is called the affine coordinate group of the variety $V(S)$.*

Systems $S = 1$ and $T = 1$ define the same variety over G iff $Rad(S) = Rad(T)$.

Let $S = 1$ be a system of equations over a torsion-free group G. Then the quotient group of $G[X]$ by the $\sqrt{S}$ is denoted by $G_{\sqrt{S}}$.

A system $S = 1$ over G is called consistent if there is a G-homomorphism $\pi : G[X] \to H \geq G$ such that $S \in ker(\pi)$. Otherwise, it is inconsistent over G. If a system $S = 1$ over G is consistent then the canonical homomorphism $G \to G[x]/Rad(S)$ is monic. Therefore, for non-empty varieties $V(S)$ we will assume that G is a subgroup of $G_{R(S)}$.

Let G be a torsion-free group, $V(S)$ is an algebraic set in G^n defined by a system $S = 1$. Then by Lemma 1 $Rad\ (S)$ contains $\sqrt{S}$.

Definition 3 *A system of equations $S = 1$ over a torsion-free group G satisfies the Nullstellensatz if*

$$Rad\ (S) = \sqrt{S}$$

Definition 4 *Let H be a group and $\mathcal{G}$ be a family of groups.*

1) A homomorphism of groups $\psi : H \longrightarrow G$ separates a nontrivial element $h \in H$ if $\psi(h) \neq 1$;

2) A family of homomorphisms $\Psi = \{\psi : H \longrightarrow G \mid G \in \mathcal{G}\}$ is called a separating (discriminating) family of homomorphisms if any nontrivial $h \in H$ (any finite number of nontrivial elements $h_1, \ldots, h_n \in H$) can be separated by some $\psi \in \Psi$. In this case H is called a residually $\mathcal{G}$ group (ω-residually $\mathcal{G}$ group or fully residually $\mathcal{G}$ group).

In the case when $\mathcal{G}$ consists of a single group G, which is also a subgroup of H and if the separating (discriminating) homomorphisms in Ψ are all G-homomorphisms we say that H is separated (discriminated) by G-homomorphisms.

Lemma 2 *[1] A system of equations $S = 1$ over a torsion-free group G satisfies the Nullstellensatz in G if and only if $G_{\sqrt{S}}$ is separated in G by G-homomorphisms.*

Denote by G_S the factor group $G[X]/ncl(S)$.

Lemma 3 *If G is torsion-free and G_S is separated in G by G-homomorphisms, then $ncl(S) = \sqrt{S} = Rad(S)$.*

The proof is straightforward.

A group G is called *Equationally Noetherian* (EN) [1] if for every system S of equations over G there is a finite subsystem S_0 such that $V(S) = V(S_0)$. For example, a free group is EN group [7].

A closed set in a topological space is called *irreducible* if it is not a union of two proper closed subsets. Zariski topology over an EN CSA-group is noetherian, and consequently, every closed subset is a finite union of its irreducible components.

Lemma 4 *[1] Let G be an EN CSA-group. Then $V(S)$ is irreducible if and only if $G_{R(S)}$ is discriminated in G by G-homomorphisms.*

Proof Suppose $V(S)$ is not irreducible and $V(S) = \bigcup_{i=1}^{n} V(S_i)$ is its decomposition into irreducible components. Then $Rad(S) = \cap_{i=1}^{n} Rad(S_i)$, and hence there exist $s_i \in Rad(S_i) \setminus \{Rad(S), Rad(S_j), j \neq i\}$. The set $s_i, i = 1, \ldots n$ cannot be separated in G by G-homomorphisms.

Suppose now $s_1, \ldots, s_n$ are elements such that for any retract $f : G_{R(S)} \longrightarrow G$ there exists i such that $f(s_i) = 1$; then $V(S) = \bigcup_{i=1}^{m} V(S \cup s_i)$. $\square$

Definition 5 *Let K be a group, $C(u)$ the centralizer of an element $u \in K$. Suppose $C(u)$ is abelian. Then the following group is called a free extension of a centralizer in K:*

$$K(u, t) = \langle K,\ t \mid [C(u), t] = 1 \rangle.$$

Note, that $K(u,t)$ can be obtained from K by an HNN-extension with respect to the identity isomorphism $C(u) \to C(u)$:

$$K(u,t) = \langle K, t \mid t^{-1}at = a, \ a \in C(u) \rangle.$$

We introduce the following notation. Let

$$G = G_0 \ \leq \ G_0(u_0, t_0) = G_1 \ \leq \ \ldots \ \leq \ G_n(u_n, t_n) = G_{n+1}$$

be a finite sequence of extensions of centralizers of elements $u_i \in G_i$. Then we denote the resulting group G_{n+1} by $G(U,T)$, where $U = \{u_0, \ldots, u_n\}$, $T = \{t_0, \ldots, t_n\}$.

Let A be an arbitrary associative ring with identity and G a group. Fix an action of the ring A on G, i.e. a map $G \times A \to G$. The result of the action of $\alpha \in A$ on $g \in G$ is written as g^α. Consider the following axioms:

1. $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$;
2. $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;
3. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;
4. $[g,h] = 1 \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha$.

Definition 6 *Groups with A-actions satisfying axioms 1)–4) are called A–*groups.

In particular, an arbitrary group G is a $\mathbf{Z}$-group. We now recall the definition of an A-completion from [12].

Definition 7 *Let G be a group . Then an A–group G^A together with a homomorphism $G \to G^A$ is called a* tensor *A–completion of the group G, if G^A satisfies the following universal property: for any A–group H and a homomorphism $\varphi : G \to H$ there exists a unique A–homomorphism $\psi : G^A \to H$ (a homomorphism that commutes with the action of A) such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & G^A \arrow[dl, "\psi"] \\ H & \end{tikzcd}$$

By $\mathbf{Z}[x]$ we denote, as usual, the ring of polynomials of one variable with integer coefficients. $\mathbf{Z}[x]$-completion of a free group F is called Lyndon's free $\mathbf{Z}[x]$-group.

Lemma 5 *[12] Every group obtained from a CSA group G by a sequence of free extensions of centralizers is embeddable into $G^{\mathbf{Z[x]}}$.*

1. Quadratic equations over groups

Notation: Let $S \in G[X]$, then the set of all variables which occur in S is denoted by $var(S)$.

Two systems $S = 1$ and $T = 1$ over G are termed to be *disjoint* if $var(S) \cap var(T) = \emptyset$. A system $S = 1$ is *splittable* if it is a union of two nonempty disjoint subsystems: $S = S_1 \cup S_2$, and $var(S_1) \cap var(S_2) = \emptyset$.

Definition 8 *A set $S \subset G[X]$ is called quadratic, if every variable from $var(S)$ occurs in S not more then twice. The set S is strictly quadratic if every letter from $var(S)$ occurs in S exactly twice.*

A system $S = 1$ over G is *quadratic (strictly quadratic)*, if the corresponding set S is quadratic (strictly quadratic).

The main result of this paper is the following

Theorem 1 *Let G be a fully residually free group and let $S = 1$ be a consistent quadratic equation over G, then $G_{R(S)}$ is G-embeddable into $G(U,T)$ for some finite U and T , and hence into $G^{\mathbf{Z}[x]}$.*

Definition 9 *Let G be a group, $\bar{c}$ a tuple of elements from G, $\bar{x}_1, \ldots, \bar{x}_n$ disjoint tuples of variables.*

A system $\bigcup_{i=1}^{m} S_i(\bar{c}, \bar{x}_i, \ldots, \bar{x}_m) = 1$ is said to be triangular quasi-quadratic if for every i the equation $S_i(\bar{c}, \bar{x}_i, \ldots, \bar{x}_m) = 1$ is quadratic in the variables from $\bar{x}_i$

Such a system is said to be nondegenerate if for each i the equation $S_i = 1$ over $G_{i-1} = G[\bar{x}_{i+1}, \ldots, \bar{x}_m]/R(\bigcup_{j=1}^{i-1} S_j)$ (with elements $\bar{x}_i$ considered as variables and elements from $\bar{c}, \bar{x}_{i+1} \ldots \bar{x}_m$ as coefficients from G_{i-1}) has a solution.

The following result is a corollary of Theorem 1.

Theorem 2 *If S is a nondegenerate triangular quasi-quadratic system over a fully residually free group G, then $G_{R(S)}$ is a subgroup of $G(U,T)$ for some U and T and hence a subgroup of $G^{\mathbf{Z}[x]}$.*

Corollary 1 *Algebraic sets corresponding to triangular quasi-quadratic systems of equations are irreducible sets in the Zariski topology on G^n for fully residually free group G.*

To formulate Theorem 3 we need a few more definitions. Every quadratic equation over G can be transformed into a standard equation:

Definition 10 *A standard quadratic equation over the group G is an equation of the one of the following forms:*

$$\prod_{i=1}^{n}[x_i, y_i] = 1, \quad n > 0; \tag{1}$$

$$\prod_{i=1}^{n}[x_i, y_i] \prod_{i=1}^{m} z_i^{-1} c_i z_i = d, \quad n, m \geq 0, m + n \geq 1; \tag{2}$$

$$\prod_{i=1}^{n} x_i^2 = 1, \quad n > 0; \tag{3}$$

$$\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1} c_i z_i = d, \quad n, m \geq 0, n + m \geq 1; \tag{4}$$

where d, c_i are nontrivial elements from G.

Lemma 6 *Let S be a strictly quadratic word over G. Then there is a G-automorphism $f \in Aut_G(G[X])$ such that S^f is a standard quadratic word over G.*

Proof See [2].

In this case we say that S is equivalent to S^f over $G[X]$.

Definition 11 *Strictly quadratic words of the type*

$$[x, y], \; x^2, \; z^{-1}cz$$

where $c \in G$, are called atomic quadratic words *or simply* atoms.

Definition 12 *Any standard quadratic equation $S = 1$ over G can be written (see above) as a product of atoms r_i:*

$$r_1 \, r_2 \ldots r_k = g.$$

The minimal such number k is called the atomic rank of S. We denote this as $k = r(S)$.

Definition 13 *A solution ϕ of a quadratic equation $r_1 r_2 \dots r_k = g$ of atomic rank $k \geq 2$ is called commutative if $[r_i^{\phi}, r_{i+1}^{\phi}] = 1$ for all $i = 1, \dots, k-1$, otherwise it is called noncommutative.*

The following theorem describes the radical of a standard quadratic equation.

Theorem 3 *Let G be a fully residually free group and let $S = 1$ be a standard quadratic equation over G. Then*

1. *If either the atomic rank of $S = 1$ is greater than one and $S = 1$ has a noncommutative solution, or $S = 1$ is one of the following two equations: $[x, y]d^{-1} = 1$, or $[x, y][z, t] = 1$, then $Rad(S) = ncl(S)$;*

2. *If either all solutions of $S = 1$ are commutative and $S \neq [x, y][z, t]$, or $S = c^z d^{-1} = 1$, then up to some linear transformation of old variables into new variables u_i's $Rad(S) = ncl\{[u_i, b_i] = 1 | i = 1, \dots, k\}$, where b_i's are constants from G.*

3. *In the following cases $S = 1$ always has a noncommutative solution, hence $Rad(S) = ncl(S)$:*

   (a) *$S = 1$ is of type 1, $n > 2$,*

   (b) *$S = 1$ is of type 2, $n > 0, n + m > 1$ (in case $n = 1, m = 0$ the notion of noncommutative solution is not defined, but $Rad(S) = ncl(S)$),*

   (c) *$S = 1$ is of type 3, $n > 3$,*

   (d) *$S = 1$ is of type 4, $n > 2$.*

The proof of Theorem 1 will be given in Sections 2-6. In the remaining part of Section 1 it will be shown that characterizing the solutions of an arbitrary quadratic system of equations can be reduced to stadying a system of standard quadratic equations with the disjoint sets of variables. In Section 2 we will show that for every standard quadratic equation either there is a so-called solution in general position, or all solutions are commutative. We will also describe the radical in the case of all commutative solutions. In Sections 3-6 it will be proved by induction on the atomic rank that for every standard quadratic equation $S = 1$ that has a solution in a general position the group G_S is embeddable into $G(U, T)$, and hence $G_S = G_{R(S)}$.

Lemma 7 *Let S be a quadratic system over G. Then there exists a system $S' = \{s_1, \dots s_n\}$ of strictly quadratic pairwise disjoint equations $s_1, \dots s_n$ such that*

$$G_S \simeq_G G_{S'}.$$

Proof Suppose a letter $x \in var(S)$ occurs in some equation $s \in S$ exactly once. Then we can rewrite the equation $s = 1$ in the form $x = s'$. Hence the group G_S can be Tietze transformed to the G-isomorphic group G_{S^*}, where S^* is obtained from S by deleting the equation $s = 1$ and replacing all other occurrences of x by s'. After a finitely many transformations of this type we arrive to a system S' with G_S G-isomorphic to $G_{S'}$ and every variable from $var(S')$ occurring in one and only one equation from S' exactly twice. It follows that S' is a system of pairwise disjoint strictly quadratic equations over G. $\square$

The following result can be deduced from Lemma 6 and Lemma 7.

Lemma 8 *Let S be a quadratic system over G. Then there exists a system $S' = \{s_1, \ldots s_n\}$ of standard quadratic pairwise disjoint equations $s_1, \ldots s_n$ and a free group F such that*

$$G_S \simeq_G G_{S'} * F.$$

A system $S' = 1$ from the lemma above splits over G in such a way that

$$G_S \simeq_G (\ldots(G_{s_1})_{s_2} \ldots)_{s_n} * F.$$

Hence we have

Corollary 2 *Let $S = 1$ be a quadratic system over G. Then*

$$G_S \simeq_G (\ldots(G_{s_1})_{s_2} \ldots)_{s_n} * F,$$

where s_i is a standard quadratic equation over G and F_1 is some finitely generated free group.

