Open problems in combinatorial group theory

Gilbert Baumslag Alexei G. Myasnikov
Vladimir Shpilrain

Introduction

This is a collection of open problems in combinatorial group theory, which is based on a similar list available on-line at our web site *http://zebra.sci.ccny.cuny.edu/web/*

In selecting the problems, our choices have been, in part, determined by our own tastes. In view of this, we have welcomed suggestions from other members of the community. We want to emphasize that many people have contributed to the present collection, especially to the Background part. In particular, we are grateful to R.Gilman, V.Remeslennikov, I.Kapovich, W.Dicks, V.Roman'kov, and D.Wise for useful comments and discussions. Still, we admit that the background we currently have is far from complete, and it is our ongoing effort to make it as complete and comprehensive as possible. We invite an interested reader to check on our on-line list for a latest update.

One more thing concerning our policy that we would like to point out here, is that we have decided to keep on our list those problems that have been solved after the first draft of the list was put on-line in *June 1997*, since we believe those problems are an important part of the list anyway, because of their connections to other, yet unsolved, problems. Solved problems are marked by a $*$, and a reference to the solution is provided in the background.

Disclaimer. We want to emphasize that all references we give and attributions we make reflect our personal opinion based on the information we have. In particular, if we are aware that a problem was raised by a specific person, we make mention of that here. We welcome any additional information and/or corrections on these issues.

OUTSTANDING PROBLEMS

(O1) The Andrews-Curtis conjecture. *Let $F = F_n$ be the free group of a finite rank $n \geq 2$ with a fixed set $X = \{x_1, ..., x_n\}$ of free generators. A set $Y = \{y_1, ..., y_n\}$ of elements of F generates the group F as a normal subgroup if and only if Y is Andrews-Curtis equivalent to X, which means one can get from X to Y by a sequence of Nielsen transformations together with conjugations by elements of F.*

1991 Mathematics Subject Classification: Primary 20-02, 20Exx, 20Fxx, 20Jxx, Secondary 57Mxx.

This problem is of interest in topology as well as in group theory. A topological interpretation of this conjecture was given in the original paper by Andrews and Curtis [*Free groups and handlebodies*, Proc. Amer. Math. Soc. **16** (1965), 192–195]. A more interesting topological interpretation arises when one allows one more transformation - "stabilization", when Y is extended to $\{y_1, ..., y_m, x_\nu\}$, where x_ν is a new free generator (i.e., $y_1, ..., y_m$ do not depend on x_ν), and the converse of this transformation. Then the Andrews-Curtis conjecture is equivalent to the following (see [P.Wright, *Group presentations and formal deformations.* Trans. Amer. Math. Soc. **208** (1975), 161–169]): two contractible 2-dimensional polyhedra P and Q can both be embedded in a 3-dimensional polyhedron S so that S geometrically contracts to P and Q. Note that this is true if 3 is replaced by 4 – this follows from a result of Whitehead. The problem is amazingly resistant; very few partial results are known. A good group-theoretical survey is [R.G.Burns, O.Macedonska, *Balanced presentations of the trivial group.* Bull. London Math. Soc. **25** (1993), 513–526]. For a topological survey, we refer to [C.Hog-Angeloni, W.Metzler, *The Andrews-Curtis conjecture and its generalizations.* Two-dimensional homotopy and combinatorial group theory, 365–380, London Math. Soc. Lecture Note Ser., **197**, Cambridge Univ. Press, Cambridge, 1993].

The prevalent opinion is that the conjecture is false; however, not many potential counterexamples are known. Two of them are given in the survey by Burns and Macedonska; a one-parameter family of potential counterexamples appears in [S.Akbulut, R.Kirby, *A potential smooth counterexample in dimension 4 to the Poincare conjecture, the Schoenflies conjecture, and the Andrews-Curtis conjecture.* Topology **24** (1985), 375–390.] Recently, a rather general series of potential counterexamples in rank 2 was reported in [C.F.Miller and P.Schupp, *Some presentations of the trivial group*, preprint].

It might be of interest that, by using the MAGNUS software package for symbolic computation in groups, we were able to show that all presentations of the trivial group with the total length of relators up to 12 satisfy the Andrews-Curtis conjecture – see *http://zebra.sci.ccny.cuny.edu/web/alex/experiments.htm*

Finally, we mention a positive solution of a similar problem for *free solvable groups* by A.G.Myasnikov [*Extended Nielsen transformations and the trivial group* (Russian). Mat. Zametki **35** (1984), 491–495.]

