FOLDINGS, GRAPHS OF GROUPS AND THE MEMBERSHIP PROBLEM

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Abstract. We use Stallings-Bestvina-Feighn-Dunwoody folding sequences to analyze the solvability of the subgroup membership problem for fundamental groups of graphs of groups.

1. Introduction

The idea of using foldings to study group actions on trees was introduced by J. Stallings in a seminal paper [41], where he applied foldings to investigate free groups. Free groups are exactly those groups that admit free actions on simplicial trees. Later J. Stallings [42] offered a way to extend these ideas to non-free actions of groups on graphs and trees. M. Bestvina and M. Feighn [5] and, independently, M. Dunwoody [18] gave a systematic treatment of Stalling's approach in the context of graphs of groups and groups acting on simplicial trees and obtained a number of interesting applications. For example, M. Bestvina and M. Feighn [5] proved a far-reaching generalization of Dunwoody's accessibility results for finitely presented groups; M. Dunwoody [19] used foldings to construct a small unstable action on an $\mathbb{R}$-tree. Some other applications of foldings in the graph of groups context can be found in [35, 37, 38, 16, 17, 24, 25, 12, 11].

In this paper we apply foldings to more computational questions, such as the subgroup membership problem. Recall that a finitely generated group

$$G = \langle x_1, \dots, x_k \,|\, r_1, r_2, \dots, \rangle$$

is said to have *solvable membership problem* (or *solvable uniform membership problem*) if there is an algorithm which, for any finite family of words $u, w_1, \dots, w_n$ in $\{x_1, \dots, x_k\}^{\pm 1}$ decides whether or not the element of G represented by u belongs to the subgroup of G generated by the elements of G corresponding to $w_1, \dots, w_n$ (it is easy to see that this definition does not depend on the choice of a finite generating set for G). Similarly, if $H \le G$ is a specific subgroup, then H is said to have *solvable membership problem in G* if there is an algorithm deciding for any word u in $\{x_1, \dots, x_k\}^{\pm 1}$ whether u represents an element of H.

Amalgamated free products, HNN-extensions and more generally, fundamental groups of graphs of groups play a very important role in group theory. However, till now there has been relatively little understanding of how these fundamental constructions affect the subgroup membership problem. One of the first results in this direction is due to K. Mihailova, who proved [33, 34] that if A and B have

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solvable membership problem then so does their free product $A * B$ (see also the subsequent work of Y. Boydron [13]). K. Mihailova [32] also produced some important counter-examples demonstrating the difficulty of the membership problem. Namely, she proved that the direct product $G = F(a,b) \times F(x,y)$ of two free groups of rank two possesses a finitely generated subgroup H with unsolvable membership problem in G. This direct product can be thought of as a double HNN-extension of $F(a,b)$:

$$G = \langle F(a,b), x, y \,|, x^{-1}fx = f, y^{-1}fy = f \text{ for any } f \in F(a,b)\rangle.$$

It is well-known that a finitely generated free group has uniform membership problem solvable in quadratic time in terms of $|u|+|w_1|+\cdots+|w_n|$. Thus even seemingly innocuous free constructions have the potential of greatly affecting the complexity of the membership problem. Another important example which to this date is not at all understood is that of the mapping torus of a free group automorphism.

Namely, let G be a group and let $\phi : G \to G$ be an automorphism of G. Then the HNN-extension of G along ϕ

$$M_\phi := \langle\, G, t \,|\, t^{-1}gt = \phi(g), \text{ for every } g \in G\,\rangle = G \rtimes_\phi \mathbb{Z}.$$

is called the *mapping torus group* of ϕ. Despite their importance in 3-dimensional topology, apart from a few obvious cases nothing is known about the solvability of the membership problem for mapping tori of free group automorphisms (or more generally, mapping tori of automorphisms of surface groups).

A substantial amount of work on the membership problem for amalgamated products and HNN-extension was done by V. N. Bezverkhnii [7, 8, 9, 10]. However, he did not use the machinery of Bass-Serre theory of graphs of groups and groups acting on trees. Consequently, all of his results have to rely on Britton's lemma and the normal form theorem for amalgamated products, which makes his proofs extremely technical and statements of most results quite special.

Our goal is to present a more geometric and unified approach to this topic which relies on Bass-Serre theory [39, 4] as well as the foldings technique of Stallings-Bestvina-Feighn-Dunwoody. When performing a sequence of folding moves, we have to keep a very careful track of what foldings of Bass-Serre trees do to quotient graphs of groups and to record what kind of conditions are necessary to be able to perform each folding step algorithmically, as well as for the process to terminate in a finite number of steps. The full list of these conditions turns out to be rather cumbersome (see Definition 6.4 and Theorem 6.13 below), so we will formulate a corollary of the main technical result instead.

Theorem 1.1. *Let $\mathbb{A}$ be a finite graph of groups such that:*

(1) *For every vertex v of A the vertex group A_v is either locally quasiconvex word-hyperbolic or polycyclic-by-finite.*
(2) *Every edge group of $\mathbb{A}$ is polycyclic-by-finite.*

Then for any vertex $v_0 \in VA$ the uniform membership problem for $G = \pi_1(\mathbb{A}, v_0)$ is solvable.