Let $S : r_1\ r_2 \ldots r_k g^{-1} = 1$ be a standard quadratic equation of atomic rank k over the group G. Denote:

$$S_i = r_1 \ldots r_i, \quad R_i = r_{i+1} \ldots r_k.$$

Let $Q_i = var(S_i)$ and $P_i = var(R_i)$; then $X = Q_i \cup P_i$.

Notation: Let $H^{(i)}$ denote the subgroup of G_S generated by G and Q_i, i.e. $H^{(i)} = gp(G, Q_i)$.

Lemma 9 *Let $S = 1$ be a standard quadratic equation over a torsion free group G, and $rk(S) = k \geq 2$. Then $H^{(i)} \simeq G * F(Q_i)$ for any $i < k$.*

Proof It is sufficient to prove the lemma just for $i = k-1$. The equation $S = 1$ has the following form

$$r_1 r_2 \dots r_k g^{-1} = 1$$

which can be written as

$$r_k = S_{k-1}^{-1} g.$$

Obviously,

$$G_S \simeq \langle G, X \mid r_k = S_{k-1}^{-1} g \rangle.$$

Now we claim that G_S is either a free product with amalgamation or an HNN-extension with respect to the form of r_k. Notice that the element S_{k-1} (and hence S_{k-1}^{-1}) is in a reduced form as an element of $G * F(Q_{k-1})$. It begins and ends with different letters from Q_{k-1}, unless either $S_{k-1} = z^{-1}cz$ or $S_{k-1} = y^2$. Therefore, we have just two possibilities for the element $S_{k-1}^{-1}g$: either it is of length more then 1 in the free product $G * F(Q_{k-1})$, or it equals either $z^{-1}c^{-1}zg$ or $y^{-2}g$. In any case, $S_{k-1}^{-1}g$ is of infinite order in the $G * F(Q_{k-1})$. Now, if $r_k = [x_k, y_k]$, or $r_k = x_k^2$, then G_S is a free product with amalgamation

$$G_S \simeq \langle F(P_k) * (G * F(Q_{k-1})) \mid r_k = S_{k-1}^{-1} g \rangle$$

over two infinite cyclic subgroups $gp(r_k)$ in $F(P_k)$, and $S_{k-1}^{-1}g$ in $G * F(Q_{k-1})$. In this event $H^{(k-1)} \simeq G * F(Q_{k-1})$. If $r_k = z_k^{-1} c_k z_k$, then G_S is an HNN-extension

$$G_S \simeq \langle G * F(Q_{k-1}), z_k \mid z_k^{-1} c_k z_k = S_{k-1}^{-1} g \rangle$$

with associated infinite cyclic groups $gp(c_k)$ in G (notice that G is a torsion free group and $c \neq 1$) and $gp(S_{k-1}^{-1}g)$ in $G * F(Q_{k-1})$. Again, from the properties of HNN-extensions we know that $H^{(k-1)} \simeq G * F(Q_{k-1})$. □

Let $S = 1$ be a standard quadratic equation of atomic rank k over group G, i.e.

$$r_1 r_2 \dots r_k = g.$$

We can rewrite it in the form $S_i R_i = g$ or $R_i = S_i^{-1} g$. The element $h = S_i^{-1} g$ belongs to the group $G * F(Q_i) \simeq H^{(i)}$. Hence we can consider the initial equation $S = 1$ also as an equation $R_i = h$ over the group $H^{(i)}$ and denote this equation by $R_i h^{-1} = 1$; it is a standard quadratic equation of atomic rank $k - i$. It turns out that the group G_S of the equation $S = 1$ over G is G-isomorphic to the group $H^{(i)}_{R_i h^{-1}}$ of $R_i = h$ over $H^{(i)}$. Indeed, we have

Proposition 1 *Let $S = 1$ be a standard quadratic equation over G with atomic rank $k > 1$. Then using the notation above $G_S \simeq_G H^{(i)}_{R_i h^{-1}}$, where $R_i = h$ is the quadratic equation of atomic rank $k - i$ over the group $H^{(i)} = G * F(Q_i)$.*

Proof The proposition follows immediately from the fact that the groups G_S and $H^{(i)}_{R_i h^{-1}}$ have exactly the same presentations. $\square$

To be able to freely manipulate quadratic equations we shall need the following

Proposition 2 *Let $H \leq G$ be arbitrary torsion-free groups, and $S = 1$ a standard quadratic equation over H. Then H_S is canonically H-embeddable into G_S.*

Proof Let $S = 1$ be a standard quadratic equation of atomic rank k; say

$$r_1 r_2 \dots r_k g^{-1} = 1.$$

As in Lemma 9 we can decompose G_S, as well as H_S, into a free product with amalgamation or an HNN-extension. Namely, if $r_k = [x_k, y_k]$ or $r_k = x_k^2$ and $S_{k-1}^{-1} g \neq 1$ (this means that either $k > 1$, or $k = 1$; but then $g \neq 1$), so G_S is a free product with amalgamation

$$G_S \simeq \langle F(P_k) * (G * F(Q_{k-1})) \mid r_k = S_{k-1}^{-1} g \rangle$$

along two infinite cyclic subgroups $gp(r_k)$ in $F(P_k)$, and $S_{k-1}^{-1} g$ in $G * F(Q_{k-1})$. If $r_k = [x_k, y_k]$, or $r_k = x_k^2$ and $S_{k-1}^{-1} g = 1$, then

$$G_S \simeq \langle F(P_k) \mid r_k = 1 \rangle * (G * F(Q_{k-1})).$$

If $r_k = z_k^{-1} c_k z_k$, then $S_{k-1}^{-1} g \neq 1$ and, consequently, G_S is an HNN-extension

$$G_S \simeq \langle G * F(Q_{k-1}), z_k \mid z_k^{-1} c_k z_k = S_{k-1}^{-1} g \rangle.$$

It is clear that H_S has a similar decomposition; one just has to replace G by H. The embedding $H \hookrightarrow G$ gives rise to a H-homomorphism $f : H_S \to G_S$ which is identical on X. Since the groups H_S and G_S have similar decompositions into free product or HNN-extensions, it follows that every reduced form in H_S will have a reduced form as its image under f. Hence f is monic. $\square$

2. Splitting

Let G be a group, X a finite set of variables, and $S \in G[X]$.

Definition 14 *An equation $S = 1$ is termed* separable *if there exists a nontrivial partition of X into $k > 1$ pairwise disjoint subsets $X = X_1 \cup \ldots \cup X_k$ and elements $w_i(X_i) \in G[X_i]$ such that S can be written as*

$$S = w_1(X_1) \dots w_k(X_k).$$

By the definition a separable equation has the form $w_1(X_1)\dots w_k(X_k) = 1$. Sometimes it is convenient to write it also in the form $w_1(X_1)\dots w_k(X_k) = g$, where $g \in G$.

Notice that any standard quadratic equation over G is separable.

We now need some more definitions.

Definition 15 *Let $\phi : H[X] \longrightarrow H[X]$ be an H-endomorphism of the group $H[X]$. We say that an element $w \in H[X]$ is* an eigenvector *of ϕ if there exists an element $g \in H$ such that*

$$w^{\phi} = w^{g}.$$

We term g an eigenvalue *of w with respect to ϕ.*

This definition of an eigenvector becomes more natural if one thinks of $H[X]$ as a noncommutative H-space over the "ground field" H, in which case H-endomorphisms play the role of linear transformations. Let $End_H(H[X])$ be the set of all such H-endomorphisms.

Definition 16 *An element $w \in G[X]$ is termed to be of* complete spectrum *if every $g \in G$ is an eigenvalue of w with respect to a suitable $\phi \in End_G(G[X])$.*

Any word $w \in F(X)$ (considered as an element from $G[X]$) is an element of complete spectrum. All quadratic atoms are elements of complete spectrum; for example, the G-homomorphism $z \rightarrow zg$ maps c^z onto $(c^z)^g$. On the other hand, the quadratic word xbx has no eigenvalue a in the group $F[x]$, where $F = F(a, b)$ is the free group on a, b. This follows from the fact that the equation $yby = a^{-1}ba$ with one variable y has no solutions in F.

Definition 17 *Let $w_1(X_1)\dots w_k(X_k) = g$ be a separable equation over the group G. We say that a solution ϕ of this equation is:*

a) in general position *if $[w_i^{\phi}, w_{i+1}^{\phi}] \neq 1$ for every $i = 1, \dots, k-1$;*

b) commutative, *if $[w_i^{\phi}, w_{i+1}^{\phi}] = 1$ for every $i = 1, \dots, k-1$;*

c) degenerate, *if $w_{i_0}^{\phi} = 1$ for some $1 \leq i_0 \leq k$.*

One may think of solutions in general position as those which give linearly independent neighbors in the sequence $w_1^{\phi}, \dots, w_k^{\phi}$ as elements in the Lie algebra associated to the group G. The commutative solutions result in all neighbors in the sequence as above being " collinear to each other".

Proposition 3 *Let* $S : w_1(X_1)\dots w_k(X_k) = g, \quad k \geq 2$, *be a nondegenerate separable equation over a CSA-group* G, *and let all elements* w_i *be of complete spectrum in* $G[X_i]$. *Then either* $S = g$ *has a solution in general position, or every nondegenerate solution of* $S = g$ *is commutative.*

Proof For $k = 2$ the statement of the proposition is obvious. Let $k = 3$ and suppose that both statements a) and b) do not hold for some equation of the type

$$S : w_1(X_1)w_2(X_2)w_3(X_3) = g.$$

That would imply that there exists a nondegenerate solution $\phi : G_S \longrightarrow G$ such that either

$$[w_1^\phi, w_2^\phi] = 1 \;\; and \;\; [w_2^\phi, w_3^\phi] \neq 1$$

or

$$[w_1^\phi, w_2^\phi] \neq 1 \;\; and \;\; [w_2^\phi, w_3^\phi] = 1.$$

Lemma 10 *Let* $S : w_1(X_1)w_2(X_2)w_3(X_3) = g$ *be a separable equation over a CSA-group* G *and the words* w_i *be of complete spectrum in* $G[X_i]$. *If* $\phi : G_S \longrightarrow G$ *is a nondegenerate solution of this equation such that either*

$$[w_1^\phi, w_2^\phi] = 1 \;\; and \;\; [w_2^\phi, w_3^\phi] \neq 1 \tag{5}$$

or

$$[w_1^\phi, w_2^\phi] \neq 1 \;\; and \;\; [w_2^\phi, w_3^\phi] = 1, \tag{6}$$

then there exists a solution $\psi : G_S \longrightarrow G$ *in general position.*

Proof Suppose we have the condition (5):

$$[w_1^\phi, w_2^\phi] = 1 \;\; and \;\; [w_2^\phi, w_3^\phi] \neq 1$$

for some nondegenerate solution ϕ. To simplify the formulas we will use the following notation $u^{-\phi} = (u^{-1})^\phi$.

Let $t = (w_2^\phi w_3^\phi)^N$, where N is a big positive integer which will be specified in due course. Define $\psi_i \in End_G(G[X_i])$ $(i = 2, 3)$to be a G- endomorphism such that w_i is the eigenvector of ψ_i with the eigenvalue t. Define $\psi_1 \in End_G(G[X_1])$ to be equal to ϕ on X_1. This ψ_i exists because w_i is of complete spectrum.

$$w_2^{\psi_2} = t^{-1}w_2t, \quad w_3^{\psi_3} = t^{-1}w_3t.$$

Since the equation $S = g$ is separable the sets of variables X_1, X_2, X_3 are pairwise disjoint, therefore we can construct a G- endomorphism $\psi^* \in End_G(G[X_1 \cup X_2 \cup X_3])$ such that the restriction of ψ^* on each $G[X_i]$ is equal to ψ_i $(i = 1,2,3)$. Now let $\psi = \phi \circ \psi^*$ be the composition of ϕ and ψ^*, in particular $\psi : G[X_1 \cup X_2 \cup X_3] \longrightarrow G$ is a G-homomorphism.

Compute the image of $w_1w_2w_3$ under ψ :

$$(w_1w_2w_3)^\psi = w_1^\phi t^{-1}(w_2^\phi w_3^\phi)t =$$

$$(w_1w_2w_3)^\phi = g$$

hence ψ induces a solution $\psi : G_S \longrightarrow G$. Moreover,

$$[w_2^\psi, w_3^\psi] = [w_2^\phi, w_3^\phi]^t \neq 1$$

since $[w_2^\phi, w_3^\phi] \neq 1$. We claim that $[w_1^\psi, w_2^\psi] \neq 1$. Suppose to the contrary, that $[w_1^\psi, w_2^\psi] = 1$. Then:

$$[w_1^\psi, w_2^\psi] = [(w_1^\phi), (w_2^\phi)^t] = 1,$$

therefore by malnormality of abelian subgroups in a CSA group G one has $[w_2^\phi, t] = 1$. This implies that $[w_3^\phi, t] = 1$, and by the transitivity of commutation $[w_2^\phi, w_3^\phi] = 1$. This is a contradiction.