(O2) The Burnside problem. *For what values of n are all groups of exponent n locally finite? Of particular interest are $n = 5$, $n = 8$, $n = 9$ and $n = 12$ – values for which, by the experts' opinion, groups of exponent n have a remote chance of being locally finite.*

In contrast with the previous problem (O1), the bibliography on the Burnside problem consists of several hundred papers. We only mention here that Golod [*On nil-algebras and finitely approximable p-groups* (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 273–276] constructed the first example of a periodic group which is not locally finite; his group however does not have bounded exponent. The first example of an infinite finitely generated group of bounded exponent is due to Novikov and Adian [*Infinite periodic groups.* I, II, III (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 212–244, 251–524, 709–731]. We refer to the book [A.Yu.Olshanskii, *Geometry of defining relations in groups.* Mathematics and its Applications (Soviet Series), **70**. Kluwer Academic Publishers Group, Dordrecht, 1991] for a survey on results up to 1988, and to the papers [S.V.Ivanov, *The free Burnside groups of sufficiently large exponents.* Internat. J. Algebra Comput. **4** (1994), 308 pp.]; [I.G.Lysenok, *Infinite Burnside groups of even period* (Russian). Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 3, 3–224] for treatment of the most difficult case where the exponent is a power of 2.

(O3) Whitehead's asphericity problem. *Is every subcomplex of an aspherical 2-complex aspherical? Or, equivalently: if* $G = F/R =< x_1, ..., x_n;\ r_1, ..., r_m, ... >$ *is an aspherical presentation of a group* G *(i.e., the corresponding relation module* $R/[R, R]$ *is a free* ZG*-module), is every presentation of the form* $\langle x_1, ..., x_n;\ r_{i_1}, ..., r_{i_k}, ...\rangle$*, aspherical as well ?*

This problem has received considerable attention in the '80s. We mention here a paper by Huebschmann [*Aspherical* 2*-complexes and an unsettled problem of J.H.C. Whitehead,* Math. Ann. **258** (1981/82), 17–37] that contains a wealth of examples of 2-complexes for which Whitehead's asphericity problem has a positive solution. J.Howie [*Some remarks on a problem of J.H.C.Whitehead,* Topology **22** (1983), 475–485] points out a connection between Whitehead's problem and some other problems in low-dimensional topology (e.g. the Andrews-Curtis conjecture). We refer to [R.Lyndon, *Problems in combinatorial group theory.* Combinatorial group theory and topology (Alta, Utah, 1984), 3–33, Ann. of Math. Stud., **111**, Princeton Univ. Press, 1987] for more bibliography on this problem. Among more recent papers, we mention a paper by Luft [*On* 2*-dimensional aspherical complexes and a problem of J.H.C. Whitehead,* Math. Proc. Cambridge Philos. Soc. **119** (1996), 493–495], where he gives a rather elementary self-contained proof of the main result of Howie's paper (see above), and strengthens the result at the same time.

(O4) *The isomorphism problem for one-relator groups.*

Sela [*The isomorphism problem for hyperbolic groups.* I. Ann. of Math. (2) **141** (1995), 217–283] has solved the isomorphism problem for torsion-free hyperbolic groups that do not split (as an amalgamated product or an HNN extension) over the trivial or the infinite cyclic group. It is not known however which one-relator groups are hyperbolic (cf. problem (O6)). Earlier partial results are [A.Pietrowski, *The isomorphism problem for one-relator groups with non-trivial centre*, Math. Z. **136** (1974), 95–106] and [S.Pride, *The isomorphism problem for two-generator one-relator groups with torsion is solvable*, Trans. Amer. Math. Soc. **227** (1977), 109–139]. We also note that two one-relator groups with relators r_1 and r_2 being isomorphic does not imply that r_1

and r_2 are conjugate by an automorphism of a free group, which deprives one from the most straightforward way of attacking this problem; see [J.McCool, A.Pietrowski, *On free products with amalgamation of two infinite cyclic groups.* J. Algebra **18** (1971), 377–383].