The above situation applies to a wide variety of situations. For example, it is applicable to a finite graph of groups where all vertex groups are virtually abelian or where all vertex groups are virtually free and edge groups are virtually cyclic. In particular, the mapping torus of an automorphism of a free abelian group of finite rank (or in fact of any virtually polycyclic group) falls into this category, as do the so called "tubular" groups (that is, multiple HNN-extensions of free abelian groups with cyclic associated subgroups). While Theorem 1.1 does not say anything about the computational complexity of the algorithm solving the membership problem, we believe that in many specific cases this complexity can be analyzed and estimated explicitly. Indeed, the intrinsic complexity of the folding algorithm is quadratic and the complexity of each individual folding move can be estimated in terms of the properties of the vertex and edge groups of $\mathbb{A}$. For example, in the case when all vertex groups are free and edge groups are cyclic, the folding algorithm provided by Theorem 1.1 appears to have polynomial complexity. This is confirmed by independent results of Paul Schupp [40] (in preparation) who used folding ideas to show that the uniform membership problem for multiple HNN-extensions of free groups with cyclic associated subgroups is solvable in polynomial time.

As an illustration of the usefulness of Theorem 1.1, we apply it to graph products and right-angled Artin groups. Recall that if Γ is a finite simple graph with a group G_v associated to each vertex of Γ then the *graph product* group $G(\Gamma)$ is defined as the free product $*_{v\in V\Gamma}G_v$ modulo the relations $[G_v, G_u] = 1$ whenever u and v are adjacent vertices in Γ. In particular, when each G_v is an infinite cyclic group, $G(\Gamma, (G_v)_{v\in V\Gamma)})$ is a *right-angled Artin group.*

Corollary 1.2. *Let T be a finite tree such that for every vertex $v \in VT$ there is an associated finitely generated abelian group G_v. Then the graph product group $G(T)$ has solvable membership problem.*

Proof. Note that for any groups K, H we can write the direct product $H \times K$ as an amalgam:

$$H \times K = H *_H H \times K *_K K.$$

Let $v_1, \dots, v_n$ be the vertices of T. Let T' be the barycentric subdivision of T. We give T' the structure of a graph of groups as follows. For each vertex v_i of T assign the vertex group $T'_{v_i} := G_{v_i}$. For each barycenter v of an edge $[v_i, v_j]$ of T assign the vertex group $T'_v := G_{v_i} \times G_{v_j}$. Also, for $e_i = [v_i, v] \in ET'$ and $e_j = [v_j, v] \in ET'$ put $T'_{e_i} := G_{v_i}$ and $T'_{e_j} := G_{v_j}$. Finally, we define the edge-monomorphisms $T'_{e_i} \to T_v$ and $T'_{e_i} \to T_{v_i}$ to be the inclusion maps $G_{v_i} \to G_{v_i} \times G_{v_j}$ and $g_{v_j} \to G_{v_i} \times G_{v_j}$. This defines a graph of groups $\mathbb{T}'$ where all the vertex groups are finitely generated abelian. Moreover, we have $G(T) \cong \pi_1(\mathbb{T}', v_1)$.

Hence by Theorem 1.1 the group $G(T)$ has solvable membership problem. □

Corollary 1.2 applies to many right-angled Artin groups, for example to any right-angled Artin group where the underlying graph is a tree. Note that if Γ is a square then the right-angled Artin group based on Γ is $G \cong F(a, b) \times F(x, y)$ and hence by Mihailova's theorem G has unsolvable membership problem. Thus if Γ is not a tree, the statement of Corollary 1.2 need not hold in general.

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2. Graphs of groups, subgroups and induced splittings

We refer the reader to the book of J.-P. Serre [39] as well as to [2, 4, 14, 36] for detailed background information regarding groups acting on trees and the Bass-Serre theory.

Convention 2.1 (Graph of groups notations)**.** For a graph of groups $\mathbb{A}$ we will denote the underlying graph by A. For each edge $e \in EA$ the initial vertex of e is denoted $o(e)$ and the terminal vertex of e is $t(e)$. The graph-of-groups $\mathbb{A}$ has vertex groups A_v for $v \in VA$, edge groups A_e for $e \in EA$ and boundary monomorphisms $\alpha_e : A_e \to A_{o(e)}$ and $\omega_e : A_e \to A_{t(e)}$ for all $e \in EA$. If e^{-1} is the inverse edge of e then we assume that $A_{e^{-1}} = A_e$, $\alpha_{e^{-1}} = \omega_e$ and $\omega_{e^{-1}} = \alpha_e$.

Recall that in Bass-Serre theory an $\mathbb{A}$-*path of length* $k \geq 0$ *from* v *to* v' is a sequence

$$p = a_0, e_1, a_1, \dots, e_s, a_s$$

where $s \geq 0$ is an integer, $e_1, \dots, e_s$ is an edge-path in A from v to v', where $a_0 \in A_v, a_s \in A_{v'}$ and $a_i \in A_{t(e_i)} = A_{o(e_{i+1})}$ for $0 < i < k$. We will denote the length k of p by $|p|$. If p is an $\mathbb{A}$-path from v to v' and q is an $\mathbb{A}$-path from v' to v'', then the *concatenation* pq *of* p *and* q is defined in the obvious way and is an $\mathbb{A}$-path from v to v'' of length $|p| + |q|$.

