CSA groups and separated free constructions

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1 Introduction

The class of hyperbolic groups and the class of groups that can act freely on a Λ-tree are two rapidly developing areas of group theory that have attracted the attention of specialists from many fields of mathematics. Their study involves an interplay of geometry and algebra. Another very active area of group theory, this time connected to mathematical logic, is the study of systems of equations in free groups and hyperbolic groups. The success of this study has inspired renewed interest in Tarski's famous problem: "is the elementary theory of a free nonabelian group algorithmically decidable?" One of the key steps in this direction is the description of $\exists$-free groups (i.e., those groups that have the same existential theory as a non-abelian free group). The purpose of this article is to study the class of CSA-groups, which contains the classes of torsion-free hyperbolic groups [?], groups acting freely on Λ-trees [?] and $\exists$-free groups [?], [?]. CSA-groups share many of the properties of the groups in the above-mentioned three classes, but have the advantage of being definable in purely group-theoretic terms.

Definition 1 *[?] We define a subgroup H of a group G to be* malnormal *(also called* conjugate separated*) if $H \cap H^x = 1$ for all $x \in G - H$.*

It is clear that the intersection of a family of malnormal subgroups is again malnormal, which allows us to define the *malnormal closure* $\mathrm{mal}_G(A)$ of a subgroup A of a group G to be the intersection of all the malnormal subgroups of G containing A.

Definition 2 *[?] A group G is called a* CSA-group *if all its maximal abelian subgroups are conjugate separated.*

Definition 3 *Let G be a group and $\phi : A \to B$ an isomorphism of subgroups of G. The HNN-extension*

$$G^* = \left\langle G, t \,\middle|\, t^{-1}at = \phi(a), a \in A \right\rangle$$

is called:

1) separated *if $A \cap g^{-1}Bg = 1$ for all $g \in G$;*

2) strictly separated *if $A \cap g^{-1} mal_G(B) g = 1$ for all $g \in G$.*

O. Kharlampovich and A. Myasnikov recently proved [?] that the class of hyperbolic groups is closed under separated HNN-extensions, subject to the condition that the two associated subgroups A and B be quasi-isometrically embedded in G and one of them be malnormal. The class of groups acting freely on Λ-trees is also closed under separated HNN-extensions, provided that A and B are abelian and satisfy some natural compatibility condition [?].

In the same spirit, we prove in section 2 that separated HNN-extensions of an arbitrary CSA*-group (CSA, without elements of order two) with associated malnormal subgroups is again CSA*. In fact, a much more general result (Theorem 1) is true: any strictly separated HNN-extension of a CSA*-group G with associated subgroups A and B is again CSA* if A is malnormal in G and B is normal in $\mathrm{mal}_G(B)$. We obtain similar results for amalgamated products of CSA*-groups (Theorem 2) and, in Theorem 3, for the fundamental groups of certain types of graphs of CSA*-groups. We show that in Theorem 1, which is the main result of the section, neither the requirement that the HNN-extension be separable, nor the requirement that one of the associated subgroups be malnormal, is a necessary condition for the HNN-extension to be CSA (this is not astonishing, because even non-abelian free group can be decomposed in very complicated ways in terms of HNN-extensions and free products with amalgamation). However, in the most important case, where the associated subgroups are abelian, we use our main result to obtain in the next section a complete description of those HNN-extensions which preserve the CSA*-property.

We start section 3 by showing that if an HNN-extension with abelian associated subgroups is CSA, then at least one of the associated subgroups must be maximal abelian. Under these assumptions there then remain four types of HNN-extensions, among which only the separated HNN-extensions and the rank 1 extensions of centralizers (described in Proposition 4) preserve the CSA*-property. Theorem 4, which states the preservation of the CSA*-property by separated HNN-extensions, has as its corollary a similar result about amalgamated products and tree products of CSA*-groups.

Gersten has conjectured that a torsion-free one-relator group is hyperbolic if and only if it does not contain any Baumslag-Solitar groups

$$B_{m,n} = \left\langle x, y \,\middle|\, yx^m y^{-1} = x^n \right\rangle, \; mn \neq 0$$

(HNN-extensions of the infinite cyclic group). We prove that a torsion-free one-relator group fails to be CSA if and only if it contains a nonabelian metabelian Baumslag-Solitar group $B_{1,n}$, $n \neq 1$, or the group $\mathcal{B} = F_2 \times \mathbf{Z}$, the product of a free group on two generators by an infinite cycle (Theorem 7); a one-relator group with torsion fails to be a CSA-group if and only if it contains the infinite dihedral group D_∞ (Theorem 8). This gives a complete description, in terms of "obstacles", of one-relator CSA-groups. Every one-relator group with torsion is hyperbolic. This has already been observed in [**?**]; it follows easily from a theorem of B. B. Newman (see [**?**] or [**?**], Proposition 5.28 on p. 109).

It is known that CSA-groups are commutative transitive (a group G is commutative transitive if the relation "a commutes with b" is transitive on the set $G - \{1\}$; i.e., if the centralizer of every nontrivial element of G is abelian, [**?**], Proposition 10). That the converse is not true was pointed out in [**?**], and is shown by the simple example of the infinite dihedral group D_∞. However, it *is* true for torsion-free one-relator groups (Theorem 9). It is not true, in general, for one-relator groups with torsion, because all these groups are commutative transitive (B.B.Newman [**?**], Theorem 2) but some of them are not CSA (see the example in Proposition 8).

In section 5 we consider exponential groups and the property of being residually of prime power order ("residually p"). Let A be an (associative) ring with identity. A group G is called an *A-group* if its elements admit exponents from the ring A. The defining axioms can be found in [**?**], [**?**], [**?**]. The *tensor completion over* A of a group is defined by the obvious universal property, and a group is said to be A-*faithful* if the canonical morphism from the group into its tensor completion over A is injective. Tensor completions of groups have been studied extensively in [?], [?] and [?]. Myasnikov and Remeslennikov proved that if G is a torsion-free CSA-group and A a ring whose underlying abelian group is torsion free, then G is A-faithful, and the tensor completion of G over A is again a torsion-free CSA-group ([?], Theorem 9). So, the class of torsion-free CSA-groups is contained in the class of $\mathbf{Q}$-faithful groups (it is clear that a $\mathbf{Q}$-faithful group must be torsion free). We prove (Proposition 9) that if, for almost all primes p, a group is residually p, then it is $\mathbf{Q}$-faithful.

Gilbert Baumslag [?] posed the problem of determining which one-relator groups are $\mathbf{Q}$-faithful. The class of torsion-free one-relator CSA-groups is strictly contained in the class of one-relator $\mathbf{Q}$-faithful groups as well as in the class of one-relator groups that are residually p for almost all primes p (Proposition 12). It would be interesting to find obstacles to $\mathbf{Q}$-faithfulness in torsion-free one-relator groups (in the sense that the groups $\mathcal{B}$ and $B_{1,n}$ of Theorem 7 are obstacles to the CSA-property, and the groups $B_{m,n}$ are conjectured to be obstacles to hyperbolicity).

In Proposition 12, we give an example of a one-relator group which is not CSA, but is nevertheless $\mathbf{Q}$-faithful, residually torsion-free nilpotent and residually p, with torsion-free pro-p-completion, for every prime p. On the other hand metabelian non-abelian Baumslag-Solitar groups provide examples of $\mathbf{Q}$-faithful groups which are non-CSA and not residually p for almost all primes p.