In case (6) we can consider the inverse equation $w_3(X_3)^{-1}w_2(X_2)^{-1}w_1(X_1)^{-1} = g^{-1}$ for which ϕ is a solution too, and in this case ϕ satisfies the condition (5), but for the new equation. Now, according to the argument above, we can construct the solution α in general position to the new equation. This α is a solution of the initial equation too, and clearly α is still in general position. $\square$

Now suppose that $k > 3$, and let $\phi : G_S \longrightarrow G$ be a nondegenerate solution which is neither in general position nor commutative. Then there exists an index $1 \leq i \leq k-3$ such that the triple w_i, w_{i+1}, w_{i+2} satisfies either (5) or (6) (after replacing index i by 1). But then, according to Lemma 10, we can construct a G-homomorphism $\psi : G[X_i \cup X_{i+1} \cup X_{i+2}] \longrightarrow G$ such that $\psi(w_i) = w_i^\phi$,$\psi(w_{i+1}) = w_{i+1}^{\phi t}, \psi(w_{i+2}) = w_{i+2}^{\phi t}$; here $t = (w_{i+1}^\phi w_{i+2}^\phi)^N$,

$$(w_{i+1}w_{i+2})^\psi = (w_{i+1}w_{i+2})^\phi$$

and moreover,

$$[w_i^\psi, w_{i+1}^\psi] \neq 1, \quad and \quad [w_{i+1}^\psi, w_{i+2}^\psi] \neq 1.$$

Extending ψ to $G[X_1 \cup \ldots \cup X_k]$ by

$$x^\psi = x^\phi, \quad for\ all\ x \in X_j, \quad where\ j \neq i, i+1, i+2;$$

we see that

$$(w_1 \dots w_k)^{\psi} = w_1^{\phi} \dots w_i^{\phi} (w_{i+1} w_{i+2})^{\phi} w_{i+3}^{\phi} \dots w_k^{\phi} = (w_1 \dots w_k)^{\phi} = g,$$

and hence ψ is a solution of $S = g$. Notice that

$$[w_j^{\psi}, w_{j+1}^{\psi}] = [w_j^{\phi}, w_{j+1}^{\phi}] \quad for \quad j \neq i, i+1, i+2,$$

and

$$[w_i^{\psi}, w_{i+1}^{\psi}] \neq 1, \quad [w_{i+1}^{\psi}, w_{i+2}^{\psi}] \neq 1.$$

We have to show that $[w_{i+2}^{\phi t}, w_{i+3}^{\phi}] \neq 1$ if $[w_{i+2}^{\phi}, w_{i+3}^{\phi}] \neq 1$. Suppose on the contrary that $[w_{i+2}^{\phi}, w_{i+3}^{\phi}] \neq 1$ and $[w_{i+2}^{\phi t}, w_{i+3}^{\phi}] = 1$.

Rewrite this equality

$$t^{-1} w_{i+2}^{-\phi} t w_{i+3}^{-\phi} t^{-1} w_{i+2}^{\phi} t w_{i+3}^{\phi} = 1.$$

If N is big enough we have to have either $[w_{i+2}^{\phi}, t] = 1$, or $[w_{i+3}^{\phi}, t] = 1$. If $[w_{i+2}^{\phi}, t] = 1$ holds, then $[w_{i+1}^{\phi}, t] = 1$ and $[w_{i+2}^{\phi}, w_{i+1}^{\phi}] = 1$, a contradiction.

If $[w_{i+3}^{\phi}, t] = 1$, then by commutation transitivity $[w_{i+2}^{\phi t}, t]$=1 and hence $[w_{i+2}^{\phi}, t] = 1$, and $[w_{i+2}^{\phi}, w_{i+3}^{\phi}] = 1$, a contradiction.

Thus we refine the solution ϕ to a solution ψ which has less then ϕ commuting neighbors of the type $w_j^{\psi}, w_{j+1}^{\psi}$. Repeating the process we arrive at a solution in general postion. It follows that the equation $S = g$ either has a solution in general postion or all nondegenerate solutions are commutative. $\square$

Now we can formulate several corollaries for standard quadratic equations and their radicals. We will consider all three cases separately: orientable of genus ≥ 1, genus $=$ 0, and non-orientable of genus ≥ 1.

Proposition 4 *Let $S : \prod_{i=1}^{i=m} [x_i, y_i] \prod_{j=1}^{j=n} c_j^{z_j} = g$ $(m \geq 1, n \geq 0)$ be a nondegenerate standard quadratic equation over a residually free group G. Then $S = g$ has a solution in general position unless $S = g$ is the equation $[x_1, y_1][x_2, y_2] = 1$ or $[x, y]c^z = 1$.*

Proof Let $n = 0$. In this event we have a standard quadratic equation of the type

$$[x_1, y_1] \dots [x_k, y_k] = g,$$

which we will sometimes write as $r_1 \dots r_k = g$, where, as before, $r_i = [x_i, y_i]$.

Lemma 11 *Let $S : [x_1, y_1][x_2, y_2] = g$ be a nondegenerate equation over a nonabelian fully residually free group G. Then $S = g$ has a solution in general position unless $S = g$ is the equation $[x_1, y_1][x_2, y_2] = 1$.*

Proof Suppose $S = g$ has a solution ϕ such that $r_1^\phi = 1$ and $r_2^\phi = 1$. Then $g = 1$ and our equation takes the form

$$[x, y][x_2, y_2] = 1. \tag{7}$$

From now on we assume that for all solutions ϕ either $r_1^\phi \neq 1$ or $r_2^\phi \neq 1$.

Suppose now that just one of the equalities $r_i^\phi = 1$ $(i = 1, 2)$ takes place, say $r_2^\phi = 1$. Write $x^\phi = a$, and $y^\phi = b$. Then the equation is in the form

$$[x, y][x_2, y_2] = [a, b] \neq 1.$$

This equation has other solutions for example,

$$\psi : \quad x \to ab^{-1}a, \; y \to aba^{-1}, \; x_2 \to ab^{-1}aba^{-1} = a[b, a^{-1}], \; y_2 \to ab^{-1}a \tag{8}$$

for which

$$r_1^\psi = [a^2, b]^{a^{-1}} \neq 1 \;\; and \;\; r_2^\psi = [b, a]^{a^{-1}} \neq 1.$$

We claim, that for this particular solution ψ we have $[r_1^\psi, r_2^\psi] \neq 1$. Indeed, $[r_1^\psi, r_2^\psi] = 1$ if and only if $[[a^2, b], [b, a]] = 1$, but the elements a, b freely generate a free subgroup in G, and $[[a^2, b], [b, a]] \neq 1$ in $F(a, b)$.

Thus, just one case is left to consider. Suppose that $[r_1^\phi, r_2^\phi] = 1$ and $r_i^\phi \neq 1$ $(i = 1, 2)$ for all solutions ϕ. To treat this case we need some facts about commutators in a fully residually free group, which will also be of use later. □

Lemma 12 *Let G be a fully residually free group. If two nontrivial commutators $[a, b]$ and $[c, d]$ commute in G, then*

$$[a, b] = [c, d] \;\; or \;\; [a, b] = [c, d]^{-1}.$$

Proof A nontrivial commutator in a free group is never a proper power, this is a result of Schutzenberger [15]. Hence, in a free group any two nontrivial commuting commutators should be in the same cyclic subgroup and, since they are not proper powers, they should be generators of the cyclic subgroup. Consequently, they satisfy one of the equalities from the lemma. Now, suppose there are two nontrivial commuting commutators in G which do not satisfy any of the equalities from the lemma; then we could approximate these inequalities into a free group – this is a contradiction. □

According to Lemma 12 a priori we have two possibilities for the solution ϕ. But if at least for one solution ϕ we have $r_1^\phi = (r_2^\phi)^{-1}$, then $g = 1$, and we have just the exceptional equation $[x_1, y_1][x_2, y_2] = 1$. So, now we can make an auxiliary assumption that $r_1^\phi = r_2^\phi$ for all solutions ϕ. In this event $g = [a, b]^2$ for some $a, b \in G$ and the equation actually takes the form

$$[x_1, y_1][x_2, y_2] = [a, b]^2.$$

Consider the map ξ :

$$x_1^\xi = ab^{-1}a, \quad y_1^\xi = aba^{-1}, \quad x_2^\xi = ab^{-1}aba^{-1}, \quad y_2^\xi = a.$$

Then:

$$r_1^\xi = [x_1, y_1]^\xi = [ab^{-1}a, aba^{-1}] = a^{-1}b^{-1}a^2ba^{-1};$$

and

$$r_2^\xi = [x_2, y_2]^\xi = [ab^{-1}aba^{-1}, a] = ab^{-1}a^{-1}b[a, b].$$

It follows, that

$$r_1^\xi r_2^\xi = a^{-1}b^{-1}a^2ba^{-1}ab^{-1}a^{-1}b[a, b] = a^{-1}b^{-1}ab[a, b] = [a, b]^2,$$

i.e. ξ is a solution of the equation. We claim, that $[r_1^\xi, r_2^\xi] \neq 1$. Indeed, as we saw $r_1^\xi r_2^\xi = [a, b]^2$, but

$$r_2^\xi r_1^\xi = ab^{-1}a^{-1}b[a, b]a^{-1}b^{-1}a^2ba^{-1}.$$

Taking a homomorphism from G into a free group in such a way that the images of a and b do not commute, we can assume from the beginning that a and b freely generate a free subgroup in G. Then the condition $[r_1^\xi, r_2^\xi] = 1$ would imply that

$$[a, b]^2 = ab^{-1}a^{-1}b[a, b]a^{-1}b^{-1}a^2ba^{-1}$$

in the free group $F(a, b)$. That is not the case since the two words are reduced and graphically different.

Thus, we have proved that either $S = 1$ has the form $[x_1, y_1][x_2, y_2] = 1$, or there is a solution ψ such that $[r_1^\psi, r_2^\psi] \neq 1$. $\square$

Lemma 13 *Let $S : [x_1, y_1] \ldots [x_k, y_k] = g$ be a nondegenerate equation over a non-abelian fully residually free group G and assume that $k \geq 3$. Then $S = g$ has a solution in general position.*

Proof The proof will follow by induction on k.

Let $k = 3$. Assume first that $g = 1$. This means we have the equation

$$[x_1, y_1][x_2, y_2][x_3, y_3] = 1.$$

It has a solution $\phi : G_S \longrightarrow G$ such that

$$x_1^\phi = a,\ y_1^\phi = b,\ x_2^\phi = b,\ y_2^\phi = a, x_3^\phi = y_3^\phi = 1,$$

where a, b are arbitrary noncommuting elements from G. Therefore the equation

$$[x_2, y_2][x_3, y_3] = [b, a]$$

is nondegenerate of atomic rank 2; hence by the lemma above it has a solution ξ such that $[r_2^\xi, r_3^\xi] \neq 1$. Now we can combine the solutions ϕ and ξ to get the solution θ, namely, let

$$x_1^\theta = a,\ y_1^\theta = b,\ x_i^\theta = x_i^\xi,\ y_i^\theta = y_i^\xi, \quad for\ i = 2, 3.$$

We have proved that the equation $S = g$ has a solution θ such that $r_i^\theta \neq 1$ $(i = 1, 2, 3)$ and $[r_2^\theta, r_3^\theta] \neq 1$. Now we are in a position to apply Proposition 3. It follows that there exists a solution ψ to $S = g$ in general position.

Assume now that $g \neq 1$. Then there exists a solution ϕ such that for at least one i we have $r_i^\phi \neq 1$. Renaming variables we can assume that exactly $r_3^\phi = [a, b] \neq 1$. Then the equation

$$r_1 r_2 = g[b, a]$$

has a solution in G. Again we have two cases: if $g[b, a] \neq 1$ then we can argue as above; if $g[b, a] = 1$ then $g = [a, b]$ and the initial equation $S = g$ actually has the form

$$r_1 r_2 r_3 = [a, b].$$

In this event, consider a solution $\xi : G_S \longrightarrow G$ such that:

$$x_1^\xi = a^2,\ y_1^\xi = b,\ x_2^\xi = b,\ y_2^\xi = a^2,\ x_3^\xi = a,\ y_3^\xi = b.$$

We see that $r_i^\xi \neq 1$ for all $i = 1, 2, 3$, and

$$[r_2^\xi, r_3^\xi] = [[b, a^2], [a, b]] \neq 1$$

in the free group generated by a, b. By Proposition 3 there exists a solution ψ to $S = g$ in general position. We are done.

Let $k > 3$. The equation

$$r_1 \dots r_k = g$$

has a solution ϕ such that at least for one i, say $i = k$ (by renaming variables we can always assume this), we have $r_k^\phi = [a, b] \neq 1$. Then the equation

$$r_1 \dots r_{k-1} = g[b, a]$$

is nondegenerate and by induction (notice that $k \geq 3$) there is a solution ξ such that $[r_i^\xi, r_{i+1}^\xi] \neq 1$ for all $i = 1, \dots, k-1$. Extend this ξ to a solution of the initial equation $S = g$ defining $x_k^\xi = a,\ y_k^\xi = b$. Now by Lemma 10 we can refine ξ on the last three atoms r_{k-2}, r_{k-1}, r_k and get a solution ψ such that $[r_i^\psi, r_{i+1}^\psi] \neq 1$ for all $i = 1, \dots, k-1$, i.e. a solution in general position.

Let $n \geq 1$.

Let $m = 1$. In the event when $n = 1$ we have the following

Lemma 14 *The equation $S : [x, y]c^z = g$ over a nonabelian fully residually free group G always has a solution in general position provided $g \neq 1$.*

Proof Let $x \to a,\ y \to b,\ z \to d$ be an arbitrary solution of $[x, y]c^z = g$, where $g \neq 1$. Then $g = [a, b]c^d$ and the equation takes the form

$$[x, y]c^z = [a, b]c^d.$$

We can assume that $[a, b] \neq 1$. Indeed, suppose $[a, b] = 1$. If $[c, d] \neq 1$, then we can write the equation as

$$[x, y]c^z = c^d = [d, c^{-1}]c$$

which has the solution $x \to d,\quad y \to c^{-1},\quad z \to 1$ such that $[x, y] \to [d, c^{-1}] \neq 1$. So we can assume now that $[c, d] = 1$, in which case we have the equation

$$[x, y]c^z = c \quad \textit{or equivalently} \quad [x, y] = [c^{-1}, z].$$

The group G is a nonabelian CSA-group; hence the center of G is trivial. In particular, there exists an element $g \in G$ such that $[c, g] \neq 1$. We see that $x \to c^{-1},\ y \to g,\ z \to g$ is a solution ϕ for which $[x, y]^\phi \neq 1$.