The following notion plays a very important role in Bass-Serre theory.

Definition 2.2 (Fundamental group of a graph of groups)**.** Let $\mathbb{A}$ be a graph of groups.

Let $\sim$ be the equivalence relation on the set of $\mathbb{A}$-path generated (modulo concatenation) by:

$$e, \omega_e(c), e^{-1} \sim \alpha_e(c), \text{ where } e \in EA, c \in A_e.$$

If p is an $\mathbb{A}$-path, we will denote the $\sim$-equivalence class of p by $\bar{p}$. Note that if $p \sim p'$ then p, p' have the same initial and the same terminal vertex in VA.

Let $v_0 \in VA$ be a vertex of A. We define the *fundamental group* $\pi_1(\mathbb{A}, v_0)$ as the set of $\sim$-equivalence classes of $\mathbb{A}$-paths from v_0 to v_0. It can be shown that G is in fact a group with multiplication corresponding to concatenation of paths.

Proposition 2.3 (Normal Form Theorem)**.** *Let* $\mathbb{A}$ *be a graph of groups. Then:*

(1) *If* $a \in A_v, a \neq 1$ *is a nontrivial vertex group element then the length zero path* a *from* v *to* v *is not* $\sim$*-equivalent to the trivial path* 1 *from* v *to* v*.*
(2) *Suppose* $p = a_0, e_1, a_1, \dots, e_k, a_k$ *is a reduced* $\mathbb{A}$*-path from* v *to* v' *with* $k > 0$*. Then* p *is not* $\sim$*-equivalent to a shorter path from* v *to* v'*. Moreover, if* p *is equivalent to a reduced* $\mathbb{A}$*-path* p' *from* v *to* v' *then* p' *has underlying edge-path* $e_1, e_2, \dots, e_k$*.*
(3) *Suppose* T *is a maximal subtree of* A *and let* $v_0 \in VA$ *be a vertex of* V*. Let* $G = \pi_1(\mathbb{A}, v_0)$*. For* $x, y \in VA$ *we denote by* $[x, y]_T$ *the* T*-geodesic edge-path in* T*. Then* G *is generated by the set* $\bar{S}$ *where*

$$S = \bigcup_{e \in E(A-Y)} [v_0, o(e)]_T e [t(e), v_0]_T \bigcup \bigcup_{v \in VA} [v_0, v]_T A_v [v, v_0]_T$$

We also need to recall the explicit construction of the Bass-Serre universal covering tree for a graph of groups.

Definition 2.4 (Bass-Serre covering tree)**.** Let $\mathbb{A}$ be a graph of groups with base-vertex $v_0 \in VA$. We define an equivalence relation $\approx$ on the set of $\mathbb{A}$-paths originating at v_0 to be generated by $\sim$ and the condition

$$p \approx pa, \text{ where } p \text{ is an } \mathbb{A}\text{-path from } v_0 \text{ to } v \in VA, \text{ and } a \in A_v.$$

Thus if p is an $\mathbb{A}$-path from v_0 to v, we shall denote the $\approx$-equivalence class of p by $\bar{p}A_v$.

We now define the Bass-Serre tree $\widetilde{(\mathbb{A}, v_0)}$ as follows. The vertices of $\widetilde{(\mathbb{A}, v_0)}$ are $\approx$-equivalence classes of $\mathbb{A}$-path originating at v_0. Thus each vertex of $\widetilde{(\mathbb{A}, v_0)}$ has the form $\bar{p}A_v$, where p is an $\mathbb{A}$-path from v_0 to a vertex $v \in VA$. (Hence we can in fact choose p to be already $\mathbb{A}$-reduced and such that the last group-element in p is equal to 1.)

Two vertices x, x' of $\widetilde{(\mathbb{A}, v_0)}$ are connected by an edge if and only if we can express x, x' as $x = \bar{p}A_v, x' = \overline{pae}A_{v'}$, where p is an $\mathbb{A}$-path from v_0 to v and where $a \in A_v$, $e \in EA$ with $o(e) = v, t(e) = v'$.

It follows from the basic results of Bass-Serre theory that $\widetilde{(\mathbb{A}, v_0)}$ is indeed a tree. This tree has a natural base-vertex, namely $x_0 = \bar{1}A_{v_0}$ corresponding to the $\approx$-equivalence class of the trivial path 1 from v_0 to v_0.

Moreover, the group $G = \pi_1(\mathbb{A}, v_0)$ has a natural simplicial action on $\widetilde{(\mathbb{A}, v_0)}$ defined as follows:

If $g = \overline{q} \in G$ (where q is an $\mathbb{A}$-path from v_0 to v_0) and $u = \overline{p}A_v$ (where p is an $\mathbb{A}$-path from v_0 to $v \in VA$), then $g \cdot u := \overline{qp}A_v$. It is not hard to check that the action is well-defined on the set of vertices of $\widetilde{(\mathbb{A}, v_0)}$ and that it preserves adjacency relation. Thus G in fact has a canonical simplicial action without inversions on $\widetilde{(\mathbb{A}, v_0)}$.