2 HNN-extensions of CSA*-groups

Our main purpose in this section is to establish some natural sufficient conditions for HNN-extensions and amalgamated products to preserve the CSA-property. We also investigate the more general problem of determining when the fundamental group of a graph of CSA-groups is again CSA.

Definition 4 *A* CSA*-group *is a CSA-group without elements of order 2.*

Throughout the paper, G^* will denote an HNN-extension of a group G relative to an isomorphism of associated subgroups $\phi : A \to B$, and t will denote the stable letter. Recall that any HNN-extension G^* of a group G is endowed with a length function ([**?**], p. 185). We denote the length of an element z of G^* by $|z|$.

Lemma 1 *Let G^* be a strictly separated HNN-extension of a group G with associated subgroups A and B such that $A = mal_G(A)$ and $B \trianglelefteq mal_G(B) = B_1$. Let $c \in G^*$.*

1. *If $1 \neq b_1 \in B_1$, $b_1^c \in G$ and $c \notin G$, then $c \in B_1 t^{-1} G$ and $b_1 \in B$.*
2. *If $B_1^c \cap B_1 \neq 1$, then $c \in B_1$.*
3. *If $1 \neq a \in A$, $a^c \in G$, then there are three possibilities: $c \in A$, $c \in tG$ or $c \in tB_1 t^{-1} G$.*
4. *If $A^c \cap A \neq 1$, then $c \in tB_1 t^{-1}$.*

Proof. Suppose that $c \notin G$. We write c in reduced form:

$$c = g_0 t^{e_1} g_1 t^{e_2} g_2 \cdots g_{n-1} t^{e_n} g_n, \ |c| = n \geq 1. \quad (1)$$

1. Suppose that $b_1 \in B_1$ and $b_1^c \in G$. We have

$$c^{-1} b_1 c = g_n^{-1} t^{-e_n} g_{n-1}^{-1} \cdots g_2^{-1} t^{-e_2} g_1^{-1} t^{-e_1} g_0^{-1} b_1 g_0 t^{e_1} g_1 t^{e_2} g_2 \cdots g_{n-1} t^{e_n} g_n. \quad (2)$$

Since the length of the left side of this equation is 0, the right side is not reduced and

$$g_0^{-1} b_1 g_0 \in \begin{cases} A \text{ if } e_1 = 1 \\ B \text{ if } e_1 = -1 \end{cases}$$

If $e_1 = 1$, then $b_1 \in A^{g_0^{-1}} \cap B_1$, which violates our hypotheses. Hence, $e_1 = -1$, $g_0^{-1} b_1 g_0 \in B$, $b_1 \in B_1 \cap B^{g_0^{-1}} \subseteq B_1 \cap B_1^{g_0^{-1}}$, which implies that $g_0 \in B_1$ and $b_1 \in B^{g_0^{-1}} \subseteq B$. Write $a = t b_1^{g_0} t^{-1}$. If $n \geq 2$, then we see from (2) that the word $t^{-e_2} g_1^{-1} a g_1 t^{e_2}$ is not reduced, and

$$g_1^{-1} a g_1 \in \begin{cases} B \text{ if } e_2 = -1 \\ A \text{ if } e_2 = 1. \end{cases}$$

If $e_2 = -1$ then $a \in B \cap A^{g_1^{-1}} \subseteq B_1 \cap A^{g_1^{-1}}$, which contradicts our hypotheses. So, $e_2 = 1$, $a \in A \cap A^{g_1^{-1}}$ and $g_1 \in A$. But then this contradicts the assumption

that (1) was in reduced form. It follows that $n = 1$ and $c = g_0 t^{-1} g_1 \in B_1 t^{-1} G$.
2. Since B_1 is malnormal in G, it is sufficient to derive a contradiction from our assumption that $c \notin G$. It follows from part 1 that if $1 \neq b_1 \in B_1 \cap c B_1 c^{-1}$, then $b_1 \in B$ and for some $g \in G$, $c \in B_1 t^{-1} g$. But then

$$b_1^c \in g^{-1} t B t^{-1} g \cap B_1 = A^g \cap B_1,$$

which contradicts our assumptions.
3. We keep our assumption that c has reduced form (1) (if $c \in G$ and $a^c \in A$, then $c \in A$, since A is malnormal). By a similar reasoning as in part 1 we find that $e_1 = 1$, $g_0 \in A$. Letting $a^{g_0 t}$ and $t^{-1} g_0^{-1} c$ play the roles of b_1 and c respectively in part 1, we find that $t^{-1} g_0^{-1} c \in G$ or $t^{-1} g_0^{-1} c \in B_1 t^{-1} G$; i.e., $c \in g_0 t G = tG$ or $c \in g_0 t B_1 t^{-1} G = t B_1 t^{-1} G$.
4. If $A^c \cap A \neq 1$, then $B_1^{c^t} \cap B_1 \neq 1$ and $c \in t B_1 t^{-1}$ by part 2.
This completes the proof of the lemma.

Lemma 2 *Let G^* be a strictly separated HNN-extension of a group G with associated subgroups A and B such that $A = mal_G(A)$ and $B \trianglelefteq mal_G(B) = B_1$. Suppose that M is a maximal abelian subgroup of G^*. Then one of the following is true.*

1. *The intersection of M with some conjugate $A^{x^{-1}}$ $(x \in G^*)$ of A is nontrivial, in which case $M \subseteq x t B_1 t^{-1} x^{-1}$.*

2. *The intersection of M with every conjugate of A is trivial, but M intersects some conjugate of G nontrivially, in which case*

$$(\forall x \in G^*)\,(M \cap G^x \neq 1 \Rightarrow M \subseteq G^x)\,.$$

3. *There exists a cyclically reduced element $z \in G^*$ such that M is a conjugate of the infinite cyclic group $\langle z \rangle$, $|z| \geq 1$.*

Proof. It follows from [**?**], Chapter 1, section 5.4, Theorem 13 and its Corollary 2 that G^* can be made to act on a tree X in such a way that the stabilizers of the edges are conjugates of A in G^* (the vertices of X are cosets yG of G in G^*, and the edges are cosets of A in G^*). If a nontrivial element of M stabilizes some edge of X, then there exists and element $x \in G^*$ such that $M \cap A^{x^{-1}} \neq 1$, and hence $M^{xt} \cap B_1 \neq 1$. Every element of M^{xt} then centralizes a nontrivial element of B_1, and it follows from Lemma 1, part 2, that M^{xt} is contained in B_1, and hence M in $xtB_1t^{-1}x^{-1}$.

Suppose now that M intersects trivially all the stabilizers of edges of the tree X. It follows from [?], Chapter 1, section 5.4 Theorem 13 (see also Example 1) of section 5.1) that M is a free product of conjugates of subgroups of G and a free group. Since M is abelian, the free product is trivial (i.e., it has only one factor). More precisely, if a nontrivial element of M fixes some vertex of X, say the coset wG (equivalently $M \cap wGw^{-1} \neq 1$), then M is entirely contained in the conjugate wGw^{-1} of G; i.e., M is a conjugate of a maximal abelian subgroup of G.