Thus we have the equation $[x, y]c^z = [a, b]c^d$, where $[a, b] \neq 1$. Consider the map ψ defined as follows:

$$x^\psi = (bc^d)^{-1}a, \quad y^\psi = (bc^d)^{-1}b(bc^d), \quad z^\psi = d(bc^d).$$

Strightforward computations show that

$$[x,y]^{\psi}=[a,b][b,c^{d}],\quad and\quad (c^{z})^{\psi}=[c^{d},b]c^{d};$$

hence

$$[x^{\psi},y^{\psi}]c^{z^{\psi}}=[a,b][b,c^{d}][c^{d},b]c^{d}=[a,b]c^{d}$$

and consequently, ψ is a solution.

We claim that $[r_1^{\psi},r_2^{\psi}]\neq 1$. Indeed, suppose $[r_1^{\psi},r_2^{\psi}]=1$; then we have

$$[[x,y]^{\psi},c^{z^{\psi}}]=1,\ [[a,b],c^{d}]=1,\quad and\ \ [x^{\psi},y^{\psi}]c^{z^{\psi}}=[a,b]c^{d}$$

which implies that in the following sequence of elements in G all neighbors commute

$$c^{z^{\psi}},\ c^{z^{\psi}}[x,y]^{\psi},\ [a,b]c^{d},\ c^{d}.$$

The group G is commutative transitive; so all elements in this sequence pairwise commute, in particular, $[c^{z^{\psi}},c^{d}]=1$. This implies $[c^{z^{\psi}d^{-1}},c]=1$, and consequently, $[z^{\psi}d^{-1},c]=1$ (the last implication comes from the malnormality of centralizers in G). Thus,

$$[z^{\phi}d^{-1},c]=1,$$

which implies that

$$1=[dbc^{d}d^{-1},c]=[dbd^{-1}c,c]=[dbd^{-1},c]\Longrightarrow[b,c^{d}]=1.$$

From transitivity of commutation we obtain that $[[a,b],b]=1$, but this contradicts the fact that the subgroup $gp(a,b)$ is freely generated by a and b. Hence, we have found the solution ψ such that $[r_1^{\psi},r_2^{\psi}]\neq 1$. $\Box$

Now suppose that $n>1$. Let $\phi:G_S\longrightarrow G$ be an arbitrary solution of $S=g$. Write

$$h=g(\prod_{j=3}^{n}c_j^{z_j})^{-\phi}$$

and consider the equation

$$[x,y]c_1^{z_1}c_2^{z_2}=h. \tag{9}$$

If this equation satisfies the conclusion of the proposition, then by Proposition 3 the equation $S=g$ will satisfy the conclusion. So we need to prove the proposition just for the equation (9). There are now just two possibilities. Case a) There exists a solution ξ of the equation (9) such that $(c_2^{z_2})^{\xi} \neq h$. In this event by Lemma 14 the equation

$$[x, y]c_1^{z_1} = h(c_2^{z_2})^{-\xi} \neq 1$$

has a solution θ in general position. Hence we can extend this θ to a solution of (9) in such a way that $r_i^{\theta} \neq 1$ for $i = 1, 2$ and $[r_1^{\theta}, r_2^{\theta}] \neq 1$. Consequently, by Proposition 3 we can construct a solution ψ in general position.

Case (b) Assume now, that $c_2^{z_2\phi} = h$ for all solutions ϕ of the equation (9). Then we actually have

$$[x, y]c_1^{z_1} = 1, \quad \textit{and} \quad c_2^{z_2} = h,$$

and this system of equations has a solution in G. It follows that $c_1 = [a, b] \neq 1$ for some $a, b \in G$. Therefore the equation (9) is in the form

$$[x, y][a, b]^{z_1} c_2^{z_2} = h,$$

and has a solution ψ of the type

$$x^{\psi} = b^f, \ y^{\psi} = a^f, \ z_1^{\psi} = f, \ z_2^{\psi} = z_2^{\phi}$$

where f is an arbitrary element in G and ϕ is an arbitrary solution of (9). The two elements $[a, b]$ and h are nontrivial in the CSA-group G hence there exists an element $f^* \in G$ such that $[[a, b]^{f^*}, h] \neq 1$. But this implies that if we take $f = f^*$ then the solution ψ will have the property $[r_2^{\psi}, r_3^{\psi}] \neq 1$. Now it is sufficient to apply Proposition 3.

The case $m = 2$. In this event we have the equation

$$[x_1, y_1][x_2, y_2] \prod_{j=1}^{j=n} c_j^{z_j} = g.$$

Again, if there exists a solution ϕ of this equation such that

$$(\prod_{j=1}^{j=n} c_j^{z_j})^{\phi} \neq g,$$

then we can write

$$h = g(\prod_{j=1}^{j=n} c_j^{z_j})^{-\phi},$$

and consider the equation

$$[x_1, y_1][x_2, y_2] = h$$

which according to Proposition 3 has a solution ξ in general position. We can extend it to a solution of $S = g$ and by Proposition 3 we can construct a solution ψ in general position.

Let assume now that

$$(\prod_{j=1}^{j=n} c_j^{z_j})^{\phi} = g$$

for all solutions ϕ of the equation $S = g$. This implies that an arbitrary map of the type

$$x_1 \to a,\ y_1 \to b,\ x_2 \to b,\ y_2 \to a$$

extends by means of any ϕ above to a solution ψ of the equation $S = g$. Choose $a, b \in G$ such that $[[b, a], r_3^{\phi}] \neq 1$ for the given solution ϕ. This is always possible, because either $[b, a]$ or $[b^2, a]$ does not commute with r_3^{ϕ} in the fully residually free group G provided $[a, b] \neq 1$. And we again just need to appeal to Proposition 3.

The case $m > 2$ is easy since if ϕ is a solution of this equation, then we can consider the equation

$$\prod_{i=1}^{i=m} [x_i, y_i] = g(\prod_{j=1}^{j=n} c_j^{z_j})^{-\phi}$$

which by Proposition 3 has a solution in general position; after that to finish the proof we need only apply Proposition 3. □

Proposition 5 *Let $S : c_1^{z_1} \dots c_k^{z_k} = g$ be a nondegenerate standard quadratic equation over a fully residually free group G. Then either $S = g$ has a solution in general position or every solution of $S = g$ is commutative.*

Proof By the definition of a standard quadratic equation $c_i \neq 1$ for all $i = 1, \dots, k$. Hence every solution of $S = g$ is a nondegenerate. Now the result follows from Proposition 3. □

The following simple lemma will be of use throughout the paper.

Lemma 15 *Let G be a group, $S = 1$ be a system over G with variables from X and $w \in G[X]$. If $w^{\phi} = 1$ for every solution $\phi : G_S \longrightarrow G$, then $S \sim S \cup \{w\}$ over G.*

Proof If $w^{\phi} = 1$ for every solution $\phi : G_S \longrightarrow G$, then w belongs to the radical $Rad(S)$, therefore $S \sim S \cup \{w\}$ over G. □

Proposition 6 *Let $S : c_1^{z_1} \dots c_k^{z_k} = g$ $(k \geq 2)$ be a nondegenerate standard quadratic equation over a CSA-group G such that all of its solutions are commutative. Then $S = g$ splits over G:*

$$S = g \sim_G \{[z_i a_i^{-1}, c_i] = 1 \mid i = 1, \dots, k\},$$

where $a_i \in G$.

Moreover, in this case, up to some linear transformations of variables $z_i \to g_i z_i h_i = u_i$, where $g_i, h_i \in G$, we can rewrite the system in the form

$$S' = \{[u_i, b_i] = 1 \mid i = 1, \dots, k\},$$

where $b_i \in G$ and $[b_i, b_j] = 1$ for every i, j.

Proof Notice, that $g \neq 1$, otherwise the equation $S = g$ is not standard. Fix an arbitrary solution, say $z_i \to a_i$, $i = 1, \dots, k$. Then $g = c_1^{a_1} \dots c_k^{a_k}$ and $[c_i^{a_i}, c_{i+1}^{a_{i+1}}] = 1$.

Let $\phi : G_S \longrightarrow G$ be an arbitrary solution of $S = g$. In the CSA-group G we have

$$[c_i^{z_i^\phi}, c_{i+1}^{z_{i+1}^\phi}] = 1, \ [c_i^{a_i}, c_{i+1}^{a_{i+1}}] = 1, \ c_1^{z_1^\phi} \dots c_k^{z_k^\phi} = c_1^{a_1} \dots c_k^{a_k}.$$

From transitivity of commutation in G we deduce that for any i, j :

$$[c_i^{z_i^\phi}, c_j^{z_j^\phi}] = 1, \ [c_i^{a_i}, c_j^{a_j}] = 1.$$

Again, from transitivity of commutation in G (notice that $g = c_1^{a_1} \dots c_k^{a_k} \neq 1$) we obtain

$$[c_i^{z_i^\phi}, c_i^{a_i}] = 1$$

and from malnormality of centralizers in G we see that

$$[z_i^\phi a_i^{-1}, c_i] = 1, \quad i = 1, \dots, k,$$

for *all solutions* ϕ. By Lemma 15

$$S = g \sim_G \{c_1^{z_1} \dots c_k^{z_k} = c_1^{a_1} \dots c_k^{a_k}, [z_i a_i^{-1}, c_i] = 1, \ i = 1, \dots, k\} = S_1.$$

The equation $[z_i a_i^{-1}, c_i] = 1$ can be rewritten as $c_i^{z_i} = c_i^{a_i}$; this allows us to eliminate the initial equation $S = g$ from S_1. Thus finally,

$$S = g \sim_G \{[z_1 a_1^{-1}, c_1] = 1, \dots, [z_k a_k^{-1}, c_k] = 1\} = S_1.$$

Moreover, $[c_1^{a_1}, c_j^{a_j}] = 1$; therefore $[c_1, c_j^{a_j a_1^{-1}}] = 1$. If we conjugate the equation $[z_j a_j^{-1}, c_j] = 1$ by $d_j = a_j a_1^{-1}$, we obtain

$$S = g \sim_G \{[d_j^{-1} z_j a_j^{-1} d_j, c_j^{d_j}] = 1 \mid j = 1, \ldots, k\} = S_1.$$

Renaming $u_j = d_j^{-1} z_j a_j^{-1} d_j$ and $b_j = c_j^{d_j}$ we can rewrite the system so that

$$S = g \sim_G \{[u_1, b_1] = 1, \ldots, [u_k, b_k] = 1\},$$

where $[b_i, b_j] = 1$. This finishes the proof. $\square$

It easy to describe the radical of the system $S' = 1$ from the proposition above.

Proposition 7 *Let $S' = \{[u_i, b_i] = 1 \mid i = 1, \ldots, k\}$ be a system (with indeterminates u_i) over a CSA-group G such that $b_i \in G$ and $[b_i, b_j] = 1$ for every i, j. Then the radical of the system S' is the normal closure of the following system*

$$S^* = \{[u_i, C] = 1, [u_i, u_j] = 1 \mid i, j = 1, \ldots, k\}$$

where $C = C_G(b_1, \ldots, b_k)$. Moreover, $G_{Rad(S)} = G_{S^*}$ is an extension of the centralizer C in G.

Proof The system S^* must follow from S' because G and, consequently $G_{R(S)}$, are CSA-groups. On the other hand, G_{S^*} is an extension of the centralizer C; hence it is fully residually G. So the radical $Rad(S)$ coincides with the normal closure of S^* in $G[u_1, \ldots, u_k]$. $\square$

Corollary 3 *Let $S : c_1^{z_1} \ldots c_k^{z_k} = g$ be a nondegenerate standard quadratic equation over a CSA-group G such that in the case $k \geq 2$ all solutions of it are commutative. Then the radical $Rad(S)$ is equal to*

$$\{[a_j^{-1} z_j, C] = 1, [a_i^{-1} z_i, a_j^{-1} z_j] = 1 \mid i, j = 1, \ldots, k\},$$

where $z_i \to a_i$ is a solution of $S = g$ and $C = C_G(c_1^{a_1}, c_2^{a_2}, \ldots, c_k^{a_k})$. Moreover, $G_{R(S)}$ is an extension of the centralizer C in G.

The proof follows from Proposition 6 and Proposition 7.

Proposition 8 *Let $S : x_1^2 \ldots x_p^2 c_1^{z_1} \ldots c_k^{z_k} = g$ be a nondegenerate standard quadratic equation over residually free group G. Then*

1. If $p \geq 2$, then there is always a nondegenerate solution.

2. If $p = 1$, then either there is a nondegenerate solution or $x_1^\psi = 1$ for any solution ψ and the radical of S is the same as the radical of system $S : c_1^{z_1} \dots c_k^{z_k} = g$.

3. If $k = 0, g = 1$ and $p \geq 4$ or $k = 0, g \neq 1$ and $p \geq 3$ or $k \neq 0$ and $p \geq 3$, then there is always a solution in general position.

4. If $p \geq 2$, then either there is a solution in general position or all solutions are commutative and $G_{R(S)}$ is an extension of a centralizer.

Proof 1) All quadratic atoms of the form $z_i^{-1}c_iz_i$ are nontrivial. Suppose $\psi(x_i) = a, \psi(x_{i+1}) = 1$. Then we can take another solution ϕ , such that $\phi(x_i) = a^2, \phi(x_{i+1}) = a^{-1}$ and $\phi(x_j) = \psi(x_j)$ for all $j \neq i$, $\phi(z_k) = \psi(z_k)$.