(O5) *The conjugacy problem for one-relator groups.*

The conjugacy problem for one-relator groups with torsion was solved by B.B.Newman [*Some results on one-relator groups*, Bull. Amer. Math. Soc. **74** (1968), 568-571]. Some other partial results are known; see e.g. [R.Lyndon, P.Schupp, *Combinatorial Group Theory.* Series of Modern Studies in Math. **89**. Springer-Verlag, 1977] for a survey.

(O6) *Is every one-relator group without Baumslag-Solitar subgroups hyperbolic?*

Note that every one-relator group with torsion is hyperbolic since the word problem for such a group can be solved by Dehn's algorithm – see the paper by B.B.Newman cited in the background to (O5). Therefore, it suffices to consider torsion-free one-relator groups. We also note that the following weak form of this problem was answered in the affirmative. A group G is called a CSA group if every maximal abelian subgroup M of G is malnormal, i.e., for any element g in G, but not in M, one has $M^g \cap M = \{1\}$. It is known that every torsion-free hyperbolic group is CSA. Now the following weak form of (O6) holds: A torsion-free one-relator group is CSA if and only if it does not contain Baumslag-Solitar metabelian groups $BS(1,p)$ and subgroups isomorpfic to $F_2 \times Z$ [D.Gildenhuys, O.Kharlampovich and A. Myasnikov, *CSA groups and separated free constructions.* Bull. Austr. Math. Soc. **52** (1995), 63–84].

There are also several partial (positive) results on this problem in a recent paper [S.V.Ivanov, P.E.Schupp, *On the hyperbolicity of small cancellation groups and one-relator groups.* Trans. Amer. Math. Soc. **350** (1998), 1851–1894].

(O7) *Let G be the direct product of two copies of the free group F_n, $n \geq 2$, generated by $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$, respectively.*
(a) *Is it true that every generating system of cardinality $2n$ of the group G is Nielsen equivalent to $\{x_1, ..., x_n, y_1, ..., y_n\}$?*
(b) *Is it true that the group G has only tame automorphisms (i.e., automorphisms induced by automorphisms of the ambient free group F_{2n})?*

The importance of this problem is in its relation to two outstanding problems in low-dimensional topology, to the Poincaré and Andrews-Curtis conjectures – see e.g. the survey [R.I.Grigorchuk, P.F.Kurchanov, *Some questions of group theory related to geometry.* Translated from the Russian by P.M.Cohn. Encyclopaedia Math. Sci., **58**, Algebra, VII, 167–240, Springer, Berlin, 1993] for details.

In particular, a combination of results of Stallings, Jaco and Waldhausen yields a purely algebraic re-formulation of the Poincaré conjecture (see [J.Hempel, *3-manifolds*, Ann. Math. Studies **86**, Princeton Univ. Press, 1976]) which implies the following: if the Poincaré conjecture is true, then every automorphism of the group G is tame. Thus, a negative answer to (O7)(b) would refute the Poincaré conjecture.

(O8) Tarski's problems. *Let $F = F_n$ be the free group of rank n, $Th(F)$ the elementary theory of F, i.e., all sentences in the language of group theory which are true in F.*
(a) *Is it true that $Th(F_2) = Th(F_3)$?*
(b) *Is $Th(F)$ decidable?*

(a) By a result of Merzlyakov [*Positive formulae on free groups* (Russian). Algebra i Logika **4** (1966), 25–42], all free groups of finite rank $n \geq 2$ satisfy the same positive sentences. Sacerdote [*Elementary properties of free groups*, Trans. Amer. Math. Soc. **178** (1973), 127–138] re-proved this result, and also proved that all free groups of finite rank $n \geq 2$ satisfy the same $(\forall\exists)$ and the same $(\exists\forall)$ sentences.

(b) Several fragments of the elementary theory of a free group of finite rank were shown to be decidable. We mention here important work of Makanin [*Equations in a free group*, Math. USSR Izv. **21** (1983), no. 3, 546–582] and Razborov [*Systems of equations in a free group*, Math. USSR Izv. **25** (1985), no. 1, 115–162] on solving equations and systems of equations in a free group. Makanin [*Decidability of the universal and positive theories of a free group*, Math. USSR Izv. **25** (1985), no. 1, 75–88] also proved decidability of the universal and positive theories of a free group.

(O9) The Hanna Neumann conjecture. *If H and K are non-trivial subgroups of a free group, then $rank(H \cap K) - 1 \leq (rank(H) - 1)(rank(K) - 1)$.*

It is convenient to write $(rank - n)(H) = max(rank(H) - n, 0)$.