On the other hand, if M acts freely on the tree X then M is free, hence cyclic, and is generated by an element w of length ≥ 1. Since w is not conjugate to an element of G (w does not fix any vertex of X), it is conjugate to a cyclically reduced element z, with $|z| \geq 1$. This completes the proof of the lemma.

Theorem 1 *Let G^* be a strictly separated HNN-extension of a CSA* group G with associated subgroups A and B such that $A = mal_G(A)$ and $B \trianglelefteq mal_G(B)$. Then G^* is a CSA*-group.*

Proof. It is clear that G^* has no elements of order 2 ([?], Theorem 2.4, p. 185). We may assume that A and B are nontrivial (else G^* is the free product of G and an infinite cyclic, hence CSA [?]). As usual we will use the notation $B_1 = \mathrm{mal}_G(B)$.

Let M be a maximal abelian subgroup of G^*. Suppose that $\exists v \in G^*$ such that $M \cap M^v \neq 1$. We must prove that $v \in M$.

We first consider the case where some element of M has nontrivial intersection with a conjugate $A^{x^{-1}}$ of A ($x \in G^*$). Then, according to Lemma 2, $M^{xt} \subseteq B_1$. The groups M^{xt} and M^{vxt} have a nontrivial element w in common, which belongs to B_1 and which is centralized by M^{vxt}. By Lemma 1, M^{vxt} (which can be written $M^{xtv^{xt}}$) is contained in B_1 and

$$1 \neq w \in B_1 \cap B_1^{v^{xt}}.$$

It follows from Lemma 1 that $v^{xt} \in B_1 \subseteq G$. Since G is CSA, and

$$1 \neq w \in M^{xt} \cap M^{xtv^{xt}},$$

we conclude that $v^{xt} \in M^{xt}$ and $v \in M$.

We now assume that M intersects trivially every conjugate of A but intersects nontrivially a conjugate G^w of G, $w \in G^*$. In this case, Lemma 2 tells us that M is entirely contained in G^w. But then M^v also intersects G^w nontrivially, and there exists a maximal abelian subgroup N of G such that $M = M^v = N^w$ (G is commutative transitive). Next, we let $v' = wvw^{-1}$, then $N^{v'} = N$, and we claim that $v' \in G$. If not, we write it in reduced form:

$$v' = v_0 t^{e_1} v_1 t^{e_2} v_2 \cdots v_{n-1} t^{e_n} v_n, \; |v'| = n \geq 1. \tag{3}$$

Let $z \in N$. We have

$$(v')^{-1} zv' = v_n^{-1} t^{-e_n} v_{n-1}^{-1} \cdots v_2^{-1} t^{-e_2} v_1^{-1} t^{-e_1} v_0^{-1} z v_0 t^{e_1} v_1 t^{e_2} v_2 \cdots v_{n-1} t^{e_n} v_n. \tag{4}$$

Since the length of the left side of this equation is 0, the right side is not reduced and

$$v_0^{-1} z v_0 \in \begin{cases} A \text{ if } e_1 = 1 \\ B \text{ if } e_1 = -1 \end{cases}$$

In both cases we find that M has a nontrivial intersection with a conjugate of A in G^*, contrary to our assumption that it intersects trivially all the stabilizers of edges of the tree X. So $v \in G$. Since G is a CSA-group, $v \in M$.

Suppose now that the third possibility of Lemma 2 applies to M. Replacing M by one of its conjugates, we may suppose that $M = \langle z \rangle$, with z cyclically reduced. All powers of z are then cyclically reduced as well. There exist integers m, n such that $v^{-1} z^m v = z^n$. By the Conjugacy Theorem for HNN-extensions ([**?**], Chapter 4, Th. 2.5),

$$|m| \, |z| = |z^m| = |z^n| = |n| \, |z| \, ;$$

hence $m = \pm n$ and v^2 lies in the center of the group $\langle z^n, v \rangle$. Suppose that $\langle z^n, v \rangle$ intersects a conjugate $A^{x^{-1}}$ of A nontrivially, then $\exists w \in \langle v, z^n \rangle^{xt} \cap B_1$, $w \neq 1$. Since $(v^{xt})^2$ centralizes w, it follows from Lemma 1 that $(v^{xt})^2 \in B_1$. Since $(z^{xt})^n$ centralizes the nontrivial element $(v^{xt})^2$ of B_1, it follows again from Lemma 1 that $(z^{xt})^n \in B_1$. But z^n, being cyclically reduced and of length ≥ 1, cannot belong to a conjugate of G. This proves that $\langle z^n, v \rangle$ is a free product of subgroups of conjugates of G and a free group. Since $\langle z^n, v \rangle$ has nontrivial center, it is indecompasable as a free product ([**?**], Section 4.1, Problem 21, p.195). Since, as already pointed out, z^n is not conjugate to an element of G, $\langle z^n, v \rangle$ is contained in a free group, hence must be cyclic. This implies that z^n is in the center of $\langle z,v\rangle$. By the same argument as before, if $\langle z,v\rangle$ intersects some conjugate of A then z is contained in a conjugate of G, which is impossible, since z is cyclically reduced of length ≥ 1. So $\langle z,v\rangle$ is a free product of subgroups of conjugates of G and a free group. It has nontrivial center, which implies as before that $\langle z,v\rangle$ is indecomposable, and hence contained in a free group. This implies that v commutes with z. By the maximality of $M=\langle z\rangle$, $v\in M$. This is what we had to prove.

Corollary 1 *A separated HNN-extension of a CSA***-group with malnormal associated subgroups is a CSA***-group.*

Similar results for amalgamated products can be easily obtained using the following lemma.

Lemma 3 *Let A and B be subgroups of groups G and H respectively, $\phi: A\to B$ an isomorphism. The groups A and B can be considered as isomorphic subgroups of the free product $G*H$. Let us denote by*

$$E\left(G,H,\phi\right)=\left\langle G*H,t\,\middle|\,t^{-1}at=\phi(a)\right\rangle$$

the HNN-extension, associated with ϕ, of the group $G*H$. The amalgamated product $G*_\phi H$ is embeddable in $E\left(G,H,\phi\right)$.

Proof. The subgroup $\langle G^t,H\rangle$ generated in $E\left(G,H,\phi\right)$ by the t-conjugate of G and H is isomorphic to $G*_\phi H$. It can be easily verified using the normal forms of elements in $E\left(G,H,\phi\right)$.

Theorem 2 *Let G and H be CSA***-groups, A and B subgroups of G and H, respectively, such that $A=mal_G(A)$ and $B\trianglelefteq mal_H(B)$, and $\phi: A\to B$ an isomorphism. Then the amalgamated product $G*_\phi H$ is CSA***.*

Proof. Let $B_1=\mathrm{mal}_H(B)$. First, we claim that A and B_1 are also malnormal in the free product $G*H$. Malnormality is transitive: if X is a malnormal subgroup of Y and Y is a malnormal subgroup of Z, then X is a malnormal subgroup of Z. So we need only point out (as Lyndon and Schupp already did on page 203 of [**?**]) that the factors of a free product are malnormal in the product. Then we need the fact that A and B are mutually conjugate separated, i.e. $A\cap B_1^x=1$ for all $x\in G*H$. It is enough to prove that G and H are mutually conjugate separated in their free product, and this is easily verified using normal forms. Next, we need the fact that the class of CSA*-groups is closed under free products ([**?**], Theorem 4). To complete the proof, we note that the amalgamated product $G *_\phi H$ is embedded in the HNN-extension of $G * H$, relative to the isomorphism ϕ (Lemma 3), and we apply Theorem 1 (note that subgroups of CSA*-groups are CSA*, [**?**], Proposition 13).