This solution has fewer trivial atoms.

Suppose $\psi(x_i) = 1, \psi(x_{i+1}) = 1$. Then again we can take another solution ϕ , such that for some a $\phi(x_i) = a, \phi(x_{i+1}) = a^{-1}$ and $\phi(x_j) = \psi(x_j)$ for all $j \neq i$, $\phi(z_k) = \psi(z_k)$.

This solution again has fewer trivial atoms.

2) Trivial.

3) Suppose there is a solution such that all the atoms commute. If there is a solution ϕ such that $x_1^{2\phi} \dots x_q^{2\phi} = s_1^2 \dots s_q^2 \neq 1$, for some $q \geq 3$, then there is another solution $\psi(x_1) = b, \psi(x_2) = b^{-1}$, where $[b, s_i] \neq 1$, $\psi(x_3) = s_1s_2s_3, \psi(x_i) = \phi(x_i)$, for $i \neq 1, 2, 3$ and $\psi(z_k) = \phi(z_k)$. For $p \geq 4$ and for the case $p \geq 3, k = 0, g \neq 1$ such a solution ϕ always exists. If $p = 3$ and $x_1^{2\phi}x_2^{2\phi}x_3^{2\phi} = 1$, then take an element b which does not commute with any conjugate of c_1 and put $\psi(x_1) = b^2, \psi(x_2) = \psi(x_3) = b^{-1}$.

4) If there is a solution with two noncommuting atoms, then by 1) there is a solution in general position. All we have to prove is that if all solutions are commutative then $G_{R(S)}$ is an extension of a centralizer.

Fix an arbitrary nontrivial solution say $z_i \to a_i,\ i = 1, \dots, k,\ x_i \to s_i,\ i = 1, \dots, p$. Then $g = s_1^2 \dots s_p^2 c_1^{a_1} \dots c_k^{a_k}$ and from transitivity of commutation in G we have $[c_i^{a_i}, c_j^{a_j}] = 1, [c_i^{a_i}, s_j] = 1, [s_k, s_j] = 1$.

Let $\phi : G_S \longrightarrow G$ be an arbitrary solution of $S = g$. Then

$$[c_i^{z_i^\phi}, c_j^{z_j^\phi}] = 1,\ [c_i^{z_i^\phi}, x_j^\phi] = 1, [x_j^\phi, x_k^\phi] = 1.$$

Again, from transitivity of commutation in G (notice that $g = s_1^2 \dots s_p^2 c_1^{a_1} \dots c_k^{a_k} \neq 1$, otherwise the equation is not standard) we obtain

$$[c_i^{z_i^\phi}, c_i^{a_i}] = 1,$$

and from malnormality of centralizers in G we see that

$$[z_i^\phi a_i^{-1}, c_i] = 1, \quad i = 1, \dots, k,$$

for *all solutions* ϕ.

The equation $[z_i^\phi a_i^{-1}, c_i] = 1$ can be rewritten as $c_i^{z_i} = c_i^{a_i}$. So we have $x_1^{2\phi} \dots x_p^{2\phi} = s_1^2 \dots s_p^2$. In a residually free group G this equality implies $x_1^\phi \dots x_p^\phi = s_1 \dots s_p$. Hence

$$S = g \sim_G \{x_1^\phi \dots x_p^\phi = s_1 \dots s_p, [a_i^{-1} z_i, c_i^{a_i}] = 1,\ i = 1, \dots, k, [c_i^{a_i}, x_j] = 1, [x_i, x_j] = 1, [x_i, s_j] = 1\} = S_1.$$

Write $u_i = a_i^{-1} z_i$. Then the radical of the system S_1 is the normal closure of the following system

$$S^* = \{x_1 \dots x_p = s_1 \dots s_p, [u_i, C] = 1, [u_i, u_j] = 1, i, j = 1, \dots, k,\ [u_i, x_j] = 1, i = 1, \dots, k,$$

$$j = 1, \dots, p, [x_i, x_j] = 1, i, j = 1, \dots, p\},$$

where $C = C_g(c_1^{a_1}, \dots, c_k^{a_k}, s_1 \dots s_p)$. The group $G_{R(S)}$ is just the extension of a centralizer C.

The proposition is proved.

3. Atomic rank 1

In this section we prove Theorem 1 for a standard quadratic equation $S = d$ of atomic rank 1.

Quadratic equations of atomic rank 1 have one of the following three forms.

Form 1. $[x, y] = d$ for some $d \in G$. By the conditions of the theorem this equation has a solution in the group G, say $x \to a,\ y \to b$, consequently, $d = [a, b]$.

Suppose, $[a, b] = 1$. Then

$$G_S = G* < x, y \mid [x, y] = 1 > \simeq G * (\mathbf{Z} \times \mathbf{Z}).$$

Definition 18 *Let $K =< H, t | a^t = a^\phi, a \in A >$ be the HNN-extension of a group H with associated subgroups A and A^ϕ. We say that we make a pinch if we replace in the word $w \in K$ a subword a^t by a^ϕ or a subword $a^{\phi t^{-1}}$ by a, $a \in A$.*

We need now a modification of a result from [1].

Lemma 16 *Let G be a nonabelian CSA-group, $G(u,t)$ an extension of a centralizer of G and $v \in G$ such that $[u,v] \neq 1$. Then for $z = (tvt)^2$ the subgroup $gp(G, u^z, t^z)$ is G-isomorphic to the free product of G and a free abelian group of rank 2 generated by u^z and t^z.*

Proof The subgroup $H = gp(u^z, t^z) \leq G(u,t)$ is free abelian of rank 2. We claim that the subgroup $N = gp(G, H)$ is G-isomorphic to $G * H$. Let

$$x = g_0(z^{-1}h_1z)g_1(z^{-1}h_2z)g_2 \ldots (z^{-1}h_nz)g_n \tag{10}$$

be an element in N such that: $1 \neq h_i \in H, i = 1, \ldots, n$ and $1 \neq g_j \in G, j = 1, \ldots, n-1$ in the case $n \neq 0$; and $g_1 \neq 1$ in the case $n = 0$. Using the method of normal forms for free products to prove that $N \simeq G * H$ one needs only prove that $x \neq 1$ in the HNN-extension $G(u,t)$. To get a contradiction, let us suppose that $x = 1$. If $n \neq 0$ then the word (10) contains a pinch. Let us consider a "typical" subword of the word (10):

$$(z^{-1}h_iz)g_i(z^{-1}h_{i+1}z). \tag{11}$$

There are only two possible types of pinches in (11), namely, $h_i \in C(u)$ and $g_{i+1} \in C(u)$. In those cases one has $t^{-1}h_it = h_i$ and $t^{-1}g_{i+1}t = g_{i+1}$. Then "making a pinch" one can rewrite the word (11) according to the reductions in HNN-extensions. In the rewritten word new pinches can occur only in subwords of the type $v^{-1}h_iv$ and $vg_{i+1}v^{-1}$. But $[u,v] \neq 1$ and $h_i \in C(u), g_{i+1} \in C(u)$ and hence $v^{-1}h_iv \notin C(u)$ and $vg_{i+1}v^{-1} \notin C(u)$ – this follows from the malnormality of centralizers in CSA-groups. This means that there are no more pinches in (11). This argument shows that pinches in (11) and hence in (10) do not affect each other and consequently after all possible reductions in (10) the remaining word represents a nontrivial element in $G(u,t)$ whenever $n \neq 0$. But if $n = 0$ then $x = g_0 \neq 1$ by the supposition. The contradiction proves the lemma. □

The lemma shows that G_S is G-embeddable in every nontrivial extension of a centralizer $G(u,t)$ of G.

Suppose now that $[a,b] \neq 1$. Then G_S is a free product with amalgamation $< G * F(x,y) \mid [a,b] = [x,y] >$.

Lemma 17 *Let $G(u,t)$ be an extension of a centralizer of a nonabelian CSA-group G, then the subgroup $< G, G^t >$ is isomorphic to the free product with amalgamation $\langle G * G^t \mid C(u) = C(u)^t \rangle$.*

The proof is similar to the proof of the lemma above, but simpler.

The element $[a, b]$ is not a proper power in G because G is a residually free group. By Lemma 17 the subgroup $< G, G^t >$ is G-isomorphic to the free product $< G * G^t \mid C([a, b]) = C([a, b]^t >$ in the extension $G([a, b], t)$ of the centralizer of $[a, b] \in G$. Any two noncommuting elements in G (hence in G^t) generate a free subgroup (this is a property of residually free groups). Therefore the subgroup $< G, a^t, b^t >$ is a free product of G and the free group $< a^t, b^t >$ with amalgamation $[a, b] = [a^t, b^t]$; so this subgroup is G-isomorphic to G_S.

Form 2. Let $S = 1$ be an equation of the type $z^{-1}cz = d$. Assume now that c is generator of $C_G(c)$.

This equation has a solution in G; so we can assume that $d = a^{-1}ca$, for some $a \in G$. This implies $[za^{-1}, c] = 1$. Hence,

$$G_S = \langle G, z \mid [za^{-1}, c] = 1 \rangle \simeq \langle G, t \mid [t, c] = 1 \rangle,$$

where t is a new letter obtained by the corresponding Tietze transformation ($t = za^{-1}$). So, in this case, G_S is an extension of a centralizer of G.

Suppose that $C_G(c) \neq < c >$; In this case by Lemma 15

$$\{S = 1\} \sim \{[za^{-1}, C_G(c)] = 1\} = S_1.$$

The group G_{S_1} is an extension of a centralizer of G; hence it is residually G, and consequently, $Rad(S)$ is the normal closure of $[za^{-1}, C_G(c)]$ in $G[z]$. Thus $G_{R(S)}$ is embeddable into an extension of a centralizer of G.

Form 3. $S = 1$ is in the form $x^2 = d$ (the case $d = 1$ is included).

If $x \to a$ is a solution of $S = 1$ in G, then $a^2 = d$ and our equation takes the form $x^2 = a^2$.

Hence $\{x^2 = d\} \sim \{x = a\}$ for some $a \in G$ such that $d = a^2$. The group $\langle G, x \mid x = a \rangle$ is isomorphic to G, and consequently, it is a residually G group. Hence the radical $R(S)$ is equal to the normal closure of xa^{-1} in $G[x]$ and $G = G_{R(S)}$ The case $k = 1$ is finished.

Remark 1 *In this proof we used just the following properties of the group G: G is a nonabelian CSA-group; every nontrivial commutator is not a proper power in G; every two noncommuting elements in G generate a free subgroup.*

4. Some auxiliary results

We shall need the following auxiliary results.

Lemma 18 *For all solutions ϕ of the equation $c^{mx}c^n = c^{lz}$ in a free group F one has $[x^\phi, c] = 1, [z^\phi, c] = 1$.*

Proof We can assume, that the absolute value of at least one of the numbers m, n, l is equal to 1. Otherwise $[x^\phi, c] = 1,\ [z^\phi, c] = 1$ (see [15]) Without loss of generality we can suppose that $l = 1$. Suppose $\phi(x) = a, \phi(z) = b$.

We can rewrite the equation in the form $c^{-m}c^{mab^{-1}}c^{nb^{-1}}c^{-n} = c^{1-m-n}$ or $[c^m, ab^{-1}][b, c^{-n}] = c^{1-m-n}$. The members of the lower central series of the free group are isolated subgroups. Hence in the case $m + n \neq 1$ this equality implies that c belongs to the intersection of the lower central series of F. This intersection is trivial; hence $c = 1$.

Consider now the case $n = 1 - m$. Let H be the subgroup generated by c and a. If $[a, c] \neq 1$, then H is free of rank 2. Elements c and $d = c^{ma}c^n$ are virtually conjugate in H (i.e., they are conjugated in a free group F) and not proper powers in H. By Theorem 4 from [4] we have: either c and d are conjugate in H (but this is not the case, because c and d are cyclically reduced in H and of different length in H) or the element $t^{-1}ctd^{-1} = t^{-1}ct(c^{ma}c^n)^{-1}$ is primitive in the free group $< a, c, t >$. But this is also impossible, because $t^{-1}ct(c^{ma}c^n)^{-1}$ belongs to the commutator subgroup of $< a, c, t >$.

This implies that $[a, c] = 1$. But then $c = c^b$ and $[c, b] = 1$. $\square$

Lemma 19 *Let H be a CSA-group and*

$$\Phi = \{\phi : H \longrightarrow H_\phi\}$$

be a separating family of homomorphisms of H. Then for any finite partition $\Phi = \bigcup_{i=1}^n \Phi_i$ there exists an index $i (1 \leq i \leq n)$ such that Φ_i is also a separating family of homomorphisms.

