Discriminating groups and c-dimension

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Abstract

We prove that every linear discriminating (square-like) group is abelian and every finitely generated solvable discriminating group is free abelian. These results follow from manipulations with c-dimensions of groups. Here c-dimension of a group G is the length of a longest strictly decreasing chain of centralizers in G.

1 Introduction

A group G *discriminates* a group H if for any finite set of nontrivial elements $h_1, \ldots, h_k \in H$ there exists a homomorphism $\phi : H \to G$ such that $h_i^{\phi} \neq 1$ for $i = 1, \ldots, k$.

This notion of discrimination plays a role in several areas of group theory; for example, in the theory of varieties of groups [12], in algorithmic group theory [8], algebraic geometry over groups [2], and in universal algebra [9], [5].

Following [3] we say that a group G is *discriminating* if G discriminates $G \times G$. A group G is called *square-like* if G is universally equivalent to $G \times G$ [5]. Every discriminating group is square-like, but there are square-like non-discriminating groups. We refer to [3], [4], and [5] for a more detailed discussion of discriminating and square-like groups. One of the aims of the current research on discriminating groups is to develop methods which for a given group G could produce a simple universal axiom (or a "nice" set of such axioms) which distinguishes the quasi-variety $qvar(G)$ generated by G from the universal closure $ucl(G)$ of G (the minimal universal class containing G).

A partial description of discriminating abelian groups was given in [3]. In [4] investigation of solvable discriminating groups was started.

In Section 3 we prove the following results which answer completely to the Questions 2D and 3D from [4]:

every linear discriminating (square-like) group is abelian;

every finitely generated solvable discriminating group is free abelian.

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To prove this we use a notion called the c-dimension of a group. Namely, a group G has finite *c-dimension* if there exists a positive integer n such that every strictly descending chain of centralizers in G has length at most n. It is not hard to see that finite c-dimension can be described by universal axioms and it prevents a non-abelian group from being discriminating (or square-like).

In Section 2 we give various examples of groups of finite c-dimension and prove some elementary properties of such groups. It follows immediately from definition that abelian groups, stable groups (from model theory), commutative-transitive groups (in which commutation is an equivalence relation on the set of non-trivial elements), torsion-free hyperbolic groups - all have finite c-dimension. It is easy to see that linear groups with coefficients in a field (or in a finite direct product of fields) have finite c-dimension, in particular, finitely generated nilpotent groups [6], polycyclic groups [1], finitely generated metabelian groups [11, 13, 14], have finite c-dimension. It turns out that finitely generated abelian-by-nilpotent groups also have finite c-dimension [7]. Moreover, the class of groups of finite c-dimension is closed under taking subgroups, finite direct products, and universal equivalence. Recall that two groups are called universally equivalent if they satisfy the same universal sentences of the first-order group theory language.

All known examples of finitely generated discriminating groups are rather special. It is unknown what types of finitely generated non-abelian discriminating groups can satisfy a non-trivial identity.

2 Groups of finite c-dimension

Definition 1 *For a group G we define a cardinal $dim_c(G)$ as the length of a longest strictly descending chain of centralizers in G. The cardinal $dim_c(G)$ is called c-dimension (or centralizer-dimension) of G. A group G is said to be of finite c-dimension if $dim_c(G)$ is finite.*

Recall that a group is called *commutative-transitive* if commutation is an equivalence relation on the set of all non-trivial elements from G. Since in commutative-transitive groups proper centralizers are maximal abelian subgroups these groups have finite c-dimension. In particular, torsion-free hyperbolic groups have finite c-dimension.

Stable groups (from model theory) provide another type of examples of groups with finite c-dimension (see, for example, [10]). The following result is known ([10]), but we give a proof for completeness.