Corollary 2 *An amalgamated product of CSA*-groups with malnormal amalgamated subgroups is again CSA*-group.*

To deal with graphs of groups we need the next Proposition, mentioned without proof in [**?**].

Proposition 1 *Let A be a malnormal subgroup of a group G, B a subgroup of a group H, $\phi : A \to B$ an isomorphism, and $P = G *_\phi H$ the associated amalgamated product. Then every malnormal subgroup of H is also malnormal in P.*

Proof. To simplify the exposition, we will suppose that G and H are subgroups of P and ϕ is the identity. Because of the transitivity of malnormality, it suffices to prove that H is malnormal in P. Every nontrivial element x of P can be written in the form

$$x = p_1 p_2 \cdots p_r,$$

where each p_i lies in one of the factors G or H, and no p_i lies in A if $r > 1$; also, if the length r of x is > 1 then p_i and p_{i+1} lie in different factors ($i = 1, \ldots, r-1$). Although this representation for x is not necessarily unique, the length r is unique, and so is the sequence of factors determined by $p_1, p_2, \ldots, p_r$. Suppose that $h \in H \cap H^x$. We shall prove by induction on r that $p_i \in H$ for all $i = 1, \ldots, r$. There exists an $h' \in H$ such that

$$h = p_r^{-1} \cdots p_2^{-1} p_1^{-1} h' p_1 p_2 \cdots p_r.$$

It follows that $p_1^{-1} h' p_1 \in A$ or $r = 1$.
Case 1: $r = 1$. If $p_1 \in H$, then $x \in H$ and we are done. If $p_1 \in G$, then $p_1^{-1} h' p_1 h^{-1} = 1$. The left side is reducible, which implies that $h' \in A$ and

$h \in H \cap G = A$. But, A is malnormal in G, so $x = p_1 \in A$. So $p_1 \in H$.
Case 2: $r > 1$. Let $a_1 = p_1^{-1} h' p_1 \in A$. Then $p_1 \in H$ by Case 1.
We can now write

$$h = p_r^{-1} \cdots p_2^{-1} a_1 p_2 \cdots p_r,$$

and apply the induction hypothesis to complete the proof.

Note that under the hypotheses of Proposition 1 the malnormal closure of A in P contains the normalizer of B in H.

Following Dicks [**?**] we define an *oriented graph of groups* as follows. It consists of an oriented graph $\Gamma = (V, E, \bar{\iota}, \bar{\tau})$ ($\bar{\iota}(e) \in V$ and $\bar{\tau}(e) \in V$ are the initial and terminal vertices respectively of an edge $e \in E$), together with a function G which assigns to each vertex $v \in V$ a group $G(v)$, and to each edge $e \in E$ a subgroup $G(e)$ of $G(\bar{\iota}(e))$ and monomorphism $t_e : G(e) \to G(\bar{\tau}(e))$. We shall say that this oriented graph of groups is *quasi-malnormal* if, for all $e \in E$, $G(e)$ is malnormal in $G(\bar{\iota}(e))$ while $t_e(G(e))$ is normal in its malnormal closure in $G(\bar{\tau}(e))$. If, in addition, for each edge e, $t_e(G(e))$ is malnormal in $G(\bar{\tau}(e))$, then we say that the graphs of groups is *malnormal*. In this case, the orientation of the graph is irrelevant. The graph of groups is said to be *separated* if for any edge e which is a loop ($v = \bar{\iota}(e) = \bar{\tau}(e)$) one has $G(e)^g \cap t_e(G(e)) = 1$ for all $g \in G(v)$.

If a quasi-malnormal separated oriented graph of CSA*-groups has only one edge, then its fundamental group is CSA* (Theorem 1, Theorem 2, [**?**], Theorem 6). We prove in Proposition 6 that the fundamental group of a quasi-malnormal oriented tree of CSA-groups need not be CSA. For every ordinal number α we define as follows an oriented graph L_α (a "line" from 1 to α). Its vertices are ordinals $\leq \alpha$, and $\{(\beta, \beta+1) \,|\, 1 \leq \beta, \beta+1 \leq \alpha\}$ is its set of edges. The edge $(\beta, \beta+1)$ has initial vertex β, and terminal vertex $\beta + 1$.

Theorem 3 *The fundamental group of an oriented graphs of CSA*-groups is again CSA* in the following two situations:*

1. *the underlying oriented graph is L_α for some ordinal α, and the oriented graph of groups is quasi-malnormal;*

2. *the underlying oriented graph is a tree and the graph of groups is malnormal.*

Proof. 1. The class of CSA*-groups is closed under direct limits ([**?**], Theorem 6) and free products (Theorem 2), hence we are reduced to proving the result for finite α. The result then follows, by a simple induction argument, from Theorem 2 and Proposition 1.
2. By a similar argument, we need only prove the result for finite trees, and the result follows from the Theorem and Proposition.

The separation conditions, necessary to ensure that the fundamental group of an arbitrary malnormal separated oriented graph of CSA*-groups is CSA*, are quite complicated and cumbersome to formulate.

Remark. The conditions of Theorem 1 are not necessary:

1) the following example G^* of an HNN-extension of a CSA*-group G is CSA*, without the HNN-extension being separated (or a centralizer extension, in the sense of Definition 6, [**?**]).

$$G^* = \left\langle x_1, x_2, x_3, t \,\middle|\, x_1^t = x_2, x_2^t = x_1 x_3 \right\rangle,$$

G is the free group on x_1, x_2, x_3, and the associated subgroups $A = \langle x_1, x_2 \rangle$, $B = \langle x_2, x_1 x_3 \rangle$, are malnormal in G, but have nontrivial intersection $\langle x_2 \rangle$. Clearly, G^* is free on two generators x_1, t, hence CSA.

2) Let

$$G^* = \left\langle x_1, x_2, x_3, x_4, x_5, t \,\middle|\, x_1^t = x_1^{x_2}, (x_1^{x_2})^t = x_3, x_4^t = x_5^2 \right\rangle.$$

Then this is a representation of G^* as a non-separated HNN-extension of the free group $G = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, with associated subgroups $A = \langle x_1, x_1^{x_2}, x_4 \rangle$ and $B = \langle x_1^{x_2}, x_3, x_5^2 \rangle$ (both A and B are free of rank 3). The subgroups A and B are not malnormal in G; moreover, neither is normal in its malnormal closure. The group G^* can be represented as a free product with amalgamation:

$$G^* = \left\langle x_1, x_2, x_3, x_4, t \,\middle|\, x_1^t = x_1^{x_2}, (x_1^{x_2})^t = x_3 \right\rangle *_\psi \langle x_5 \rangle,$$

where $\psi : \langle x_4^t \rangle \to \langle x_5^2 \rangle$ is the obvious isomorphism of infinite cycles. Let us denote the left factor of the above decomposition by G_1. The subgroup $\langle x_4^t \rangle$ is maximal abelian in G_1, therefore, to prove that G^* is a

CSA-group it is enough to prove that G_1 is a CSA-group (Theorem 2). Writing $s = tx_2^{-1}$, one can represent G_1 as follows:

$$G_1 = \langle x_1, x_2, x_4, s \,|[x_1, s] = 1 \rangle \,.$$

Hence the group G_1 is CSA (see Proposition 4).