Proof A family of homomorphisms Φ separates H if and only if the diagonal homomorphism $\eta : H \longrightarrow \prod_{\phi \in \Phi} H_\phi$ is an embedding. For every $i (i = 1, \ldots, n)$ we have the diagonal homomorphism

$$\eta_i : H \longrightarrow H_i = \prod_{\phi \in \Phi_i} H_\phi.$$

Now we can concoct the diagonal homomorphism

$$\eta : H \longrightarrow \prod_{i=1}^n H_i,$$

which is an embedding because Φ is a separating family for H. To prove the proposition it is sufficient to prove that at least one of homomorphisms is an embedding. Let K_i be the kernel of the map η_i. If all these kernels are non-trivial, then we can choose nontrivial elements $k_i \in K_i$. Then, according to the CSA-property, there are some elements $x_i \in H$ such that the commutator

$$c = [[[k_1, k_2^{x_2}], k_3^{x_3}], \ldots, k_n^{x_n}]$$

is non-trivial. But the image of c under η is trivial, since every η_i maps c onto 1 (because η_i maps k_i onto 1). This contradicts the fact that η is an embedding. □

Lemma 20 *Let G be a fully residually free group, $a, b \in G$, $[a, b] \neq 1$, and let $C_G(ab)$ not be conjugate to either $C_G(a)$ or $C_G(b)$. Let A consist of alternating products of nontrivial elements $p_i \in C_G(a)$ and $q_j \in C_G(ab)$, such that the alternating product does not belong to $< b >$. Let $B = C_G(b) \setminus < b >$, and $h = h_0 g_1 h_1 \ldots g_n h_n$ be an alternating product of elements $g_i \in B$ and $h_j \in A$, which does not begin and end with an element from B. Then $h \notin C_G(b)$.*

Proof For each homomorphism $\phi : G \to F$ if $[a, b]^\phi = 1$, then either a^ϕ or b^ϕ or $(ab)^\phi$ is not a proper power, see [15]. The family Φ of discriminating homomorphisms for G can be subdivided into three subfamilies Φ_1, Φ_2 and Φ_3 corresponding to these three possibilities (we can assume $[a, b]^\phi = 1$ for each $\phi \in \Phi$.

By Lemma 19 one of these three families Φ_j is a separating family for G. According to the CSA property for any nontrivial elements $k_1, \ldots k_n \in G$ there are some elements $x_i \in G$ such that the commutator

$$c = [[[k_1, k_2^{x_2}], k_3^{x_3}], \ldots, k_n^{x_n}]$$

is non-trivial. As a separating family Φ_j contains a homomorphism ϕ such that $c^\phi \neq 1$. Hence ϕ separates the elements $k_1, \ldots, k_n$. Hence Φ_j is discriminating. Suppose for definiteness that Φ_3 is a discriminating family. Then for any $c \in C_G(ab)$ and any $\phi \in \Phi_3$ $c^\phi = (a^\phi b^\phi)^n$, for some n.

We show now, that for an element $s \in A$ $[s, b] \neq 1$. Indeed, by Lemma 21 s belongs to the free product of the centralizers $C_G(a)$ and $C_G(ab)$; b also belongs to this free product, has length 2 as a reduced word in the free product, and is not a proper power. Hence the centralizer of b in this free product is $< b >$.

We will show that for $h_1, \ldots, h_k \in A$ and $g_1, \ldots, g_{k-1} \in B$, $h_1 g_1 \ldots h_k b h_k^{-1} \ldots g_1^{-1} h_1^{-1} b^{-1} \neq 1$.

Let $h_i = b^{t_{i1}} p_{i1} q_{i1} \ldots p_{im_i} q_{im_i} b^{t_{i2}}$ and $c_i = [p_{i1} q_{i1}, b] \neq 1 and d_i = [p_{im_i} q_{im_i}, b] \neq 1$. Such representation exists because $[h_i, b] \neq 1$. There is a homomorphism $\phi \in \Phi_3$ separating the elements $[h_i, b], c_i, d_i, b^{t_{i2}+\alpha} g_i b^{t_{i+1,1}+\beta}$, where $\alpha, \beta \in \{0, 1, -1\}$.

Consider

$$h_1^\phi g_1^\phi \dots h_k^\phi b h_k^{-\phi} \dots g_1^{-\phi} h_1^{-\phi} b^{-1}. \tag{12}$$

Let $a^\phi = a_0, b^\phi = b_0$. Consider this element as an element in the free product of $C_F(a_0)$ and $C_F(b_0)$, and suppose that all h_i^ϕ's are in the reduced form in this product, hence containing a_0, then (12) will be in the reduced form, because there are no pinches in $h_i^\phi g_i^\phi h_{i+1}^\phi$ or in $(h_k b h_k^{-1})^\phi$. $\square$

5. Atomic rank 2

In this section we shall prove the following result.

Proposition 9 *For every fully residually free group G and every nondegenerate standard quadratic equation $S = 1$ of atomic rank 2 that has a solution in general position over G there is an extension of centralizers $G(U,T)$ and a G-embedding $\psi : G_S \longrightarrow G(U,T)$.*

Let $S = 1$ be a standard nondegenerate quadratic equation of atomic rank 2. In this case the equation $S = 1$ can be written as

$$r_1 \, r_2 = g, \quad g \in G,$$

where r_1, r_2 are standard quadratic atoms of the types:

$$[x,y], \quad z^{-1}cz, \quad x^2.$$

With respect to the different forms of atoms we consider four cases.

1) If $r_1 = [x,y]$, then the equation takes form

$$[x,y]r_2 = d.$$

There exists a solution $\phi : G_S \to G$ of the equation $S = 1$ over G such that $[r_1^\phi, r_2^\phi] \neq 1$. Let

$$\phi : \quad x \to a, \quad y \to b, \quad r_2 \to c.$$

This implies that $[a,b]c = d$, $[a,b] \neq 1$, and that the centralizers $C_G([a,b])$ and $C_G(c)$ have trivial intersection (because G is a CSA-group).

Let us consider a group $G(U,T)$ which is obtained from G by two extensions of centralizers:

$$G(U,T) = \langle G, s, u \mid [C_G([a,b]), s] = 1, [C_{G([a,b],s)}(d), u] = 1 \rangle.$$

Define a G-map ψ which depends on the form of the atom r_2: if $r_2 = [x_2, y_2]$, then

$$\psi : \quad x \to a^{su}, \quad y \to b^{su}, \quad x_2 \to (x_2^\phi)^u, \quad y_2 \to (y_2^\phi)^u;$$

if $r_2 = z^{-1}cz$, then

$$\psi : x \to a^{su}, \quad y \to b^{su}, z \to (z^\phi)u.$$

The map ψ can be extended to a G-homomorphism $\psi : G_S \longrightarrow G(U,T)$. Indeed,

$$(r_1 r_2)^\psi = ((r_1)^\phi)^{su} (r_2^\phi)^u = (r_1^\phi r_2^\phi)^u = d^u = d.$$

Proposition 10 *Let $\phi : G_S \to G$ be a solution of the equation $S = 1$ over G such that $[r_1^\phi, r_2^\phi] \neq 1$. Then the G-homomorphism $\psi : G_S \to G(U,T)$ defined above is monic on the subgroup $H = \langle G, x, y \rangle$.*

Proof From Lemma 9 we know that $H \simeq G * F(x,y)$. Hence to prove the proposition we need to show that ψ is monic on $F(x,y)$ and $H^\psi \simeq G * F(x,y)^\psi$ (we have $G^\psi = G$ since ψ is a G-map). The homomorphism ψ is monic on $F(x,y)$ if and only if

$$F(x,y)^\psi \simeq F(x^\psi, y^\psi) \simeq F(x^\phi, y^\phi)^{su} \simeq F(x^\phi, y^\phi) = F(a,b),$$

which is equivalent to say that a, b freely generate a free subgroup of rank 2. Notice, that $r_1^\phi = [a,b] \neq 1$ in G and the group G is residually free; so the subgroup $\langle a, b \rangle$ is indeed free of rank 2. Now it is sufficient to prove that $H^\psi \simeq G * F(x,y)^\psi$.

Choose an arbitrary nontrivial element $h \in H$. It can be written in the form

$$h = g_1 \ v_1(x,y) \ g_2 \ v_2(x,y) \ g_3 \ldots v_n(x,y) \ g_{n+1},$$

where $1 \neq v_i(x,y) \in F(x,y)$ are words in x, y and $1 \neq g_i \in G$ (with the possible exception of g_1 and g_{n+1}, they could be trivial). Then

$$h^\psi = g_1 \ v_1(a,b)^{su} \ g_2 \ v_2(a,b)^{su} \ g_3 \ldots v_n(a,b)^{su} \ g_{n+1}. \tag{13}$$

The group $G(U,T)$ is obtained from G by two HNN-extensions (extensions of centralizers), so every element in $G(U,T)$ can be rewritten in reduced form by making finitely many pinches. In the case of $G(U,T)$, to make a pinch means to rewrite the element according to the following rules: (here $e \in \{1, -1\}$, $p \in C_G([a,b])$, $q \in C_{G([a,b],s)}(d)$)

$$s^e p s^{-e} \to p, \quad u^e q u^{-e} \to q,$$

which allows us to cancel s or u.

Claim: The leftmost occurrence of u in the product (13) occurs in the reduced form of h^ψ uncancelled. Moreover, the element h^ψ can be written in the following reduced form

$$h^\psi = g_1 u^{-1} w f, \tag{14}$$

where :

a) either w is an alternating product $p_1 q_1 \dots p_k q_k$ of nontrivial elements (perhaps, with the exception of q_k) such that $p_i \in C_G([a,b])$ and $q_i \in C_G(d)$ or w is trivial;

b) f begins with either the letter u or s or their inverses, or f is trivial. In particular, if $w = 1$ then f does not begin with $u^{\pm 1}$ because the form (14) is reduced.

We prove the claim by induction on n. If $n = 1$, and h^ψ is already a reduced form, then the claim is obviously correct. Suppose now, there is a pinch there. Since u does not commute with any element of $G(U,T)$ containing s in reduced form $s^{-1}v(a,b)s$, then we should have a pinch $s^{-1}v(a,b)s$. This means $v(a,b) \in C_G([a,b])$. After cancelling s we have

$$h^\psi = g_1 u^{-1} v(a,b) u g_2;$$

this is a reduced form , since the centralizers of d and $[a,b]$ have trivial intersection. In this case $w = v(a,b) = p_1$, $f = ug_2$ and all conditions of the claim are satisfied.

Assume now that h^ψ is in the general form (13) and $n > 1$. Let

$$h_1^\psi = g_2\ v_2(a,b)^{su}\ g_3 \dots v_n(a,b)^{su}\ g_{n+1}.$$

By induction we can write h_1^ψ in a reduced form which satisfies the conditions of the claim:

$$h_1^\psi = g_2 u^{-1} w_1 f_1.$$

In particular, if $w \neq 1$, then $w = p_2 q_2 \dots$ is an alternating product of nontrivial elements from the corresponding centralizers. It follows that

$$h^\psi = g_1 u^{-1} s^{-1} v(a,b) s u g_2 u^{-1} w_1 f_1. \tag{15}$$

If this form is reduced, then it satisfies the statement of the claim. If (15) is not reduced then either $s^{-1}v(a,b)s$ or ug_2u^{-1} (or both) should be a pinch; indeed, $g_2u^{-1}w_1f_1$ is already reduced by induction; so either the pinch is inside $g_1u^{-1}s^{-1}v(a,b)sug_2$ (and this case was already discussed for $n = 1$), or it should be of the type ug_2u^{-1}. If $s^{-1}v(a,b)s$ is the only pinch in (15) then cancelling s we have a reduced form which evidently satisfies the claim. If ug_2u^{-1} is a pinch, but $s^{-1}v(a,b)s$ is not a pinch, then we have $g_2 = q_1 \in C_G(d)$ and

$$h^{\psi} = g_1u^{-1}s^{-1}v(a,b)sq_1w_1f_1 = g_1u^{-1}s^{-1}v(a,b)sq_1p_2q_2\dots f_1.$$

If we have both possible pinches then $v(a,b) = p_1 \in C_G([a,b])$ and

$$h^{\psi} = g_1u^{-1}p_1q_1p_2q_2\dots f_1.$$

In the last two cases to meet the conditions of the claim, we need to prove that the rewritten forms are reduced; it is sufficient to prove that any alternating product $w = p_1q_1\dots$ of length at least two, in which all factors are nontrivial, does not belong to either of centralizers $C_G([a,b])$, $C_G(d)$. Now we need

Lemma 21 *Let G be a fully residually free group. If centralizers $A = C_G(g)$ and $B = C_G(h)$ in G do not coincide, then the subgroup $C = \langle A, B\rangle$ generated by them in G is isomorphic to the free product $A * B$.*

Proof Consider an alternating product of the type

$$c = c_1c_2\dots c_n, \quad c_i \in A\cup B,$$

where neighbors c_i and c_{i+1} are in different centralizers. The group G is fully residually free; so there exists a homomorphism $\lambda : G \longrightarrow F$ which separates all the elements c_i in F and also separates the commutator $[g,h]$. The inequality $[g^{\lambda}, h^{\lambda}] \neq 1$ ensures that the centralizers $C_F(g^{\lambda})$ and $C_F(h^{\lambda})$ do not coincide in F, and, consequently, they generate a free subgroup of rank two, which is exactly their free product. Hence,

$$c^{\lambda} = c_1^{\lambda}c_2^{\lambda}\dots c_n^{\lambda} \neq 1.$$

This implies $c \neq 1$ and therefore $C \simeq A * B$. □

We have almost completed the proof of Proposition 10; indeed, the element $w = p_1q_1\dots$ has length at least 2 in the free product $C = C_G([a,b]) * C_G(d)$; hence, it does not belong to either of the factors. The proposition has been proved.

Case 1a) Let the equation $r_1r_2 = d$ be of the form $[x,y][z,w] = d$. Then by Proposition 1 $G_{r_1r_2=d} \simeq H_{r_2=r_1^{-1}d}$ and by Proposition 10 H is G-embeddable into some $G(U,T)$. Hence, by Proposition 2 $H_{r_2=r_1^{-1}d}$ is embeddable into $G(U,T)_{r_2=r_1^{-1}d}$. According to Section 3, $G(U,T)_{r_2=r_1^{-1}d}$ is embeddable into $G(U,T)(U_1,T_1) = G(U\cup U_1, T\cup T_1)$ for some U_1 and T_1.

Case 1b) Let the equation $S = 1$ have the form

$$[x, y]z_1^{-1}c_1z_1 = d,$$

and suppose there is a solution ϕ of this equation such that $[r_1^\phi, r_2^\phi] \neq 1$. Denote $z_1^\phi = a_1, x^\phi = a, y^\phi = b$.