Proposition 2.1 *General linear groups $GL(m, K)$ over a field K have finite c-dimension.*

Proof. Let $A_i \in GL(m, K), i \in I$, be a finite set of matrices. Then the system $T = 1$ of matrix equations $[X, A_i] = 1$ $(i \in I)$, where X is an indeterminate matrix, is equivalent to a system $S_T = 0$ of linear equations over K with m^2 variables (the entries of X). The system $S_T = 0$ has at most m^2 independent equations, hence the system $T = 1$ is equivalent to its own subsystem of at most m^2 equations. This implies that the length of any strictly descending chain of centralizers in $GL(m,K)$ is at most m^2+1, so $dim_c(GL(m,K)) \leq m^2+1$.

Lemma 2.2 *Let G and H be groups. Then the following holds:*

1) *If $H \leq G$ then $dim_c(H) \leq dim_c(G)$;*

2) *If $dim_c(G) < \infty$ and $dim_c(H) < \infty$ then*

$$dim_c(G \times H) = dim_c(G) + dim_c(H) - 1.$$

Proof. To show 1) it suffices to notice that if

$$C_H(A_1) > C_H(A_2) > \ldots > C_H(A_\alpha) \ldots$$

is a strictly descending chain of centralizers in H then

$$C_G(A_1) > C_G(A_2) > \ldots > C_G(A_\alpha) \ldots$$

is also a strictly descending chain of centralizers in G.

2) Let

$$C_G(A_1) > C_G(A_2) > \ldots > C_G(A_d),$$

$$C_H(B_1) > C_H(B_2) > \ldots > C_H(B_c)$$

be strictly descending chains of centralizers in G and H, correspondingly. Then

$$C(A_1 \times B_1) > C(A_2 \times B_1) > \ldots > C(A_d \times B_1) >$$

$$C(A_d \times B_2) > C(A_d \times B_3) > \ldots > C(A_d \times B_c)$$

is a strictly descending chain of centralizers in $G \times H$, which has length $d+c-1$. Hence

$$dim_c(G \times H) \geq dim_c(G) + dim_c(H) - 1.$$

We prove the converse by induction on $dim_c(G) + dim_c(H)$. If $dim_c(G) = dim_c(H) = 1$ then G and H are abelian. Hence $G \times H$ is abelian and

$$dim_c(G \times H) = 1 = dim_c(G) + dim_c(H) - 1.$$

Now, let

$$C_{G\times H}(Z_1) > C_{G\times H}(Z_2) > \ldots > C_{G\times H}(Z_k)$$

be a strictly descending chain of length k of centralizers in $G \times H$. Then

$$C_{G\times H}(Z_2) = C_G(B) \times C_H(D)$$

for suitable subsets $B \subset G, D \subset H$. Strict inequality

$$C_{G\times H}(Z_1) > C_{G\times H}(Z_2)$$

implies

$$G \geq C_G(B), \quad H \geq C_H(D)$$

where at least one of these inclusions is proper. By induction

$$dim_c(C_{G\times H}(Z_2)) = dim_c(C_G(B)\times C_H(D)) = dim_c(C_G(B)) + dim_c(C_H(D)) - 1$$

$$\leq dim_c(G) + dim_c(H) - 2.$$

Clearly,

$$k \leq dim_c(C_{G\times H}(Z_2)) + 1 \leq dim_c(G) + dim_c(H) - 1,$$

as required. This proves the result.

Combining Proposition 2.1 and Lemma 2.2 we get the following result.

Corollary 2.3 *Let G be a linear group, or a subgroup of a stable group, or a finite direct product of such groups. Then G has a finite c-dimension.*

The next result is a generalization of Proposition 2.1.