It is interesting that both examples above can be constructed using only "admissable" HNN-extensions and free products with amalgamation (i.e. those mentioned in our theorems as sufficient conditions).

3 HNN extensions of CSA*-groups with abelian associated subgroups

In this section we give a complete description of HNN-extensions, with abelian associated subgroups, that preserve the CSA*-property.

Remark. An abelian subgroup of a CSA-group is malnormal iff it is maximal abelian.

Proposition 2 *Let G^* be an HNN-extension of a CSA-group G, relative to an isomorphism $\phi : A \to B$ of nontrivial abelian subgroups A and B of G. Then the HNN-extension G^* is strictly separated if and only if it is separated.*

Proof. Suppose that $A^s \cap B_1 \neq 1$, $s \in G$; then the CSA-property of G implies that the maximal subgroups A^s and B_1 of G are equal. Hence, $A^s \cap B \neq 1$. The result follows.

Proposition 3

1. *Suppose that $\phi : A \to B$ is an isomorphism of two nontrivial abelian subgroups A and B of a group G. If neither A nor B is maximal abelian in G then the associated HNN-extension G^* is not CSA.*
2. *Suppose that $\phi : A \to B$ is an isomorphism of two nontrivial abelian subgroups of two groups G and H respectively. If A is not maximal abelian in G and B is not maximal abelian in H, then the associated amalgamated product is not CSA.*

Proof. 1. Suppose that G^* is CSA. Let A_1 and B_1 be maximal abelian subgroups of G containing A and B respectively. Let $a_1 \in A_1 - A$, $b_1 \in B_1 - B$ and let t denote as usual the stable letter of the HNN-extension G^*, so that $a^t = \phi(a)$ for all $a \in A$. Let $1 \neq b \in B$. Then a_1^t commutes with b, and so does b_1. Since CSA-groups are commutation transitive,

$$1 = \left[a_1^t, b_1\right] = t^{-1}a_1^{-1}tb_1^{-1}t^{-1}a_1tb_1.$$

This is impossible, since the word on the right cannot be reduced.
2. A similar argument can be used in this case.

Suppose as before that $\phi : A \to B$ is an isomorphism of two nontrivial abelian subgroups A and B of a CSA-group G, with A maximal abelian in G. Then one of the following four possibilities must apply:

1) $(\forall s \in G)\,(A^s \cap B = 1)$;

2) $(\exists s \in G)\,(A^s = B \,\wedge\, (\forall a \in A)\,(\phi(a) = a^s))$;

3) B is maximal abelian in G, $(\exists v \in G)\,(A^v \cap B \neq 1)$ and

$$(\forall s \in G)\,[A^s \cap B \neq 1 \Rightarrow A^s = B \,\wedge\, (\exists a_0 \in A)\,(a_0^s \neq \phi\,(a_0))]\,;$$

4) B is not maximal abelian in G, and $(\exists s \in G)\,(A^s \cap B \neq 1)$ (in this case, $B \subset A^s$).

In the HNN-extension

$$G^* = \left\langle G, t \,\middle|\, t^{-1}at = \phi(a), a \in A \right\rangle,$$

$\phi(a) = a^t$ for all $a \in A$. In cases 3) and 4), there exist $s \in G$ and $a_0 \in A$ such that

$$A^{ts^{-1}} \subseteq A \text{ but } \left[a_0, ts^{-1}\right] \neq 1, \text{ and hence } ts^{-1} \notin A.$$

This shows that in the cases 3) and 4) the group G^* is not CSA.

Proposition 4 *Let A be a maximal abelian subgroup of a CSA*-group G, B a subgroup of G and $\phi : A \to B$ an isomorphism. Suppose that there exists an $s \in G$ such that $B = A^s$ and $\phi(a) = a^s$ for all $a \in A$. Then the HNN-extension $G^* = \langle G, t \,|\, t^{-1}at = \phi(a)\rangle$ is a CSA*-group.*

Proof. It is clear that if we put $v = ts^{-1}$, then

$$G^* = \langle G, v \,|\, [a, v] = 1, \forall a \in A \rangle \,.$$

I.e., we get a presentation for G^* from a presentation for G by taking as generators for G^* those of G together with v, and as defining relations those of G together with $[a, v] = 1$ for all $a \in A$. Since A is maximal abelian, we have $A = C_G(a)$ for some fixed $a \in A$, and G^* is the direct, rank 1, extension of the centralizer of the element a (Definition 6, [**?**]). Since $A \times \langle v \rangle$ has no element of order 2, it follows from Theorem 5 [**?**] that G^* is a CSA*-group.

Theorem 4 *Let G be a CSA*-group and G^* a separated HNN extension of G, relative to an isomorphism $\phi : A \to B$ of abelian subgroups of G, with A maximal abelian in G. Then G^* is a CSA*-group.*

Proof. We may assume that A and B are nontrivial (else G^* is the free product of G and an infinite cyclic, hence CSA* [**?**]). The maximal abelian subgroup B_1 of G containing B is the malnormal closure of B. Clearly, B is normal in B_1, and the result follows from Theorem 1 and Proposition 2.

Combining Proposition 4, Theorem 4 and the remarks at the beginning of this section, we obtain

Theorem 5 *Let $\phi : A \to B$ be an isomorphism of abelian subgroups of a CSA*-group G, with A maximal abelian in G. Then the corresponding HNN-extension of G is again CSA* if and only if it is a separated extension or there exists an $s \in G$ such that $B = A^s$ and $\phi(a) = a^s$ for all $a \in A$.*

Theorem 6 *Let G and H be CSA*-groups and $\phi : A \to B$ an isomorphism of abelian subgroups of G and H respectively. Then the amalgamated product $G *_\phi H$ is CSA* if and only if at least one of the subgroups A or B is maximal abelian in G or H respectively.*

Proof. It follows directly from Proposition 3 and Theorem 2.

Remark. An oriented graph Γ of CSA-groups, with abelian edge groups, is quasi-malnormal if and only if each edge group $G(e)$ is maximal abelian in $G(\bar{\iota}(e))$; and Γ is malnormal if and only if each edge group $G(e)$ is maximal abelian in both vertex groups (i.e., $G(e)$ is maximal abelian in $G(\bar{\iota}(e))$ and its image $t_e(G(e))$ is maximal abelian in $G(\bar{\tau}(e))$). Theorem 3 gives examples of types of oriented graphs of CSA*-groups, with abelian edge groups, whose fundamental groups are again CSA*. We have, in particular,

Proposition 5 *If the edge groups $G(e)$ of an oriented tree of CSA*-groups are maximal abelian in $G(\bar{\iota}(e))$ and also have maximal abelian images in the target groups $G(\bar{\tau}(e))$, then the fundamental group of the tree of groups is again a CSA*-group.*

Proof. The result follows immediately from Theorem 3 and the Remark at the beginning of this section.