Consider the following group which is obtained from G by two extensions of centralizers:

$$G(U, T) = \langle G, s, u \mid [C_G(c_1^{a_1}), s] = 1, \quad [C_G(d), u] = 1\rangle.$$

Notice that the centralizers $C_G(c_1^{a_1})$ and $C_G(d)$ have trivial intersection in G. This shows that the centralizers of s and u in $G(U, T)$ have trivial intersection (see [MR]); moreover:

$$C_{G(U,T)}(s) = \langle C_G(c_1^{a_1}), s\rangle, C_{G(U,T)}(u) = \langle C_G(d), u\rangle. \tag{16}$$

Define a G-map $\beta : G_S \longrightarrow G(U, T)$ by:

$$z_1 \longrightarrow a_1\ s\ u, \quad x \longrightarrow a^u, \quad y \longrightarrow b^u.$$

By straightforward verification we see that

$$(r_1r_2)^\beta = [a, b]a_1^{-1}c_1a_1 = d;$$

hence, ψ defines the G-homomorphism which we denote again by $\psi : G_S \longrightarrow G(U, T)$.

Proposition 11 *The restriction of β onto the subgroup $H = \langle G, z_1\rangle$ is monic.*

The proof of this proposition is the same as the proof of Proposition 12 below. Now to finish Case 1b) it is sufficient to repeat the argument from the end of 1a).

Case 2) Let the equation $S = 1$ have the form

$$z_1^{-1}c_1z_1\ z_2^{-1}c_2z_2 = d,$$

and suppose there is a solution ϕ of this equation such that $[r_1^\phi, r_2^\phi] \neq 1$. Denote $z_i^\phi = a_i,\ i = 1, 2$. Then $d = c_1^{a_1}\ c_2^{a_2}$, and $[c_1^{a_1}, c_2^{a_2}] \neq 1$. We can also assume that the solution ϕ also satisfies an auxiliary condition $[a_1, c_1] \neq 1$. In fact, if $[a_1, c_1] = 1$, instead of ϕ we can consider the solution

$$\phi' : z_1 \longrightarrow a_1d, \quad \phi' : z_2 \longrightarrow a_2d$$

in the group G (clearly, ϕ' is a solution of our equation). Now, if $[a_1 d, c_1] = 1$ then $[d, c_1] = 1$, but $d = c_1 c_2^{a_2}$, therefore $[c_1, c_2^{a_2}] = 1$ – a contradiction to the condition $[r_1^\phi, r_2^\phi] \neq 1$.

In this event we have also

$$[c_1^{a_1}, c_1] \neq 1.$$

Otherwise $C_G(c_1)^{a_1} \cap C_G(c_1) \neq 1$, but the group G is a CSA-group; hence every stabilizer is malnormal in G and consequently $a_1 \in C_G(c_1)$, which implies that $[c_1, a_1] = 1$ – a contradiction to the choice of ϕ.

In the CSA-group G the centralizers $C_G(c_1),\ C_G(c_2),\ C_G(d)$ are either conjugate to each other or at least one of them, say $C_G(d)$, is not conjugate to any of the others. Suppose $[c_i^{b_i}, d] = 1$ for some $b_i \in G,\ i = 1,2$. Then $(c_1^{b_1})^{b_1^{-1} z_1} (c_2^{b_2})^{b_2^{-1} z_2} = d$, where $[c_1^{b_1}, c_2^{b_2}] = 1$. By Lemma 18 for every solution ϕ of this equation we have $[(b_i^{-1} z_i)^\phi, c_i^{b_i}] = 1,\ i = 1,2$. This implies $[r_1^\phi, r_2^\phi] = 1$ - contradiction with conditions of the proposition. Hence, every conjugate of $C_G(d)$ intersects trivially with $C_G(c_1)$ and $C_G(c_2)$.

Consider the following group which is obtained from G by two extensions of centralizers:

$$G(U,T) = \langle G, s, u \mid [C_G(c_1^{a_1}), s] = 1, [C_G(d), u] = 1 \rangle.$$

Since every conjugate of $C_G(d)$ has trivial intersection in G with $C_G(c_1)$, then (see [MR]):

$$C_{G(U,T)}(s) = \langle C_G(c_1^{a_1}), s \rangle, C_{G(U,T)}(u) = \langle C_G(d), u \rangle. \tag{17}$$

Define a G-map $\psi : G_S \longrightarrow G(U,T)$ by:

$$z_1 \longrightarrow a_1\ s\ u, \quad z_2 \longrightarrow z_2\ u.$$

By straightforward verification, we see that

$$(r_1 r_2)^\psi = a_1^{-1} c_1 a_1\ a_2^{-1} c_2 a_2 = d;$$

hence, ψ defines a G-homomorphism which we denote again by $\psi : G_S \longrightarrow G(U,T)$.

Proposition 12 *The restriction of ψ to the subgroup $H = \langle G, z_1 \rangle$ is monic.*

Proof We know that $H \simeq G * F(z_1)$. It is sufficient to prove then that $H^\psi \simeq G * F(z_1^\psi)$. Let

$$h = g_1 z_1^{\ell_1}\ g_2\ z_1^{\ell_2} \dots z_1^{\ell_n} g_{n+1}$$

be an element in H such that $h^\psi = 1$. Then

$$h^\psi = g_1 (a_1 s u)^{\ell_1} \; g_2 \; (a_1 s u)^{\ell_2} \dots (a_1 s u)^{\ell_n} g_{n+1} = 1. \tag{18}$$

The group $G(U,T)$ is obtained from G by two HNN-extensions (extensions of centralizers); so every element in $G(U,T)$ can be rewritten in its reduced form by making finitely many pinches. In our case to make a pinch means to rewrite the element according to the following rules: (here $e \in \{1,-1\}$, $p \in C_G(c_1^{a_1})$, $q \in C_G(d)$)

$$s^e p s^{-e} \to p, \qquad u^e q u^{-e} \to q,$$

which allow us to cancel s or u.

Claim: In the product (18) the leftmost occurrence of s, in the case when $\ell_1 > 0$, or u, in the case when $\ell_1 < 0$, occurs in the reduced form of h^ψ uncancelled . Moreover, the element h^ψ can be written in one of the following two reduced form:

a) $\ell_1 > 0$:

$$h^\psi = g_1 a_1 s w f, \tag{19}$$

where either w is trivial or w is an alternating product of nontrivial elements (with a possible exception of p_k, q_k) of the type $q_1 p_1 \dots q_k p_k$, such that $p_i \in C_G(c_1^{a_1})$ and $q_i \in C_G(d)$; and $f = s^{-1} a_1^{-1} (a_1 s u)^{\ell_{2k}+1} g_{2k+1} \dots$, if $q_k \neq 1$ and $f = u (a_1 s u)^{\ell_{2k-1}-1} g_{2k} \dots$, if $q_k = 1$.

b) $\ell_1 < 0$:

$$h^\psi = g_1 u^{-1} w f, \tag{20}$$

where either w is trivial or w is an alternating product of nontrivial elements (maybe, with a possible exception of p_k, q_k) of the type $p_1 q_1 \dots p_k q_k$ such that $p_i \in C_G(c_1^{a_1})$ and $q_i \in C_G(d)$; and $f = u (a_1 s u)^{\ell_{2k}-1} g_{2k+1} \dots$, if $q_k = 1, p_k \neq 1$ and $f = s^{-1} a_1^{-1} (a_1 s u)^{\ell_{2k+1}+1} g_{2k+2} \dots$, if $p_k = 1$.

We prove the claim by induction on n in (18). For $n = 1$ we have

$$h^\psi = g_1 (a_1 s u)^{\ell_1} \; g_2.$$

This form is reduced and the claim is obviously correct.

Assume now that h^ψ is in the general form 18 and $n > 1$. Let

$$h_1^\psi = g_2 \; (a_1 s u)^{\ell_2} \dots (a_1 s u)^{\ell_n} g_{n+1}.$$

By induction we can write h_1^ψ in a reduced form which satisfies the conditions of the claim. The arguments we use depend on the signs of exponents ℓ_1 and ℓ_2.

Subcase 1) Let $\ell_2 > 0$. In this event by induction the element h_1^ψ can be written in the reduced form as

$$h_1^\psi = g_2 a_1 s w f,$$

where $w_1 = q_1 p_2 \dots q_k p_k$ is an alternating product of $p_i \in C_G(c_1^{a_1})$, $q_i \in C_G(d)$ such that all of them are not trivial except, perhaps, p_k.

If $\ell_1 > 0$ then

$$h^\psi = g_1 (a_1 s u)^{\ell_1} \ g_2 a_1 s w_1 f_1,$$

and this form is evidently reduced.

Suppose $\ell_1 < 0$. Then we have

$$h^\psi = g_1 \ (u^{-1} s^{-1} a_1^{-1})^{|\ell_1|} \ g_2 a_1 s w_1 f_1.$$

According to the description of the centralizers of s, u in (17) the element u commutes just with powers of u and some elements from G; the same is true for s. Therefore to have a pinch we must have $a_1^{-1} g_2 a_1 \in C_G(c_1^{a_1})$. Write $p_1 = a_1^{-1} g_2 a_1$. Now, we see that by cancelling s we get

$$h^\psi = g_1 \ (u^{-1} s^{-1} a_1^{-1})^{|\ell_1|-1} u^{-1} \ p_1 w_1 f_1,$$

and this form is already reduced, since w_1 begins with $q_1 \in C_G(d)$, and consequently $w = p_1 w_1 = p_1 q_1 p_2 \dots q_k p_k$ does not belong to the centralizer $C_G(d)$ (this means that we will not have a pinch even if f_1 begins with letter u). Moreover this reduced form of h^ψ satisfies the condition a) of the claim.

Subcase 2) Let $\ell_2 < 0$. In this case by induction the element h_1^ψ can be written in reduced form as

$$h_1^\psi = g_2 u^{-1} w_1 f_1,$$

where $w_1 = p_1 q_2 \dots p_k q_k$.

If $\ell_1 < 0$, then we have no pinches in the product

$$h^\psi = g_1 (u^{-1} s^{-1} a_1^{-1})^{|\ell_1|} g_2 u^{-1} w_1 f_1.$$

So this is a reduced form of h^ψ and it satisfies the condition b) of the claim.

If $\ell_1 > 0$, then we have

$$h^\psi = g_1 (a_1 s u)^{\ell_1} g_2 u^{-1} w_1 f_1.$$

To have a pinch we must have $g_2 \in C_G(d)$, and by writing $q_1 = g_2$ we obtain

$$h^\psi = g_1 (a_1 s u)^{\ell_1 - 1} a_1 s q_1 w_1 f_1.$$

As above, this form is reduced and satisfies the conditions of the claim.

The outcome of this discussion is that for every nontrivial $h \in H$ there exists a reduced form of h^{ψ} in which the leftmost letter (u or s) is uncancelled. Consequently, $h^{\psi} \neq 1$.$\square$

Let, as above, the equation $S = 1$ have the form

$$z_1^{-1} c_1 z_1 \ z_2^{-1} c_2 z_2 = d,$$

where $c_1, c_2 \in G$. Let there be a solution ϕ of this equation such that $[r_1^{\phi}, r_2^{\phi}] \neq 1$. Let $z_i^{\phi} = a_i, \ i = 1, 2$. Then $d = c_1^{a_1} \ c_2^{a_2}$, and $[c_1^{a_1}, c_2^{a_2}] \neq 1$. As above, we can assume that the solution ϕ also satisfies an auxiliary condition $[a_1, c_1] \neq 1$. In this event we have

$$[c_1^{a_1}, c_1] \neq 1.$$

Conjugating the equation by a_2^{-1} and changing variables we can rewrite the equation in the form $z_1^{-1} c_1 z_1 \ z_2^{-1} c_2 z_2 = c_1^a c_2 = d$. If some conjugate of c_1 commutes with c_2, then instead of c_1 we can take this conjugate.

We can suppose that $C_G(c_1^a c_2)$ is not conjugated to $C_G(c_1)$ and not conjugated to $C_G(c_2)$. Indeed, either all the three centralizers are mutually non-conjugated or two of them are conjugated; then changing variables, we can suppose that c_1 and c_2 commute. In the latter case, using approximations and Lemma 18 one shows that $C_G(c_1^a c_2)$ is not conjugated to $C_G(c_1)$.

Consider the following group which is obtained from G by four extensions of centralizers:

$$G(U,T) = \langle G, s, u, t, r \mid [C_G(c_1^a), s] = 1, [C_G(d), u] = 1, [t, C_G(c_2)] = 1, [r, C_{G(d,u)} c_2^u] = 1 \rangle.$$

$$C_{G(U,T)}(s) = \langle C_G(c_1^a), s \rangle, C_{G(U,T)}(u) = \langle C_G(d), u \rangle, C_{G(U,T)}(r) = \langle C_G(c_2)^u, t^u, r \rangle. \quad (21)$$

In the case when $[c_1, c_2] = 1$ we have

$$C_{G(U,T)}(s) = \langle C_G(c_1^a), s, r^{u^{-1}a}, t^a \rangle, C_{G(U,T)}(u) = \langle C_G(d), u \rangle. \quad (22)$$

Define a G-map $\psi : G_S \longrightarrow G(U,T)$ by:

$$z_1 \to a \ s \ u, \quad z_2 \to t \ u \ r.$$

By straightforward verification we see that

$$(r_1 r_2)^{\psi} = d;$$

hence, ψ defines the G-homomorphism which we denote again by $\psi : G_S \longrightarrow G(U,T)$.