It can be shown that if p is an $\mathbb{A}$-path from v_0 to v then the map $A_v \to G$, $a \to \overline{pap^{-1}}$ is an embedding. Moreover, in this case the G-stabilizer of the vertex $\bar{p}A_v$ of $\widetilde{(\mathbb{A}, v_0)}$ is equal to the image of the above map, that is to $\overline{pA_vp^{-1}}$. Similarly, the G-stabilizer of an edge in $\widetilde{(\mathbb{A}, v_0)}$ connecting $\bar{p}A_v$ to $\overline{pae}A_{v'}$ is equal to $\bar{p}(a\alpha_e(A_e)a^{-1})\bar{p}^{-1}$.

The following well-known statement lies at the foundation of Bass-Serre theory and provides a duality between group actions on trees and fundamental groups of graphs of groups.

Proposition 2.5. *Let U be a group acting on a simplicial tree X without inversions. Suppose Y is a U-invariant subtree with base-vertexv_0. Then the graph $B = Y/U$ has a natural graph-of-groups structure $\mathbb{B}$ such that U is canonically isomorphic to $\pi_1(\mathbb{B}, v_0')$ and Y is U-equivariantly isometric to the universal covering Bass-Serre tree of $\mathbb{B}$ (here v_0' is the image of v_0 in B).*

Remark 2.6. We want to remind an explicit construction of $\mathbb{B}$. Let $T_2 \subseteq Y$ be a subtree of Y which is a fundamental domain for the action of U on Y. Namely, $UT_2 = Y$ and no two distinct edges of T_2 are U-equivalent. Therefore for some subtree $T_1 \subseteq T_2$ we have:
(1) No two vertices of T_1 are U-equivalent and $U(VT_1) = VY$.
(2) For every vertex $v \in T_2 - T_1$ the vertex v is connected by an edge to a vertex of T_1.
(3) For every vertex of $v \in T_2 - T_1$ there is a unique vertex $x(v) \in VT_1$ which is U-equivalent to v.

For each vertex $v \in T_2 - T_1$ choose an element $t_v \in U$ such that $t_v v = x(v)$. The graph of groups $\mathbb{B}$ is then defined as follows.

(1) The graph $B = Y/U$ is obtained from T_2 by identifying v with $x(v)$ for each vertex $v \in T_2 - T_1$. Thus we can assume that T_1 is a subgraph of B (in fact a spanning tree of B) and that $v_0' = v_0$. Similarly, we assume that

$EB = ET_2$. If an edge $e = [z, v]$ of T_2 has $z \in T_1$ and $v \in T_2 - T_1$, we set $o_B(e) = z$ and $t_B(e) = x(v)$.

(2) For each vertex $v \in VT_1$ we set $B_v := Stab_U(v)$, where $Stab_U(v)$ is the U-stabilizer of $v \in X$.
(3) For each edge $e = [z, v] \in ET_2$ we set $B_e := Stab_U(e)$.
(4) For each edge $e = [z, v] \in ET_1$ the boundary monomorphisms $\alpha_e^B : B_e \to B_z$ and $\omega_e^B : B_e \to B_v$ are defined as inclusions of $Stab_U(e)$ in $Stab_U(z)$ and $Stab_U(v)$ accordingly.
(5) Suppose $e = [z, v]$ is an edge of T_2 with $z \in T_1$, $v \in T_2 - T_1$. We set the boundary monomorphism $\alpha_e^B : B_e \to B_z$ to be the inclusion of $Stab_U(e)$ in $Stab_U(z)$. We set the boundary monomorphism $\omega_e^B : B_e \to B_{x(v)}$ to be the map $g \mapsto t_v g t_v^{-1}, g \in B_{x(v)}$.

Definition 2.7 (Induced splitting)**.** Let $\mathbb{A}$ be a graph of groups with a base-vertex v_0. Let $G = \pi_1(\mathbb{A}, v_0)$ and let $X = \widetilde{(\mathbb{A}, v_0)}$ be the universal Bass-Serre covering tree of the based graph-of-groups $(\mathbb{A}, v_0)$. Thus X has a base-vertex x_0 mapping to v_0 under the natural quotient map.

Suppose $U \le G$ is a subgroup of G and $Y \subset X$ is a U-invariant subtree. Then the graph-of-groups splitting $\mathbb{B}$ of U obtained as in Proposition 2.5 on the quotient graph $B = Y/U$ is said to be *an induced splitting of $U \le G$ with respect to Y corresponding to the splitting $G = \pi_1(\mathbb{A}, v_0)$.*

If U without fixed point then here is a preferred choice of a U-invariant subtree of X, namely the smallest U-invariant subtree containing x_0, which will be denoted X_U:

$$X_U := \cup_{u \in U}[x_0, ux_0]$$

Notice that because of the explicit construction of $\mathbb{B}$ each vertex groups of $\mathbb{B}$ fixes a vertex of X and hence is conjugate to a subgroup of a vertex group of $\mathbb{A}$. Similarly, edge groups of $\mathbb{B}$ are conjugate to subgroups of edge groups of $\mathbb{A}$. In practice we will often choose Y to be the smallest among U-invariant subtrees of X which contain x_0.