Proposition 6 *Let $T = (V, E)$ be the tree with two edges e_1 and e_2, having a common initial point v, and endpoints w_1 and w_2 respectively. Suppose that the vertex groups are CSA* and the edge groups $G(e_i)$ are maximal abelian in $G(v)$, with nontrivial intersection (hence coincident). If their images are not maximal abelian in $G(w_1)$ and $G(w_2)$ respectively, then the fundamental group of this oriented graph of groups is not CSA.*

Proof. The fundamental group is the amalgamated product of the CSA-groups $G(v) *_{G(e_1)} G(w_1)$ (Theorem 2) and $G(w_2)$, with amalgamated subgroup $G(e_2)$. The maximal abelian subgroup of $G_v *_{G(e_1)} G(w_1)$, containing the image of $G(e_2)$, contains the image of the maximal abelian subgroup of $G(w_1)$ containing the image of $G(e_1)$, hence cannot coincide with the image of $G(e_2)$. The result now follows from Proposition 3.

4 One-relator CSA-groups

In this section we completely characterise, in terms of "obstacles", all one-relator groups that are CSA, and we show that all the obstacles are realized. It follows from the characterization of one-relator CSA-groups that in the torsion-free case the CSA-property is equivalent to the transitivity of commutation; however, this equivalence fails for one-relator groups with torsion.

Proposition 7

1. *The group $\mathcal{B} = F_2 \times \mathbf{Z}$ is not commutative transitive (hence not CSA).*
2. *The non-abelian Baumslag-Solitar groups*

$$B_{m,n} = \left\langle x, y \,\middle|\, yx^m y^{-1} = x^n \right\rangle, \; mn \neq 1,$$

are not CSA; furthermore, the non-metabelian Baumslag-Solitar groups ($|m| \neq 1 \neq |n|$) contain $\mathcal{B}$ as a subgroup (hence are not commutative transitive and not CSA)

3. The one-relator group

$$G = \langle x, y \,|\, [[x, y], y] = 1 \rangle$$

contains $\mathcal{B}$, but does not contain any non-abelian Baumslag-Solitar groups.

Proof. 1. If we write $\mathcal{B} = F(x, y) \times \langle z \rangle$, then we see immediately that the elements x and y commute with z, but not with each other.
2. A metabelian non-abelian BS-group contains a nontrivial abelian normal subgroup, therefore it is not a CSA-group. See the proof of Theorem 7 for a verification of the statement that the non-metabelian Baumslag-Solitar groups contain $\mathcal{B}$.
3. The proof that $\mathcal{B}$ is a subgroup of G is contained in the proof of Proposition 12 of the next section. Since G is residually p (Proposition 12) for all primes p but the non-abelian Baumslag-Solitar groups fail to be residually p for almost all p (Proposition 11), G does not contain any non-abelian Buamslag-Solitar groups.

Lemma 4 *Suppose that a group T is a nonabelian semidirect product $A^+ \rtimes \langle t \rangle$ of the underlying abelian group A^+ of a finitely generated subring A of $\mathbf{Q}$ and an infinite cycle $\langle t \rangle$. Then every epimorphism ψ from T onto a torsion-free group W, with ψ nontrivial on A and $\langle t \rangle$, is an isomorphism.*

Proof. It is easily seen that $A = \mathbf{Z}\left[\frac{1}{k}\right]$ for some natural number k, and that the restriction of ψ to A^+ is injective. The action of t on A^+ is multiplication by some rational number $\frac{m}{n}$, where both m and n divide a power of k. Clearly, $\psi(T) = \psi(A^+) \langle \psi(t) \rangle$. Identify $\psi(A^+)$ with A^+. The action (through conjugation) by $\bar{t}$ on A^+ is multiplication by $\frac{m}{n}$. If no power of $\bar{t}$ is in A^+, then ψ is clearly an isomorphism. Suppose now that some power $\bar{t}^i$ is in A^+. Then, since A^+ is abelian, $m^i = n^i$ and $m = \pm n$. Since T is nonabelian, $\frac{m}{n} = -1$. The $\langle t \rangle$-module A^+ is the union of an ascending chain of submodules $\langle a_i \rangle$; therefore, T is the union of an ascending chain of groups, each isomorphic to $H = \langle a, t \,|\, t^{-1}at = a^{-1} \rangle$. If the restriction of ψ to each of these groups is injective then ψ is injective; so we may assume, without loss of generality, that T is in fact equal to H. If ψ is not injective then $\psi(T)$ admits a presentation with generators a, t, and

$$t^{-1}at = a^{-1}, \; t^r = a^s \; (r, s \in \mathbf{Z})$$

among its defining relations. Then

$$t^r = t^{-1}a^s t = \left(t^{-1}at\right)^s = a^{-s} = t^{-r}$$

and $t^{2r} = 1$. This contradicts our assumption that $\psi(T)$ is torsion free.

Theorem 7 *A nonabelian torsion-free one-relator group G is CSA if and only if it does not contain a copy of $\mathcal{B}$ or one of the (nonabelian) metabelian Baumslag-Solitar groups $B_{1,n} = \langle x, y \,|\, yxy^{-1} = x^n \rangle$, $n \in \mathbf{Z} - \{0, 1\}$.*

The proof of Theorem 7 will use the following

Lemma 5 *Let H be a subgroup of a torsion-free one-relator group G. Then one of the following is true:*

- *H is locally cyclic;*
- *H contains a nonabelian free group of rank 2;*
- *H is isomorphic to $B_{1,m}$ for some $m \in \mathbf{Z} - \{0\}$.*

Proof of the lemma. By [**?**], Chapter II, Proposition 5.27, H is either solvable or contains a free group of rank 2. Suppose that H is solvable. The result then follows from Moldavanskii's Theorem ([**?**], Chapter II, Prop. 5.25, p. 109) or from the classification of solvable groups of cohomological dimension ≤ 2 [**?**] (G has cohomological dimension ≤ 2, and the same must be true of its subgroups).

Proof of the Theorem. Clearly G cannot be CSA if it contains a copy of $\mathcal{B}$ or $B_{1,n}$. If G is not CSA, then there exists a maximal abelian subgroup A of G, and elements $a_1, a_2 \in A$, $z \in G - A$, such that $a_1^z = a_2 \neq a_1$. By Lemma 5, $\langle a_1, a_2 \rangle$ is either free abelian of rank two or it is cyclic.