Proposition 13 *The homomorphism ψ is monic.*

Proof Let

$$\bar{h} = \bar{g}_1 z_2^{i_1}\ \bar{g}_2\ z_2^{i_2} \dots z_2^{i_k} \bar{g}_{k+1} \tag{23}$$

be an element in G_S in reduced form such that $\bar{g}_i \in G * F(z_1) \simeq H$ and $h^\psi = 1$. Then

$$\bar{h}^\psi = \bar{g}_1^\psi (tur)^{i_1}\ \bar{g}_2^\psi\ (tur)^{i_2} \dots (tur)^{i_k} \bar{g}_{k+1}^\psi = 1. \tag{24}$$

The element $\bar{h}$ was in reduced form in G_S; hence in the case $i_j < 0, i_{j+1} > 0$ $\bar{g}_{j+1} \notin < c_2 >$, and in the case $i_j > 0, i_{j+1} < 0$, $\bar{g}_{j+1} \notin < c_2 >^u$.

The group $G(U,T)$ is obtained from G by four HNN-extensions (extensions of centralizers); so every element in $G(U,T)$ can be rewritten in its reduced form by making finitely many pinches.

From the description of the centralizer of r and elements from H^ψ in the proof of Proposition 12, it follows that in (24) in the product $(r^{-1}u^{-1}t^{-1})g(tur)$, where $g \in H^\psi$, r can only be cancelled if $g \in C_G(c_2)$. Notice that $g \notin < c_2 >$ because $\bar{h}$ is in the reduced form. In that case $(r^{-1}u^{-1}t^{-1})g(tur) = g^u$. Direct verification shows that in the product $(tur)g(r^{-1}u^{-1}t^{-1})$, where $g \in H^\psi$,the element r cannot be cancelled.

Replace all products $(r^{-1}u^{-1}t^{-1})g(tur)$, where $g \in C_G(c_2) \backslash < c_2 >$ in the element (24) by g^u. There are two possibilities.

1. The element $\bar{h}^\psi$ does not contain r anymore. Then it is an alternating product of elements in $H^\psi \backslash < c_2 >^u$, and in $(C_G(c_2) \backslash < c_2 >)^u$.

2. The element $\bar{h}^\psi$ contains r.

We start with the second case. The nontriviality of $\bar{h}^\psi$ in the first case will be proved in the process of considering the second case.

Consider the product

$$(tur)g(r^{-1}u^{-1}t^{-1}), \tag{25}$$

where g is an alternating product of elements in $H^\psi \backslash < c_2 >^u$, and in $(C_G(c_2) \backslash < c_2 >)^u$, g does not begin and end with a syllable in $(C_G(c_2) \backslash < c_2 >)^u$. Such restrictions on the alternating product follow from the fact that (23) was in reduced form in G_S. In this product we can make an r-pinch if and only if g belongs to a subgroup generated by $C_G(c_2)^u, t^u$ and possibly $s^{a^{-1}u}$ (if $c_1 \in C_G(c_2)$). This would only be possible if g had the form of g_1^u, where g_1 does not contain u. From the Claim in the proof of Proposition 12, it follows that g_1 must be an alternating product of nontrivial elements from $C_G(c_2) \backslash < c_2 >$ and

$$A = \{p_1 q_1 \dots p_k | p_i \in C_G(c_1^a), q_j \in C_G(c_1^a c_2), i, j = 1, \dots, k;$$

$$p_i \neq 1, q_j \neq 1, i = 2, \dots k-1, j = 1 \dots k-1\},$$

and g_1 begins and ends with elements from A. But by Lemma 20 such an element g_1 belongs to G and does not belong to $C_G(c_2)$; hence g_1^u does not belong to the centralizer of r. So there is no r-pinch in (25).

Consider the product

$$(r^{-1}u^{-1}t^{-1})g(tur), \tag{26}$$

where g is an alternating product of elements in H^ψ, and in $(C_G(c_2) \setminus < c_2 >)^u$, $g \notin H^\psi$. This product can be rewritten as

$$(u^{-1}t^{-1}ur^{-1}u^{-1})g(uru^{-1}tu). \tag{27}$$

Then g must not contain s and must be in the form $g = g_1^u$, where g_1 is an alternating product of elements from $C_G(c_2) \setminus < c_2 >$ and A. But r does not commute with $u^{-2}g_2u^2$, where $g_2 \in G$; so we cannot have r-pinch in the product (27).

Finally, in the element (24) either u or r cannot disappear. $\square$

Case 3) If $r_1 = x^2$ and $r_2 = z^{-1}cz$, there exists a solution $\phi : G_S \to G$ of the equation S over G such that $[r_1^\phi, r_2^\phi] \neq 1$. Let

$$\phi : x \to a, \quad z \to b.$$

This implies that $a^2b^{-1}cb = d$, $a \neq 1$, the centralizers $C_G(a)$ and $C_G(c^b)$ have trivial intersection (because G is a CSA-group), and, consequently, that the centralizers $C_G(a)$ and $C_G(d)$ have trivial intersection.

Let us consider the group $G(U,T)$ obtained from G by three extensions of a centralizer:

$$G(U,T) = \langle G, u, s, r \mid [C_G(d), u] = 1, [C_G(c^b), s] = 1, [C_{G(d,u)}(c^{bu}), r] = 1 \rangle$$

Define a G-map ψ

$$\psi : x \to a^u, \ , \ z \to bsur.$$

The map ψ can be extended to a G-homomorphism $\psi : G_S \longrightarrow G(U,T)$.

Proposition 14 *Let $\phi : G_S \to G$ be a solution of the equation S over G and such that $[r_1^\phi, r_2^\phi] \neq 1$. Then the G-homomorphism $\psi : G_S \to G(U,T)$ defined above is monic.*

The proof is very similar to the proposition above, but much simpler.

Case 4) Suppose $r_1 = x^2, r_2 = y^2$. In this case the equation takes the form

$$x^2y^2 = d.$$

There exists a solution $\phi : G_S \to G$ of the equation S over G such that $[r_1^\phi, r_2^\phi] \neq 1$. Let

$$\phi : x \to a, \quad y \to b.$$

This implies $a^2b^2 = d$, $a \neq 1$, the centralizers $C_G(a)$ and $C_G(b)$ have trivial intersection (because G is a CSA-group), and, consequently, the centralizers $C_G(a)$ and $C_G(d)$ have trivial intersection.

Let us consider a group $G(U,T)$ which is obtained from G by a single extension of a centralizer:

$$G(U,T) = \langle G, u \mid [C_G(d), u] = 1 \rangle.$$

Define a G-map ψ

$$\psi : x \to a^u, \ , \ y \to b^u,$$

Proposition 15 *Let $\phi : G_S \to G$ be a solution of the equation S over G such that $[r_1^\phi, r_2^\phi] \neq 1$. Then the G-homomorphism $\psi : G_S \to G(U,T)$ defined above is monic.*

6. Induction step and the proof of Theorems 1,2,3

We have shown that for every fully residually free group G and every nondegenerate standard quadratic equation $S_0 = 1$ of atomic rank 2 that has a solution in general position over G there is an extension of centralizers $G(U_0, T_0)$ and a G-embedding $\psi_0 : G_{S_0} \longrightarrow G(U_0, T_0)$.

Theorem 4 *For every fully residually free group G and an arbitrary nondegenerate quadratic equation $S = 1$ over G that has a solution in general position, there exists an extension of centralizers $G(U,T)$ and a G-embedding $\psi : G_S \longrightarrow G(U,T)$.*

Proof First we formulate the following simple lemma.

Lemma 22 *Let G be a group and $S(X)$ an arbitrary system of equations over G. Suppose $X = X_1 \cup X_2$ is an arbitrary partition of the set X. Then any specialization map $\xi : F(X_2) \longrightarrow G$ gives rise to a unique G-epimorphism $\xi : G[X] \longrightarrow G[X_1]$ which can be extended to the unique G-epimorphism*

$$\phi_\xi : G_S \longrightarrow G_{S^\xi}$$

In particular, $S^\xi(X_1) = 1$ is a system of equations over G with variables from X_1.

The proof is straightforward.

We will now prove the theorem by induction on the atomic rank. The basis of the induction is given in the previous section. Suppose that the theorem is true for atomic rank less than k. Let S be a nondegenerate standard quadratic equation of atomic rank $k > 2$ over a group G, which is a subgroup of $F(U_1, T_1)$. As was mentioned above we can rewrite it in the form

$$S_2 = gR_2^{-1},$$

where $g \in G$, $S_2 = r_1 r_2$ and $R_2 = r_3 \ldots r_k$. Let $\phi : G_S \to G$ be a solution of S in G. Then in G we have the equality

$$(S_2)^\phi = (gR_2^{-1})^\phi.$$

If we denote $h = (S_2)^\phi \in G$, then the equation

$$S_2 = h$$

has a solution in G. Notice that if we consider the restriction ξ of ϕ on the set P_2 of all variables from R_2, then the equation $S_2 = h$ is exactly of the form $S^\xi = 1$. Hence by the lemma above we have a canonical G-epimorphism

$$\phi_\xi : G_S \longrightarrow G_{S^\xi}.$$

By Lemma 9 the epimorphism ϕ_ξ is monic on $H = H^{(1)} = gp(G, F(Q_1))$. The system $S_2 = h$ over G is of atomic rank 2 having a solution in a general position; hence there exists a G-embedding $\psi_0 : G_{R(S_2=h)} \to G(U_0, T_0)$, which is monic on H. We identify now H with H^ψ via ψ_0 (then G is identified with itself, because ψ_0 is a G-map). Thus $H \leq G(U_0, T_0)$. Denote by η the canonical epimorphism $\eta : G_{S_2=h} \longrightarrow G_{R(S_2=h)}$. By the conditions of the theorem η is an H-homomorphism. Now, the equation $S = 1$ can be written as

$$R_1 = r_1^{-1} g,$$

which is an equation over H and is of atomic rank $k - 1$. By Proposition 1 we have $G_S \simeq H_{R_1}$ (here, as everywhere above, we omit the constant h in the equation $R_1 = h$ and write simply R_1). Let ψ be the composition of ϕ_ξ, η and ψ_0. Then $\psi : H_{R_1} \longrightarrow G(U_0, T_0)$ is an H -homomorphism; in particular, ψ is a solution of the equation $R_1 = h$ (which originally is an equation over H, and consequently, over the group $G(U_0, T_0)$. Hence, $R_1 = h$ is a non-degenerate standard quadratic equation of atomic rank $k - 1$ over $G(U_0, T_0)$. It has a solution in general position. Indeed, $[\psi(r_2), \psi(r_3)] \neq 1$ since

$[\phi(r_2), \phi(r_3)] \neq 1$, and ϕ is a specialization of ψ. The group $G(U_0, T_0)$ is fully residually G and therefore it is fully residually free. By induction there exists an embedding $\lambda : G(U_0, T_0)_{R(R_1)} \longrightarrow G(U_0, T_0)(U_2, T_2)$. We have that $G(U_0, T_0)(U_2, T_2) = G(U_0 \cup U_2, T_0 \cup T_2) = G(U', T')$. Hence, λ gives an $G(U_0, T_0)$-embedding of $G(U_0, T_0)_{R(R_1)}$ into some extension of centralizers of G. $\square$

Theorems 1 and 3 now follow from Theorem 4, Propositions 3–8, Section 3 and by considering case 2) in Section 5. Theorem 2 follows by induction from Theorem 1.

References

[1] G. Baumslag, A. Myasnikov, and V. Remeslennikov. Algebraic geometry over groups. 1996.

[2] L.P. Comerford and C.C. Edmunds. Quadratic equations over free groups and free products. *Journal of Algebra*, 68:276–297, 1981.

[3] L. P. Comerford Jr. and C. C. Edmunds. *Solutions of equations in free groups.* Walter de Gruyter, Berlin, New York, 1989.

[4] B. Fine, A.M. Gaglione, A. Myasnikov, G. Rosenberger, and D. Spellman. A classification of fully residually free groups. 1994. accepted by the Journal of Algebra.

[5] R. I. Grigorchuk and P. F. Kurchanov. Some questions of group theory related to geometry. In *Itogi Nauki i Techniki, Sovremennye problemy matematiki. Fundamental'nye napravlenia, VINITI, 58. Encyclopedia of math. sciences*, 1990. English translation in 1993.

[6] R.I. Grigorchuk and P.F. Kurchanov. On quadratic equations in free groups. *Contemp. Math.*, 131(1):159–171, 1992.

[7] V. Guba. Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems. *Mat. Zametki*, 40(3):321–324, 1986.

[8] R. C. Lyndon. Groups with parametric exponents. *Trans. Amer. Math. Soc.*, 96:518–533, 1960.

[9] R.C. Lyndon. Equations in groups. *Bol. Soc. Bras. Mat.*, 11:79–102, 1980.

[10] Roger C. Lyndon and Paul E. Schupp. *Combinatorial Group Theory*. Springer, 1977.

[11] G.S. Makanin. Equations in a free group (Russian). *Izv. Akad. Nauk SSSR, Ser. Mat.*, 46:1199–1273, 1982. transl. in Math. USSR Izv., V. 21, 1983; MR 84m:20040.

[12] A. G. Myasnikov and V. N. Remeslennikov. Exponential groups 2: extension of centralizers and tensor completion of csa-groups. *Interntional Journal of Algebra and Computation*, 6(6):687–711, 1996.

[13] A. Razborov. On systems of equations in a free group. *Math. USSR, Izvestiya*, 25(1):115–162, 1985.

[14] A. Razborov. *On systems of equations in a free group*. PhD thesis, Steklov Math. Institute, Moscow, 1987.

[15] M. P. Schutzenberger. Comptes rendus des seances de l'academie des sciences. 248:2435–2436, 1959.

[16] J.R. Stallings. Fineteness of matrix representation. *Ann. Math.*, 124:337–346, 1986.