3. $\mathbb{A}$-graphs

In this section we introduce the combinatorial notion of an $\mathbb{A}$-graph. These $\mathbb{A}$-graphs will approximate induced splittings of subgroups of $\pi_1(\mathbb{A}, v_0)$. In good situations, namely when an $\mathbb{A}$-graph is "folded", an induced splitting can be directly read off a $\mathbb{A}$-graph.

Definition 3.1 ($\mathbb{A}$-graph)**.** Let $\mathbb{A}$ be a graph of groups. *An $\mathbb{A}$-graph $\mathcal{B}$* consists of an underlying graph B with the following additional data:

(1) A graph-morphism $[\,.\,] : B \to A$.
(2) Each vertex $u \in VB$ has an associated group B_u, where $B_u \le A_{[u]}$.
(3) To each edge $f \in EB$ there are two associated group elements $f_o \in A_{[o(f)]}$ and $f_t \in A_{[t(f)]}$ such that $(f^{-1})_o = (f_t)^{-1}$ for all $f \in EB$.

Convention 3.2. If $f \in EB$ and $u \in VB$, we shall refer to $e = [f] \in EA$ and $v = [u] \in VA$ as *the type of f and u* accordingly. Also, especially when representing $\mathbb{A}$-graphs by pictures, we will sometimes say that an edge f of an $\mathbb{A}$-graph $\mathcal{B}$ has *label* $(f_o, [e], f_t)$. Similarly, we will say that a vertex $u \in VB$ has *label* $(B_u, [u])$.

To any $\mathbb{A}$-graph we can associate in a natural way a graph of groups:

Definition 3.3 (Graph of groups defined by an $\mathbb{A}$-graph)**.** Let $\mathcal{B}$ be a $\mathbb{A}$-graph. The associated graph of groups $\mathbb{B}$ is defined as follows:

(1) The underlying graph of $\mathbb{B}$ is the graph B.
(2) For each $u \in VB$ we put the vertex group of u to be B_u.
(3) For each $f \in EB$ we define the edge group of f in $\mathbb{B}$ as
$$B_f := \alpha_{[f]}^{-1}(f_o^{-1} B_{o(f)} f_o) \cup \omega_{[f]}^{-1}(f_t B_{o(f)} f_t^{-1}) \leq A_{[f]}.$$
(4) For each $f \in EB$ the vertex monomorphism $\alpha_f : B_f \to B_{o(f)}$ is defined as
$$\alpha_f(g) = f_o\big(\alpha_{[f]}(g)\big) f_o^{-1} \text{ for every} g \in B_f.$$

Convention 3.4. Suppose $u, u' \in VB$ and p is a $\mathbb{B}$-path from u to u'. Thus p has the form:
$$p = b_0, f_1, b_1, \dots, f_s, b_s$$
where $s \geq 0$ is an integer, $f_1, \dots, f_s$ is an edge path in B from u to u', where $b_0 \in B_u$, $b_s \in B_{u'}$ and $b_i \in B_{t(f_i)} = B_{o(f_{i+1})}$ for $0 < i < s$. Recall that each edge f_i has a label $g_i e_i k_i$ in $\mathcal{B}$, where $e_i = [f_i]$, $g_i = (f_i)_o$ and $k_i = (f_i)_t$.

Hence the $\mathbb{B}$-path p determines the $\mathbb{A}$-path $\mu(p)$ from $[u]$ to $[u']$ in $\mathbb{A}$ defined as follows:
$$\mu(p) = (b_0 g_1), e_1, (k_1 b_1 g_2), e_2, \dots, (k_{s-1} b_{s-1} g_s), e_s, (k_s b_s)$$
Notice that $|p| = |\mu(p)|$.

We also want to think about an $\mathbb{A}$-graph as an "automaton" over $\mathbb{A}$ which "accepts" a certain subgroup of the fundamental group of $\mathbb{A}$.

Definition 3.5. Let $\mathcal{B}$ be an $\mathbb{A}$-graph with a base-vertex $u_0 \in VB$.

We define the *language* $L(\mathcal{B}, u_0)$ as
$$L(\mathcal{B}, u_0) := \{\mu(p) | \, p \text{ is a reduced } \mathbb{B} - \text{path from } u_0 \text{ to } u_0 \text{ in } \mathcal{B}\}$$
Thus $L(\mathcal{B}, u_0)$ consists of $\mathbb{A}$-paths from $v_0 := [u_o]$ to v_0.

A simple but valuable observation states that the language of an $\mathbb{A}$-graph represents a subgroup in the fundamental group of $\mathbb{A}$.

Proposition 3.6. *Let $\mathcal{B}$ be an $\mathbb{A}$-graph, $u_0 \in VB$, $v_0 = [u_0]$ and $G = \pi_1(\mathbb{A}, v_0)$.*

Then:

(1) If p, p' are $\sim$-equivalent $\mathbb{B}$-paths, then $\mu(p) \sim \mu(p')$ as $\mathbb{A}$-paths.
(2) The map μ restricted to the set of $\mathbb{B}$-paths from u_0 to u_0 factors through to a homomorphism $\nu : \pi_1(\mathbb{B}, u_0) \to G$.
(3) We have $\overline{L(\mathcal{B}, u_0)} = \nu(\pi_1(\mathbb{B}, u_0))$. In particular, $\overline{L(\mathcal{B}, u_0)}$ is a subgroup of G.
(4) There is a canonical ν-equivariant simplicial map $\phi : \widetilde{(\mathbb{B}, u_0)} \to \widetilde{(\mathbb{A}, v_0)}$ respecting the base-points.