Consider first the case where $\langle a_1, a_2 \rangle$ is cyclic. Then $\exists a_0 \in A$, $m, n \in \mathbf{Z}$ such that $a_1 = a_0^m$, $a_2 = a_0^n$. If $|m| = 1$, then, by Lemma 4, $\langle a_0, z \rangle$ is isomorphic to the semidirect product of the additive group of the ring $\mathbf{Z}\left[\frac{1}{n}\right]$ and the infinite cycle $\langle z \rangle$, where the action by z is multiplication by n. Similarly, if $|n| = 1$ then $\langle a_0, z \rangle$ is isomorphic to the semidirect product of $\mathbf{Z}\left[\frac{1}{m}\right]$ and the infinite cycle $\langle z \rangle$, where the action of z is now multiplication by m. So, if $|n|$ or $|m|$ is 1, then $\langle a_0, z \rangle$ is isomorphic to $B_{1,n}$ or $B_{1,m}$. It cannot be isomorphic to the free abelian group of rank 2, $B_{1,1}$, since we have supposed that $a_1 \neq a_2$. Suppose now that $|m| \neq 1 \neq |n|$. Consider, for every natural number i, the group

$$H_i = \left\langle a_0^{z^{-i}}, a_0^{z^{-i+1}}, \ldots, a_0^{z^{-1}}, a_0, a_0^z, \ldots, a_0^{z^i} \right\rangle .$$

These groups cannot all be abelian, since in that case $\langle a_0, z\rangle$ would be isomorphic to a homomorphic image of the semidirect product of $\mathbf{Z}\left[\frac{1}{mn}\right]$ and $\langle z\rangle$, where the action of z is multiplication by $\frac{m}{n}$; hence, by Lemma 4, $\langle a_0, z\rangle$ would be isomorphic to this (metabelian) semidirect product. But this solvable group is not among those that appear in the statement of Lemma 5 (it has cohomological dimension 3, [**?**]). It follows from the Lemma that either some H_i is isomorphic to $B_{1,m}$, $m \notin \{0, 1\}$, or H_i contains a free group of rank 2. It is easily proved, by induction on $i \geq 1$, that

$$a_0^{(mn)^i} = \left(a_0^{m^{2i}}\right)^{z^i} = \left(a_0^{n^{2i}}\right)^{z^{-i}} . \tag{5}$$

So, $a_0^{(mn)^i}$ centralizes H_i, and it follows that, if H^i contains a free group of rank 2, then $\langle a_0, z\rangle$ contains a copy of the group $\mathcal{B}$. In particular, the above arguments show that $B_{m,n}$ contains a copy of $\mathcal{B}$.

We now consider the case where $\langle a_1, a_2\rangle$ is not cyclic. Suppose that G does not contain any nonabelian Baumslag-Solitar group. Then every subgroup of G is either abelian or contains a free group of rank 2. Let K_i be the subgroup of G generated by $a_1^{z^j}$, $-i \leq j \leq i$. If all K_i are abelian, then their union K is abelian and normal in the nonabelian solvable subgroup $\langle K, z\rangle$ of G. By the Lemma and our assumptions, G cannot contain a nonabelian solvable subgroup. Hence, we may assume that K_i is abelian and K_{i+1} is not. Then K_{i+1} contains a free subgroup of rank 2, and we claim that K_0 centralizes it. If $i = 0$, this is clear since $\langle a_1, a_1^z\rangle$ and $\left\langle a_1^{z^{-1}}, a_1\right\rangle$ are abelian. If $i > 0$, then

$$\left\langle a_1, a_1^z, \ldots, a_1^{z^{i+1}} \right\rangle, \ \left\langle a_1^{z^{-i-1}}, \ldots, a_1^{z^{-1}}, a_1 \right\rangle$$

are contained in conjugates of K_i, hence are abelian. The result follows.

Proposition 8 *The infinite dihedral group*

$$D_\infty = \left\langle x, y \,\middle|\, yxy^{-1} = x^{-1}, y^2 = 1 \right\rangle$$

is commutative transitive, but not CSA. Moreover, D_∞ is a subgroup of the one-relator group

$$G = \left\langle x, y \,\middle|\, x^2 = 1 \right\rangle . \tag{6}$$

Proof. Clearly, $\langle x \rangle \cap \langle x \rangle^y = \langle x \rangle$, $y \notin \langle x \rangle$, which shows that D_∞ cannot be CSA.

The normal subgroup of G generated by x is the free product of countably many copies of a cyclic group of order 2, hence contains D_∞.

We can represent G as a free product of $\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$. To prove that G, and hence D_∞, is commutative transitive, it suffices to observe that a free product P of commutative transitive groups is commutative transitive. Indeed, if x commutes with y and z in P, then $H = \langle x, y, z \rangle$ is a free product of conjugates of subgroups of the factors of P and (possibly) a free group F. Since the center of H is nontrivial, the decomposition of H as a free product is trivial, hence H is isomorphic to a subgroup of F or of one of the factors of P. Thus, y commutes with z.

Theorem 8 *Let G be a one-relator group with torsion. Then G is CSA if and only if it does not contain the infinite dihedral group*

$$D_\infty = \left\langle x, y \,\middle|\, yxy^{-1} = x^{-1}, y^2 = 1 \right\rangle .$$

Proof. If G contains D_∞, it cannot be CSA, since the class of CSA-groups is closed under taking subgroups and D_∞ is not CSA, by Proposition 8.

Conversely, suppose that G is not CSA and does not contain a copy of D_∞. By a result of Karass and Solitar, every subgroup of a one-relator group with torsion is either cyclic, D_∞ or contains a free group of rank 2 ([**?**], Chapter II, Prop. 5.27). Let A be a maximal abelian subgroup of G. Then A is cyclic.

Case 1: A is infinite.
Suppose that $1 \neq a_1 \in A = \langle a_0 \rangle$, $a_1 = a_0^m$, $z \in G$, $a_2 = a_1^z = a_0^n$, $z \notin A$, then there are two possibilities for

$$K = \left\langle a_0^{z^i} \, | i \in \mathbf{Z} \right\rangle :$$

it contains a free subgroup of rank 2 or is cyclic. If K is cyclic, then $\langle K, z \rangle$ does not contain a nonabelian free group. Hence is abelian. It contains the

maximal abelian subgroup A of G, hence $z \in A$. Contradiction. If K contains a free subgroup F of rank 2, then F is contained in the subgroup generated by a finite number of the displayed generators of K; hence is centralized by a power $a_0^{(mn)^i}$ of a_0 (see the proof of Theorem 7). Since the center of F is trivial, any nontrivial element of F, together with $a_0^{(mn)^i}$, generate a noncylic abelian subgroup, and we again have a contradiction.
Case 2: A is finite.
We apply the usual Magnus treatment to the one-relator group ([**?**], Chapter 2, section 6, Chapter 4, section 5), and argue by induction on the length of the defining relator. If only one letter appears in the defining relator, then the one-relator group is finite cyclic, or the free product of a free group and a finite cyclic group. It is easily verified that in this case the one-relator group is again CSA (the base of the induction). If the exponentsum of no generator in the relator is zero, we can adjoin a root of a generator to our one-relator group G, and take a different system of generators, so that the relator, when expressed in these new generators, has exponentsum zero in one of the generators, and is an HNN-extension of a one-relator group H, with shorter defining relator. The associated subgroups, the co-called Magnus subgroups, are free, hence have trivial intersection with A. It follows that A is a free product of subgroups of conjugates of H and a free group. But A, being finite cyclic, is indecomposable as a free product, and must be contained in a conjugate of H. As in the proof of Theorem 1 (see the discussion around (3) and (4)), we find that z belongs to the same conjugate of H. By the induction hypothesis, H is CSA. But then $z \in A$, the desired contradiction.

Theorem 9 *A torsion-free one-relator group is CSA if and only if it is commutative transitive. The class of one-relator groups with torsion that are CSA is strictly contained in the class of commutation-transitive one-relator groups with torsion.*

Proof. In the torsion-free case, the obstacles $\mathcal{B}$ and $B_{1,n}$ $(n \neq 1)$ to the CSA-property are not commutative transitive. The result follows. In the torsion case the group G of Proposition 8 is commutative transitive but not CSA.