Proposition 2.4 *Let $R = K_1 \times \ldots \times K_n$ be a finite direct product of fields K_i. Then the general linear group $GL(m, R)$ has finite c-dimension.*

Proof. For every $i = 1, \ldots, n$ denote by π_i the canonical projection $\pi_i : R \to K_i$. Then the homomorphism π_i gives rise to a homomorphism

$$\phi_i : GL(m, R) \to GL(m, K_i).$$

Clearly, for each non-trivial element $g \in GL(m, R)$ there exists an index i such that $\phi_i(g) \neq 1$. Therefore, the direct product of homomorphisms $\phi = \phi_1 \times \ldots \times \phi_n$ gives an embedding

$$\phi : GL(m, R) \to GL(m, K_1) \times \ldots \times GL(m, K_n).$$

Hence $GL(m, R)$ has finite c-dimension as a subgroup of a finite direct product of groups of finite c-dimensions (Proposition 2.4, 2.1, Lemma 2.2). This proves the proposition.

V. Remeslennikov proved in [11] that a finitely generated metabelian group (under some restrictions) is embeddable into $GL(n, K)$ for a suitable n and a suitable field K. In [13], see also [14], B. Wehrfritz showed that any finitely generated metabelian group is embeddable into $GL(n, R)$ for a suitable n and a suitable ring $R = K_1 \times \ldots \times K_n$ which is a finite direct product of fields K_i. This, together with Proposition 2.4 implies the following

Corollary 2.5 *Every finitely generated metabelian group has finite c-dimension.*

The following result provides another method to construct groups of finite c-dimension.

Proposition 2.6 *Let G be a group with $dim_c(G) < \infty$. Then the following holds:*

1) *If a group H discriminates G then $dim_c(H) \geq dim_c(G)$;*

2) *If a group H is universally equivalent to G then $dim_c(H) = dim_c(G)$;*

Proof. Let

$$C(A_1) > C(A_2) > \ldots > C(A_d)$$

be a strictly descending finite chain of centralizers in G. There are elements $g_i \in C(A_i)$ and $a_{i+1} \in A_{i+1}$ such that $[g_i, a_{i+1}] \neq 1$ for $i = 1, \ldots, d-1$. Since H discriminates G there exists a homomorphism $\phi : G \to H$ such that $[g_i, a_{i+1}]^\phi \neq 1$. This shows that the chain of centralizers

$$C(A_1^\phi) > C(A_2^\phi) > \ldots > C(A_d^\phi)$$

is strictly descending in H. This proves 1).

To prove 2) one needs only to verify that the argument in 1) can be described by an existential formula, which is easy.

3 Main results

Theorem 3.1 *Let G be a group of finite c-dimension. If G is discriminating or square-like, then G is abelian.*

Proof. Let G be a group of finite c-dimension. If G is discriminating then G discriminates $G \times G$. Hence by Proposition 2.6 and Lemma 2.2

$$dim_c(G) \geq dim_c(G \times G) = 2dim_c(G) - 1.$$

This implies that $dim_c(G) = 1$, i.e., the group G is abelian.

If G is square-like, then G is universally equivalent to $G \times G$ and hence by Proposition 2.6

$$dim_c(G) = dim_c(G \times G) = 2dim_c(G) - 1.$$

Again, it follows that $dim_c(G) = 1$, and the group G is abelian. Theorem has been proven.

Combining Theorem 3.1 and Corollaries 2.3 and 2.5 we obtain the following theorem.

Theorem 3.2 1) *Every linear discriminating (square-like) group is abelian;*

2) *Every finitely generated metabelian discriminating (square-like) group is abelian.*

The following notion allows one to argue by induction when dealing with discriminating groups. We fix a group G and a normal subgroup N of G.