Hanna Neumann proved that $(rank - 1)(H \cap K) \leq 2(rank - 1)(H)(rank - 1)(K)$ and conjectured that the coefficient 2 could be removed. R.Burns [*On the intersection of finitely generated subgroups of a free group*, Math. Z. **119** (1971), 121–130] showed $(rank - 1)(H \cap K) \leq (rank - 1)(H)(rank - 1)(K) + max((rank - 1)(H)(rank - 2)(K), (rank - 2)(H)(rank - 1)(K))$, thus proving the conjectured inequality when both subgroups have rank two. W.Neumann [*On intersections of finitely generated subgroups of free groups*, in: Groups -Canberra 1989, 161–170, Lecture Notes Math. **1456**, Springer, Berlin, 1990] formulated a stronger version of the Hanna Neumann conjecture, and proved the stronger version of Burns' bound. All subsequent results have applied to the stronger version.

G.Tardos [*On the intersection of subgroups of a free group*, Invent. Math. **108** (1992), 29–36] proved the conjectured inequality when one subgroup has rank two. W.Dicks [*Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture,* Invent. Math. **117** (1994), 373–389] translated the stronger version into a graph-theoretic conjecture.

G.Tardos [*Towards the Hanna Neumann conjecture using Dicks' method,* Invent. Math. **123** (1996), 95–104] improved Burns' bound showing $(rank - 1)(H \cap K) \leq$

$(rank-1)(H)(rank-1)(K) + max((rank-2)(H)(rank-2)(K)-1, 0)$, thus proving the conjectured inequality when both subgroups have rank three. W.Dicks and E. Formanek [*The rank three case of the Hanna Neumann conjecture,* preprint] improved Tardos' bound showing $(rank-1)(H \cap K) \leq (rank-1)(H)(rank-1)(K) + (rank-3)(H)(rank-3)(K)$, thus proving the conjectured inequality when one subgroup has rank three. Their preprint is available at *http://manwe.mat.uab.es/dicks/Rankthree.html*

A complete proof of the conjecture was claimed by W.S.Jassim [*On the intersection of finitely generated subgroups of free groups*, Rev. Mat. Univ. Complut. Madrid **9** (1996), 67–84], but W.Dicks [Rev. Mat. Univ.Complut. Madrid **11** (1998), to appear] has given an example which he feels makes it appear likely that the argument is not valid, although Jassim is at this time (December 1998) not in agreement.

(O10) *Is the automorphism group of a free group of rank 2 linear? Or, equivalently, is the braid group B_4 linear?*

Formanek and Procesi [*The automorphism group of a free group is not linear,* J. Algebra **149** (1992), 494–499] proved that the automorphism group of a free group of rank n is not linear if $n \geq 3$. The "Or, equivalently" statement is due to Dyer, Formanek and Grossman [*On the linearity of automorphism groups of free groups*, Arch. Math. **38** (1982), 404–409].

(O11) *Is there an infinite finitely presented periodic group?*

Needless to say, infinite periodic groups constructed by Golod, Novikov-Adian, and Olshanskii (see the background to (O2)) are infinitely related.

(O12) (I.Kaplansky) *Can the (integral) group ring of a torsion-free group have zero divisors?*

The bibliography on this problem consists of about a hundred papers. The highest point here is a result of Kropholler, Linnell and Moody [*Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. **104** (1988), 675–684] which implies, in particular, that the integral group ring of a torsion-free virtually solvable group has no zero divisors. We refer to [D.S.Passman, *The algebraic structure of group rings*, John Wiley and Sons, New York, 1977] for a survey on results up to 1977.

FREE GROUPS

These are problems about free groups, their automorphisms and related issues. See also problems (O1), (O7), (O8), (O9), (O10).