Proof. Part (1) follows directly from the definitions of $\sim$ and $\mathbb{B}$. Part (1) immediately implies parts (2) and (3).

To establish (4) we will provide a direct construction of ν which will rely on the explicit definition of Bass-Serre tree for a graph of groups given earlier. Denote $X = \widetilde{(\mathbb{A}, v_0)}$ and $Y = \widetilde{(\mathbb{B}, u_0)}$. Let $y = \overline{p} B_u$ be a vertex of Y, where p is a $\mathbb{B}$-path from u_0 to $u \in VB$. Denote $v = [u] \in VA$. We put $\phi(y) := \overline{\mu(p)} A_v \in VX$. First, note that this definition does not depend on the choice of p. Indeed, suppose p' is another $\mathbb{B}$-path from u_0 to u. Then by the normal form theorem $\overline{p'} = \overline{pb}$ for some $b \in B_u \le A_v$. Hence $\overline{\mu(p)}A_v = \overline{\mu(p)}bA_v = \overline{\mu(pb)}A_v = \overline{\mu(p')}A_v$. Thus ϕ is well-defined on the vertex set of Y.

It remains to check that ϕ preserves the adjacency relation. Let $y = \overline{p}B_u \in VY$ be as above and let $y' = \overline{pbf}B_u \in VY$ be an adjacent vertex of Y, where $b \in B_u \le A_v$ and where $f \in EB$ is an edge of type $e \in EA$ with $o(f) = u$. Thus $o(e) = v \in VA$. We already know that $\phi(y) = \overline{\mu(p)}A_v$. Denote $u' = t(f)$ and $v' = t(e)$, so that $[u'] = v'$. Also denote $g = f_o \in A_v$ and $h = f_t \in A_{v'}$. Then pbf is an $\mathbb{B}$-path from u_0 to u'.

Therefore

$$\phi(y') = \overline{\mu(pbf)}A_{v'} = \overline{\mu(p)bgeh}A_{v'} = \overline{\mu(p)bge}A_{v'}$$

is an adjacent vertex of $\phi(y) = \overline{\mu(p)}A_v$ since $bg \in A_v$. Thus indeed ϕ is a well-defined simplicial map from Y to X. We leave checking the equivariance properties of ϕ to the reader. $\square$

We shall see that every subgroup of $G = \pi_1(\mathbb{A}, v_0)$ arises in this fashion. Moreover, for an "efficient" choice of $\mathcal{B}$ the corresponding graph-of-groups $\mathbb{B}$ represents the induced splitting of the subgroup $H = \overline{L(\mathcal{B}, u_0)} \le G$ with respect to the action of H on the Bass-Serre covering tree of $\mathbb{A}$.

The following lemma is an immediate corollary of Proposition 2.3 and Proposition 3.6:

Lemma 3.7. *Let $\mathcal{B}$ be an $\mathbb{A}$-graph with a base-vertex u_0 of type v_0. Let $T \subseteq B$ be a spanning tree. For any two vertices $u, u' \in T$ denote by $[u, u']_T$ the T-geodesic path from u to u'.*

Then $\pi_1(\mathbb{B}, u_0)$ is generated by $\overline{S_T}$ where S_T is the following set:

$$S_T := \cup_{u \in VB}\big([u_0, u]_T B_u [u, u_0]_T\big) \bigcup \{[u_0, o(e)]_T e [t(e), u_0]_T \mid e \in E(B - T)\}.$$

In particular, $\overline{L(\mathcal{B}, u_0)} \le \pi_1(\mathbb{A}, v_0)$ is generated by $\overline{\mu(S_T)} = \nu(\overline{S_T})$.

4. Folding moves and folded graphs

Definition 4.1 (Folded $\mathbb{A}$-graph)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph.

We will say that $\mathcal{B}$ is *not folded* if at least one of the following applies:

(1) There are two distinct edges f_1, f_2 with $o(f_1) = o(f_2) = z$ and labels (a_1, e, b_1), (a_2, e, b_2) accordingly, such that z has label (A', u) and $a_2 = a'a_1\alpha_e(c)$ for some $c \in A_e$ and $a' \in A'$.
(2) There is an edge f with label (a, e, b), with $o(f)$ labeled (A', u) and $t(f)$ labeled (B', v) such that $\alpha_e^{-1}(a^{-1}A'a) \ne \omega_e^{-1}(bB'b^{-1})$.

Otherwise we will say that $\mathcal{B}$ is *folded*.

It is easy to see that if $\mathcal{B}$ is folded then any reduced $\mathbb{B}$-path translates into a reduced $\mathbb{A}$-path.