Definition 2 *We say that G is* discriminating modulo N *(or N-*discriminating*) if for any finite set X of elements from $G \times G$, but not in $N \times N$, there exists a homomorphism $\phi : G \times G \to G/N$ such that $x^\phi \neq 1$ for any $x \in X$.*

For a subset $A \subset G$ denote by $C_G(A, N)$ the centralizer of A modulo N:

$$C_G(A, N) = \{g \in G \mid [g, A] \subseteq N\},$$

which is the preimage of the centralizer $C_{G/N}(A^\nu)$ in G/N under the canonical epimorphism $\nu : G \to G/N$. We define a c-dimension $dim_{c,N}(G)$ of G as the length of the longest chain of strictly descending centralizers in G modulo N. Obviously,

$$dim_{c,N}(G) = dim_c(G/N). \tag{1}$$

The same argument as in Proposition 2.6 shows that if G is N-discriminating then

$$dim_{c,N\times N}(G \times G) \leq dim_c(G/N). \tag{2}$$

Now if N is a normal subgroup of G and K is a normal subgroup of a group H then

$$dim_{c,N\times K}(G \times H) = dim_c(G \times H/N \times K) = dim_c(G/N \times H/K).$$

By Lemma 2.2

$$dim_c(G/N \times H/K) = dim_c(G/N) + dim_c(H/K) - 1,$$

hence

$$dim_{c,N\times K}(G \times H) = dim_{c,N}(G) + dim_{c,K}(H) - 1. \tag{3}$$

The following is a slight generalization of Theorem 3.1.

Lemma 3.3 *Let G be N-discriminating. If G/N has finite c-dimension then G/N is abelian.*

Proof. It readily follows from (2) and (3) that if G is N-discriminating then

$$dim_{c,N\times N}(G \times G) = dim_{c,N}(G) + dim_{c,K}(H) - 1 \leq dim_c(G/N),$$

hence $dim_c(G/N) = 1$ and G/N is abelian, as required.

Lemma 3.4 *Let G be an N-discriminating group, $v(G)$ be a verbal subgroup of G, and $C = C_G(v(G), N)$. Then G is C-discriminating.*

Proof. Observe that

$$C_G(v(G), N)^\nu = C_{G/N}(v(G)^\nu) = C_{G/N}(v(G/N)).$$

Now if

$$(g_1, h_1), \ldots, (g_k, h_k) \in G \times G \smallsetminus C \times C$$

then there exist elements

$$(a_1, b_1), \dots, (a_k, b_k) \in v(G) \times v(G)$$

such that $[(g_i, h_i), (a_i, b_i)] \notin N$ for $i = 1, \dots, k$. Since G is N-discriminating there exists a homomorphism $\phi : G \times G \to G/N$ such that

$$[(g_i, h_i), (a_i, b_i)]^\phi \neq 1 \quad (i = 1, \dots, k).$$

Notice that $(a_i, b_i)^\phi \in v(G/N)$. It follows that

$$(g_1, h_1)^\phi, \dots, (g_k, h_k)^\phi \notin C_{G/N}(v(G/N))$$

and their canonical images are non-trivial in G/C, as desired.

Lemma 3.5 *Let G be a finitely generated N-discriminating group. If G/N is solvable then it is abelian.*

Proof. Let $G^{(0)} = G$ and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ be the i-term of the derived series of G. Denote by k the derived length of G/N. We assume that $k \geq 2$ and proceed by induction on k. Set $C = C_G(G^{(k-1)}, N)$. Then $G^{(k-1)} \leq C$ and the derived length of G/C is at most $k - 1$. By Lemma 3.4 G is C-discriminating. By induction G/C is abelian so $G^{(1)} \leq C$ and $[G^{(k-1)}, G^{(1)}] \leq N$. Now put $D = C_G(G^{(1)}, N)$. Then $G^{(k-1)} \leq D$ and so G/D has derived length less than k. By Lemma 3.4 G is D-discriminating and by induction G/D is abelian. Therefore $G^{(1)} \leq D$. This implies that $G^{(2)} \leq N$ and the group G/N is finitely generated and metabelian. By Corollary 2.5 G/N has finite c-dimension. Now in view of Lemma 3.3 we conclude that G/N is abelian.

Now from Lemma 3.5 (for $N = 1$) and the description of finitely generated discriminating abelian groups from [3], we deduce

Theorem 3.6 *Every finitely generated discriminating solvable group is free abelian.*

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