(F1) *Is there an algorithm for deciding if a given automorphism of a free group has a non-trivial fixed point ?*

S.Gersten [*On fixed points of automorphisms of finitely generated free groups.* Bull. Amer. Math. Soc. **8** (1983), 451–454; *Fixed points of automorphisms of free groups.* Adv. in Math. **64** (1987), 51–85] proved that the fixed point group $Fix(\phi)$ of any automorphism ϕ of a free group F_n of finite rank is finitely generated. A simpler proof was given by D.Cooper [*Automorphisms of free groups have finitely generated fixed point sets.* J. Algebra **111** (1987), 453–456], and R.Goldstein and E.Turner [*Fixed subgroups of homomorphisms of free groups.* Bull. London Math. Soc. **18** (1986), 468–470] obtained a similar result for arbitrary endomorphisms of a free group. In [M.Bestvina and M.Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. **135** (1992), 1–53], it is shown that the rank of $Fix(\phi)$ cannot exceed n. In [W.Imrich, E.Turner, *Endomorphisms of free groups and their fixed points.* Math. Proc. Cambridge Philos. Soc. **105** (1989), 421–422], this was generalized to arbitrary endomorphisms.

All these results however do not give an effective procedure for detecting fixed points of a given automorphism. Cohen and Lustig [*On the dynamics and the fixed subgroup of a free group automorphism.* Invent. Math. **96** (1989), 613–638] obtained several useful partial results and, in particular, solved the problem for *positive* automorphisms (i.e., for those that take every free generator to a positive word.)

***(F2)** (H.Bass) *Does the automorphism group of a free group satisfy the "Tits alternative" ?*

A preprint (by M.Bestvina, M.Feighn, M.Handel) with a positive solution of this problem is available on-line at *http://www.math.utah.edu/~bestvina*

(F3) (V.Shpilrain) *If an endomorphism ϕ of a free group F of finite rank takes every primitive element to another primitive, is ϕ an automorphism of F?*

This problem was solved in the affirmative for $n = 2$ by V.Shpilrain [*Generalized primitive elements of a free group*, Arch. Math. **71** (1998), 270–278] and by S.Ivanov [*On endomorphisms of a free group that preserve primitivity*, Arch. Math., to appear]. S.Ivanov also showed that the answer is positive in the general case under an additional assumption on ϕ to have a *primitive pair* in the image.

(F4) Denote by $Orb_\phi(u)$ the orbit of an element u of the free group F_n under the action of an automorphism $\phi \in Aut(F_n)$. That is, $Orb_\phi(u) = \{v \in F_n,\ v = \phi^m(u)\ for\ some\ m \geq 0\}$. If an orbit like that is finite, how many elements can it possibly have if u runs through the whole group F_n, and ϕ runs through the whole group $Aut(F_n)$?

It is known that the number of elements in an orbit is bounded by a function depending only on n – this observation is due to G.Levitt (informal communication). Here is his argument. Suppose that for some automorphism ϕ of $F = F_n$, we have $\phi^k(g) = g$ and $\phi^l(g) \neq g$ for $0 < l < k$. Consider the action of ϕ on the subgroup $H = Fix(\phi^k)$ consisting of all elements fixed by ϕ^k. (This subgroup is clearly invariant under ϕ.) Then ϕ has order k as an element of $Aut(H)$. Since H has rank at most n by [M.Bestvina and M.Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. **135** (1992), 1–53], this gives a bound for k in terms of n, since there is a bound for the order of a torsion element in $GL_n(Z)$, hence also for the order of a torsion element in $Aut(F_n)$ because the kernel of the map from $Aut(F_n)$ to $GL_n(Z)$ is torsion-free.

(F5) (H.Bass) *Is the automorphism group of a free group "rigid", i.e., does it have only finitely many irreducible complex representations in every dimension?*

(F6) *The conjugacy problem for the automorphism group of a free group of finite rank.*

An outer automorphism Φ of a free group F of finite rank is said to be reducible if there is a free factorization $F = F_1 \star \cdots \star F_k \star F'$ such that Φ permutes the conjugacy classes of the subgroups $F_1, \cdots, F_k$; otherwise, Φ is irreducible. Z.Sela [*The isomorphism problem for hyperbolic groups.* I. Ann. of Math. (2) **141** (1995), 217–283] and J.Los [*On the conjugacy problem for automorphisms of free groups*, Topology **35** (1996), 779–808] obtained algorithms which decide if two irreducible outer automorphisms are conjugate in the group of outer automorphisms of F.

(F7) (V.Shpilrain) *Denote by $Epi(n, k)$ the set of all homomorphisms from a free group F_n onto a free group F_k; $n, k \geq 2$. Are there 2 elements $g_1, g_2 \in F_n$ with the following property: whenever $\phi(g_i) = \psi(g_i)$, $i = 1, 2$, for some homomorphisms $\phi, \psi \in Epi(n, k)$, then $\phi = \psi$? (In other words, every homomorphism from $Epi(n, k)$ is completely determined by its values on just 2 elements.)*

S.Ivanov [*On certain elements of free groups*, J. Algebra **204** (1998), 394–405] has proved that every *injective* homomorphism from $Epi(n, k)$ is completely determined by its values on just 2 elements.