Lemma 4.2. *Let $\mathcal{B}$ be a folded $\mathbb{A}$-graph defining the graph of groups $\mathbb{B}$. Suppose p is a reduced $\mathbb{B}$-path. Then the corresponding $\mathbb{A}$-path $\mu(p)$ is $\mathbb{A}$-reduced.*

Proof. Suppose p is a $\mathbb{B}$-reduced $\mathbb{B}$-path and $\mu(p)$ is the corresponding $\mathbb{A}$-path. Assume that $\mu(p)$ is not reduced. Then p has a subsequence of the form f, a_1, f' where f^{-1}, f are edges of B of the same type $e \in EA$ such that the label of f^{-1} is aeb, the label of f' is $a'eb'$, where $v \in VA$ is the type of $o(f') = t(f) \in VB$, $a, a' \in A_v$, $a_1 \in B_{t(f)} \le A_v$ and the $\mathbb{A}$-path $e^{-1}, a^{-1}a_1a', e$ is not $\mathbb{A}$-reduced.

This means that for some $c \in A_e$ we have $a^{-1}a_1a' = \alpha_e(c)$, that is $a_1a' = a\alpha_e(c)$. If f^{-1} and f' are two distinct edges of B, this contradicts our assumption that $\mathcal{B}$ is folded. Thus $f^{-1} = f'$, so that $a = a', b = b'$. Therefore $a^{-1}a_1a = \alpha_e(c)$. Recall that since $\mathcal{B}$ is folded, part (2) of Definition 4.1 does not apply. Therefore the edge group in $\mathbb{B}$ is $B_{f'} = \alpha_e^{-1}(a^{-1}A_1a)$ and so $c \in B_{f'}$. Moreover, the edge-monomorphism of f' in $\mathbb{B}$ was defined as $\alpha^B_{f'}(c) = a\alpha_e(c)a^{-1}$. Thus $a_1 = a\alpha_e(c)a^{-1} \in \alpha^B_{f'}(B_{f'})$. Hence f, a_1, f' is not $\mathbb{B}$-reduced, contrary to our assumptions. □

The above lemma easily implies the following important fact:

Proposition 4.3. *Let $\mathcal{B}$ be a folded $\mathbb{A}$-graph defining the graph of groups $\mathbb{B}$. Let u_0 be a vertex of B of type $v_0 \in VA$. Denote $G = \pi_1(\mathbb{A}, v_0)$ and $U = \overline{L(\mathcal{B}, u_0)} \leq G$.*

Then the epimorphism $\nu : \pi_1(\mathbb{B}, u_0) \to U$ is an isomorphism and the graph map ϕ between the Bass-Serre covering trees $\phi : \widetilde{(\mathbb{B}, u_0)} \to \widetilde{(\mathbb{A}, v_0)}$ is injective.

The above Proposition essentially says that if $\mathcal{B}$ is a folded $\mathbb{A}$-graph defining a subgroup $U \leq G$, then $U = \pi_1(\mathbb{B}, u_0)$ is an induced splitting for $U \leq G = \pi_1(\mathbb{A}, v_0)$.

We will now describe certain moves, called *folding moves* on $\mathbb{A}$-graphs, which preserve the corresponding subgroups of the fundamental group of $\mathbb{A}$. These folding moves are a more combinatorial version of the folding moves of M.Bestvina-M.Feighn [5] and M.Dunwoody [18].

Whenever we make changes to the label of an edge f of an $\mathbb{A}$-graph we assume that the corresponding changes are made to the label of f^{-1}.

4.1. **Auxiliary moves.**

Definition 4.4 (Roll-over move $A0$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Suppose f is an edge of $\mathcal{B}$ with the label (a, e, b) such that the edge e of A is not a loop. Let the label of $o(f)$ be (A_1, y) and the label of $t(f)$ be (B_1, v). Thus $A_1 \leq A_y$, $B_1 \leq A_v$, $a \in A_y, b \in A_v$, where $y, v \in VA$.

Let $\mathcal{B}'$ be the $\mathbb{A}$-graph obtained from $\mathcal{B}$ as follows:

1. replace the label of the edge f by $(a, e, 1)$;
2. replace the label of $t(f)$ by (bB_1b^{-1}, v);
3. for each non-loop edge $f' \neq f^{-1}$ with origin $t(f)$ in $\mathcal{B}$ and label (b', e', b'') replace the label of f' by (bb', e', b'').
4. for each loop f' with origin $t(f)$ in $\mathcal{B}$ and label (b', e', b'') we replace the label of f' by $(bb', e', b''b^{-1})$.

In this case we will say that $\mathcal{B}'$ *is obtained from $\mathcal{B}$ by a folding move of type $A0$.*

Suppose $u_0 \in B$ is a fixed base-vertex. If f is an edge as above and $t(f) \neq u_0$, we will say that the roll-over move $A0$ along f is *admissible with respect to u_0.*

Definition 4.5 (Roll-over move $A0$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Suppose f is a non-loop edge of $\mathcal{B}$. Let $b = f_t$

Let $\mathcal{B}'$ be the $\mathbb{A}$-graph obtained from $\mathcal{B}$ as follows:

1. replace f_t by 1.
2. replace $B_{t(f)}$ by $bB_{t(f)}b^{-1}$;
3. for each non-loop edge $f' \neq f^{-1}$ of $\mathcal{B}$ with $o(f') = t(f)$ replace f_o with ff_o.
4. for each loop f' with origin $t(f)$ in $\mathcal{B}$ replace f_o with ff_o and f_t with f_tb^{-1}.