5 Q-faithfulness, residual properties and one-relator groups

In the context of G. Baumslag's problem [**?**] of describing the class of **Q**-faithful one-relator groups, we prove here that this class strictly contains the union of the class of one-relator CSA*-groups and the class of one-relator groups that are residually p for almost all primes p.

Proposition 9 *If, for almost all primes p, a group G is residually p then it is* **Q***-faithful.*

Proof. Let S be the set of prime numbers and Φ an ultrafilter on S, containing all cofinite subsets of S. The ultraproduct $\prod_{\Phi}^{*} \mathbf{Z}_p$ of the rings of p-adic numbers contains $\mathbf{Q}$ as a subring (every integer is divisible by only finitely many primes). Moreover, the componentwise action by exponentiation of this ultraproduct on the ultraproduct $\prod_{\Phi}^{*} \hat{G}_p$ of pro-p-completions $\hat{G}_p$ of G is faithful. The restriction to $\mathbf{Q}$ of this action is faithful. Since G is embeddable in the $\mathbf{Q}$-group $\prod_{\Phi}^{*} \hat{G}_p$, G is $\mathbf{Q}$-faithful.

Proposition 10 *If a group G is residually torsion-free nilpotent, then it is faithful over* **Q***.*

Proof. By hypothesis, G is embeddable in a product of torsion-free nilpotent groups. Since torsion-free nilpotent groups are faithful over $\mathbf{Q}$ (see [**?**] or [**?**]), each one is embeddable in its $\mathbf{Q}$-completion, and it follows that there is a monomorphism from G into a product of $\mathbf{Q}$-groups. The canonical homomorphism from G into its $\mathbf{Q}$-completion factors through this monomorphism, hence is itself a monomorphism, which means that G is $\mathbf{Q}$-faithful.

Proposition 11 *Every nonabelian Baumslag-Solitar group $B_{m,n}$ is not residually p for almost all primes p. If $B_{m,n}$ is metabelian (i.e. $|m|$ or $|n| = 1$), then it is* **Q***-faithful.*

Proof. The defining relation of a nonabelian Baumslag-Solitar $G = B_{m,n}$ can be written in the form

$$x^{n-m} = \left[x^m, y^{-1}\right]. \tag{7}$$

Consider first the case were $n \neq m$. If p is a prime not dividing $m-n$, then G is not residually p. Indeed, the pro-p-completion $\hat{G}_p$ is a one-relator pro-p-group with the same defining relation, and if x belongs to the k-th term of the central descending series of $\hat{G}_p$, then (7) shows that x^{n-m}, and hence x, belongs to the $(k+1)$-st term. This proves that the image of x in $\hat{G}_p$ is 1, and the canonical map from G into $\hat{G}_p$ cannot be injective, which means that G is not residually p. If $m = n$ then y commutes with x^n but not with x. However, for every prime p not dividing n the pro-p-completion $\hat{G}_p$ of G is an abelian pro-p-group. This shows that the canonical map:$G \to \hat{G}_p$ is not injective, and G is not residually p.

Clearly, the nonabelian metabelian groups are semidirect products of the form $\mathbf{Z}\left[\frac{1}{n}\right] >\!\!\lhd \mathbf{Z}$, where the action of a generator of $\mathbf{Z}$ on $\mathbf{Z}\left[\frac{1}{n}\right]$ is multiplication by $n > 1$. This semidirect product is naturally embedded in the semidirect product $E = \mathbf{R} >\!\!\lhd \mathbf{Q}$, where the action

$$\theta : \mathbf{Q} \longrightarrow \operatorname{Aut} \mathbf{R}$$

is given by $\theta\left(\frac{s}{t}\right)(r) = n^{\frac{s}{t}} r$. For every $(c,d) \in E$ and natural number $m > 1$, the element (c,d) has a unique m-th root (a,b). Indeed, let $b = \frac{d}{m}$, and solve the equation

$$\left(1 + b + \cdots + b^{n-1}\right) a = c$$

for a, then clearly $(a,b)^m = (c,d)$.

Proposition 12 *The one-relator group*

$$G = \langle x, y \,|\, [[x,y], y] = 1 \rangle$$

is not a CSA-group. However, it is $\mathbf{Q}$*-faithful, residually torsion-free nilpotent, residually* p *and with torsion-free pro-*p*-completion for every prime* p *(hence does not contain a nonabelian Baumslag-Solitar group).*

Proof. Let X be the normal subgroup of the free group $F(x,y)$ generated by x, and let $x_i = y^{-i} x y^i$ for all integers i. Then

$$[[x,y], y] = \left[x_0^{-1} x_1, y\right] = x_1^{-1} x_0 x_1^{-1} x_2.$$

We see that in G $x_i^{-1} x_{i-1} = x_{i+1}^{-1} x_i$ for all $i \in \mathbf{Z}$. Let $d = x_1^{-1} x_0$, then $d = x_{i+1}^{-1} x_i$ for all $i \in \mathbf{Z}$. It is easy to prove, by induction on $|i|$, that

$x_i = xd^{-i}$ for all integers i. It follows that X is freely generated by d and x, and G is the semidirect product of the free group $F(d,x)$ and the infinite cycle $\langle y \rangle$, with y acting trivially on d, and $x^{y^i} = xd^{-i}$.

Let F_k be the k-th term of the central descending series of $F(d,x)$. Then G/F_{k+1} is torsion free for all k (it is a semi-direct product of F/F_{k+1} and $\langle y \rangle$) We claim that it is also nilpotent. We have $[y,d] = 1$ and

$$[x,d]^y = \left[xd^{-1}, d\right] = [x,d]^{d^{-1}}.$$

Suppose that c is a commutator of weight $k \geq 2$ in d and x. It is "multilinear" modulo F_{k+1}; hence it follows from the relations $d^y = d$ and $x^y = xd^{-1}$, that if x appears only once in c, then $c^y \equiv c \pmod{F_{k+1}}$ and $[y,c] \equiv 1 \pmod{F_{k+1}}$. By the same reasoning, if x appears $j \geq 2$ times in the commutator c, then $[y,c]$ is congruent, modulo F_{k+1}, to a product of commutators in which x appears at most $j-1$ times. If we commutate the element c j times by y, it drops into F_{k+1}. This shows that G/F_{k+1} is nilpotent. By Proposition 10, G is **Q**-faithful.

For every prime p, the pro-p-completion of G is the semi-direct product of the free pro-p-group on x, d by the free pro-p-group on the single generator y. Clearly, the canonical map from G into its pro-p-completion is injective. Thus, G is residually p.

That G is **Q**-faithful follows from Proposition 10 or Proposition 9.

To see that G is not CSA, we write $y_i = x^{-i}yx^i$, and we find that $[y_0, y_1] = 1$. Hence,

$$1 \neq \langle y_1 \rangle = \langle y_0, y_1 \rangle \cap \langle y_1, y_2 \rangle = \langle y_0, y_1 \rangle \cap \langle y_0, y_1 \rangle^x, x \notin \langle y_0, y_1 \rangle.$$

This completes the proof of the Proposition.