(F8) (W.Dicks, E.Ventura) *Let ϕ be an endomorphism of a free group F_n, S a subgroup of F_n having finite rank. Is it true that $rank(Fix(\phi) \cap S) \leq rank(S)$?*

This is true if $S = F_n$, although the only known proof of this fact is highly nontrivial (see [M.Bestvina and M.Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. **135** (1992), 1–53] for the case where ϕ is an automorphism, and [W.Imrich and E.C.Turner, *Endomorphisms of free groups and their fixed points*, Math. Proc. Cambridge Phil. Soc. **105** (1989), 421–422] for an extension of this result to arbitrary endomorphisms). For S an arbitrary finite rank subgroup of F_n, the result was established in the case where ϕ is injective [W.Dicks, E.Ventura, *The group fixed by a family of injective endomorphisms of a free group.* Contemporary Mathematics, **195**. American Mathematical Society, Providence, RI, 1996].

(F9) (A.I.Kostrikin) *Let F be the free group of rank 2 generated by x, y. Is the commutator $[x, y, y, y, y, y, y]$ a product of fifth powers in F? (If not, then the Burnside group*

$B(2,5)$ *is infinite.)*

(F10) (A.I.Mal'cev) *Can one describe the commutator subgroup of a free group by a first order formula in the language of group theory ?*

We remark here that a positive answer to this problem would imply that elementary theory of a free non-abelian group F (with constants from F in the language) is undecidable, since there is no algorithm for deciding if a given equation in a free group F has solutions from $[F,F]$ [V.G.Durnev, *A generalization of Problem 9.25 in the Kourovka notebook*, Math. Notes USSR **47** (1990), 117–121].

(F11) (G.Bergman) *Let S be a subgroup of a free group F, and R a retract of F. Is it true that the intersection of R and S is a retract of S?*

We can only remark here that the intersection of two retracts of a free group is itself a retract, but a proof of this fact is much harder than one would expect – see [G.Bergman, *Supports of derivations, free factorizations, and ranks of fixed subgroups in free groups*, Trans. Amer. Math. Soc., to appear].

(F12) (G.Baumslag) *Let $F = F_n$ be a free group generated by $\{x_1, ..., x_n\}$, and let F^Q be the free Q-group, i.e., the free object of rank n in the category of uniquely divisible groups. Consider the map $x_i \longrightarrow (1 + x_i)$ from the generators of F^Q into the formal power series ring $Q\langle\langle x_1, ..., x_n\rangle\rangle$ with coefficients in Q. It is known that this map induces a homomorphism $\lambda : F^Q \longrightarrow Q\langle\langle x_1, ..., x_n\rangle\rangle$ (the Magnus homomorphism). Is λ injective? Or, equivalently, is the group F^Q residually torsion-free nilpotent?*

A construction of the group F^Q in terms of free products with amalgamation is given in [G.Baumslag, *Some aspects of groups with unique roots*, Acta Math. **104** (1960), 217–303].

The best known result about the Magnus homomorphism of the group F^Q is due to G.Baumslag [*On the residual nilpotence of certain one-relator groups*, Comm. Pure Appl. Math. **21** (1968), 491–506]. He proved that the Magnus homomorphism is one-to-one on any subgroup of F^Q of the form $\langle F, t | u = t^n\rangle$.

This problem can be re-formulated in a more general form, where the ring Q of rationals is replaced by some other associative ring A. In [A.Myasnikov and V.Remeslennikov, *Exponential groups.II. Extensions of centralizers and tensor completion of CSA-groups.* Internat. J. Algebra Comput. **6** (1996), 687–711], it was shown how to construct a free group F^A for an arbitrary unitary associative ring A of characteristic 0. In particular, if $A = Z[X]$ is a ring of polynomials with integral coefficients, then F^A is Lyndon's free group.

In [A.M.Gaglione, A.G.Myasnikov, V.N.Remeslennikov, D.Spellman, *Formal power series representations of free exponential groups.* Comm. Algebra **25** (1997), 631–648], it was shown that the Magnus homomorphism of $F^{Z[x]}$ into the corresponding power series ring is an emebedding. Moreover, the Magnus homomorphism is an embedding for *every* unitary associative ring A of characteristic 0 if and only if it is an embedding in the case where $A = Q$.