In this case we will say that $\mathcal{B}'$ *is obtained from* $\mathcal{B}$ *by a folding move of type* $A0$.

Suppose $u_0 \in B$ is a fixed base-vertex. If f is an edge as above and $t(f) \neq u_0$, we will say that the roll-over move $A0$ along f is *admissible with respect to* u_0.

Definition 4.6 (Bass-Serre move $A1$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Suppose f is an edge of $\mathcal{B}$ with the label (a, e, b). Let the type of $o(f)$ be $u \in VA$ and the type of $t(f)$ be $v \in VA$. Suppose $a = a'c'$ where $c' = \alpha_e(c)$ for some $c \in A_e$. Put $c'' = \omega_e(c)$.

Let $\mathcal{B}'$ be the $\mathbb{A}$-graph obtained from $\mathcal{B}$ by replacing the label of f with $(a', e, c''b)$. In this case we will say that $\mathcal{B}'$ *is obtained from* $\mathcal{B}$ *by a folding move of type* $A1$.

Definition 4.7 (Bass-Serre move $A1$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Suppose f is an edge of $\mathcal{B}$ and that $c \in A_{[f]}$.

Let $\mathcal{B}'$ be the $\mathbb{A}$-graph obtained from $\mathcal{B}$ by replacing f_o with $f_o\alpha_e(c)^{-1}$ and f_t with $\omega_e(c)f_t$.

In this case we will say that $\mathcal{B}'$ *is obtained from* $\mathcal{B}$ *by a folding move of type* $A1$.

Definition 4.8 (Simple adjustment $A2$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Suppose f is an edge of $\mathcal{B}$ with the label (a, e, b). Let the label of $o(f)$ be (A_1, u) and suppose $a_1 \in A_1$.

Let $\mathcal{B}'$ be the $\mathbb{A}$-graph obtained from $\mathcal{B}$ by replacing the label of f with (a_1a, e, b). In this case we will say that $\mathcal{B}'$ *is obtained from* $\mathcal{B}$ *by a folding move of type* $A2$.

Definition 4.9 (Simple adjustment $A2$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Suppose f is an edge of $\mathcal{B}$ and that $a' \in B_{o(f)}$.

Let $\mathcal{B}'$ be the $\mathbb{A}$-graph obtained from $\mathcal{B}$ by replacing f_o with $a'f_o$.

In this case we will say that $\mathcal{B}'$ *is obtained from* $\mathcal{B}$ *by a folding move of type* $A2$.

Definition 4.10 (Simple conjugation $A3$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Let $u_0 \in V$ be a vertex of type $v_0 \in VA$ with the vertex group $A_1 \leq A_v$. Suppose $x \in A_v$.

Let $\mathcal{B}'$ be obtained from $\mathcal{B}$ as follows. The underlying graphs of $\mathcal{B}$ and $\mathcal{B}'$ are the same: $B = B'$. For $u_0 \in VB$ replace the vertex group $B_{u_0} = A_1$ by $B'_{u_0} = x^{-1}A_1x \leq A_v$. For each non-loop edge $f \in EB$ with $o(f) = u_0$ replace the label (a, e, b) of f with $(x^{-1}a, e, b)$. For each loop-edge $f \in EB$ with $o(f) = t(f) = u_0$ replace the label (a, e, b) of f by $(x^{-1}a, e, bx)$. The remaining vertex groups and edge labels of $\mathcal{B}$ remain unchanged. We will say that the resulting $\mathbb{A}$-graph is obtained from $\mathcal{B}$ by a *folding move of type* $A3$ *corresponding to the element* $x \in A_v$.

Definition 4.11 (Simple conjugation $A3$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Let $u_0 \in VB$ be a vertex of $\mathcal{B}$ and $x \in A_{[u_o]}$

Let $\mathcal{B}'$ be obtained from $\mathcal{B}$ as follows:

(1) replace the group B_{u_0} by $x^{-1}B_{u_0}x$.
(2) For each non-loop edge $f \in EB$ with $o(f) = u_0$ replace f_o with $x^{-1}f_o$.
(3) For each loop-edge $f \in EB$ with $o(f) = t(f) = u_0$ replace f_o with $x^{-1}f_o$ and f_t with f_tx.

We will say that the resulting $\mathbb{A}$-graph is obtained from $\mathcal{B}$ by a *folding move of type* $A3$ *corresponding to the element* $x \in A_v$.

4.2. **Main folding moves.**

Definition 4.12 (Simple fold $F1$)**.** Let $\mathcal{B}$ be an $\mathbb{A}$-graph. Suppose f_1 and f_2 are two distinct non-loop edges of $\mathcal{B}$ with labels (a_1, e, b_1) and (a_2, e, b_2) accordingly and such that $o(f_1) = o(f_2)$, $t(f_1) \neq t(f_2)$. Let (A', u) be the label of $o(f_1)$. Let (B', v) be the label of $t(f_1)$ and let (B'', v) be the label of $t(f_2)$.