(F13) (I.Kapovich) *Is the group F^Q in the previous problem linear?*

We note that Lyndon's free $Z[x]$-group $F^{Z[x]}$ (see the background to (F12)) is linear. Indeed, the group $F^{Z[x]}$ is discriminated by F [R.Lyndon, *Groups with parametric exponents*, Trans. Amer. Math. Soc. **96** (1960), 518–533], hence it is universally equivalent to F, therefore it is embeddable into an ultrapower of F, which is linear.

(F14) *Let F be a non-cyclic free group of finite rank, and G a finitely generated residually finite group. Is G isomorphic to F if it has the same set of finite homomorphic images as F does?*

We note that the answer is "yes" for a *free metabelian* group of finite rank – see [G. A.Noskov, *The genus of a free metabelian group* (Russian). Preprint 84-509. Akad. Nauk SSSR Sibirsk. Otdel., Vychisl. Tsentr, Novosibirsk, 1984. 18 pp.].

(F15) (V.Shpilrain) *Let F be a non-cyclic free group, and R a non-cyclic subgroup of F. Suppose that the commutator subgroup $[R, R]$ is a normal subgroup of F. Is R necessarily a normal subgroup of F?*

This question was motivated by the following result of [M.Auslander, R.C.Lyndon, *Commutator subgroups of free groups.* Amer. J. Math. **77** (1955), 929–931]: if R and S are normal subgroups of F, and $[R, R] \subseteq [S, S]$, then $R \subseteq S$. Dunwoody [*On verbal subgroups of free groups.* Arch. Math. **16** (1965), 153–157] showed that the condition on R being normal cannot be dropped, but it is not known whether or not the condition on S being normal can be dropped.

(F16) (V.Remeslennikov) *Let R be the normal closure of an element r in a free group F with the natural length function, and suppose that s is an element of minimal length in R. Is it true that s is conjugate to one of the following elements: $r, r^{-1}, [r, f]$, or $[r^{-1}, f]$ for some element f ?*

This question was motivated by a well-known result of Magnus (see e.g. [R.Lyndon, P.Schupp, *Combinatorial Group Theory*, Series of Modern Studies in Math. **89**. Springer-Verlag, 1977]: if two elements, r and s, of a free group F have the same normal closure in F, then s is conjugate to r or r^{-1}.

(F17) (M.Wicks) *Let F be a non-cyclic free group of rank n, and $P(n, k)$ the number of its primitive elements of length k. What is the growth of $P(n, k)$ as a function of k, with n fixed ?*

We just note here that the function $P(n, k)$ is recursive, i.e., its values can be actually computed.

(F18) (C.Sims) *Is the c-th term of the lower central series of a free group of finite rank the normal closure of basic commutators of weight c ?*

This is known to be true for $c \leq 5$.

(F19) (A.Gaglione, D.Spellman) *Let F be a non-cyclic free group, and G the Cartesian (unrestricted) product of countably many copies of F. Is the group $G/[G,G]$ torsion-free?*

***(F20)** (L.Comerford) *If an equation over a free group F has no solutions in F, is there a finite quotient of F in which the equation has no solutions? (If so, this provides another proof of Makanin's theorem).*

The answer was recently shown to be negative [T.Coulbois, A.Khelif, *Equations in free groups are not finitely approximable*, Proc. Amer. Math. Soc., to appear].

(F21) (P.M.Neumann) *Let G be a free product amalgamating proper subgroups H and K of A and B, respectively. Suppose that A, B, H, K are free groups of finite ranks. Can G be simple?*

It cannot if H and K are of infinite index – see [R.Camm, *Simple free products.* J. London Math. Soc. **28** (1953), 66–76].

(F22) (A.Olshanskii) *Does the free group of rank 2 have an infinite ascending chain of fully invariant subgroups, each being generated (as a fully invariant subgroup) by a single element?*

(F23) (A.Myasnikov, V.Remeslennikov) *Let G be a free product of two isomorphic free groups of finite ranks amalgamated over a finitely generated subgroup.*
(a) *Is the conjugacy problem solvable in G?*
(b) *Is there an algorithm to decide if G is free ?*
(c) *Is there an algorithm to decide if G is hyperbolic?*

If the amalgamated subgroup is cyclic then the first two problems have affirmative answers:

(a) is due to S.Lipschutz [*Generalization of Dehn's result on the conjugacy problem*, Proc. Amer. Math.Soc. **17** (1966), 759–762]. See also [S.Lipschutz, *The conjugacy problem and cyclic amalgamations*, Bull. Amer. Math.Soc. **81** (1975), 114–116], and

(b) is due to Whitehead since a one-relator group is free if and only if the relator is part of a basis of the ambient free group.

(F24) (G.Baumslag, A.Myasnikov, V.Remeslennikov) *Is a free product of two equationally noetherian groups equationally noetherian? (A group is called equationally noetherian if every system of equations in finitely many variables in this group is equivalent to a finite subsystem.)*

R.Bryant [*The verbal topology of a group.* J.Algebra **48** (1977), 340–346] and V.Guba [*Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems.* Math. Notes USSR **40** (1986), 688–690] proved that free groups are equationally noetherian. See also [J.Stallings, *Finiteness properties of matrix representations.* Ann. of Math. (2) **124** (1986), 337–346].

For a general discussion on this and related problems, we refer to [G.Baumslag, A.Myasnikov, V.Remeslennikov, Algebraic geometry over groups I: Algebraic sets and ideal theory, preprint].

ONE-RELATOR GROUPS

(OR1) (G.Baumslag) *Are all one-relator groups with torsion residually finite?*

For a background to this problem, see the survey [G.Baumslag, *Some open problems.* Summer School in Group Theory in Banff, 1996, 1–9. CRM Proceedings and Lecture Notes. **17**. Amer. Math. Soc., Providence, 1999].

(OR2) Is the isomorphism problem solvable for one-relator groups with torsion?

See the background to the problem (O4).

(OR3) (A.Myasnikov) *Is the complexity of the word problem for every one-relator group quadratic, i.e., is there for every one-relator group an algorithm solving the word problem in quadratic time with respect to the length of a word? In polynomial time?*

(OR4) *Is the generalized word problem solvable for one-relator groups? That is, is there an algorithm for deciding if a given element of the group belongs to a given finitely generated subgroup?*

(OR5) *Is it true that if the relation module of a group G is cyclic, then G is a one-relator group?*

J.Harlander [*Solvable groups with cyclic relation module*, J. Pure Appl. Algebra **90** (1993), 189–198] showed that the answer is "yes" in the case where G is finitely generated and solvable.

(OR6) (G.Baumslag) *Let $H = F/R$ be a one-relator group, where R is the normal closure of an element $r \in F$. Then, let $G = F/S$ be another one-relator group, where S is the normal closure of $s = r^k$ for some integer k. Is G residually finite whenever H is ?*

See the survey [G.Baumslag, *Some open problems.* Summer School in Group Theory in Banff, 1996, 1–9. CRM Proceedings and Lecture Notes. **17**. Amer. Math. Soc., Providence, 1999].

(OR7) (G.Baumslag) *Let $G = F/R$ be a one-relator group with the relator from $[F, F]$.*
(a) *Is G hopfian ?*
***(b)** *Is G residually finite ?*

***(c)** *Is G automatic ?*

A solution of the problems (b) and (c) was communicated to us by A.Olshanskii. In fact, the commutator subgroup $[F, F]$ can be replaced here by *any* non-cyclic subgroup of a free group F; the answer will still be negative. It follows from a result of [A.Olshanskii, *SQ-universality of hyperbolic groups*, Mat. Sb. **186** (1995), no. 8, 119–132] that for any m, every non-cyclic subgroup H of F contains a subgroup K, which is a free group of rank m, with the following property: for any normal subgroup U of K, the intersection of K and the normal closure of U in F is again U.

To apply this result to our situation, take two elements, x and y, that generate a subgroup $K = F_2$ of $H = [F, F]$ with the property described above. Let r be a Baumslag-Solitar relator built on these two elements; for example, take $r = xyx^{-1}y^{-2}$. Let U be the normal closure (in K) of r. Then, from what is said in the previous paragraph, it follows that the normal closure of U in F (call it V) intersects K in U. Therefore, the (one-relator) group F/V contains a subgroup KV/V which is isomorphic to a Baumslag-Solitar group, hence F/V can be neither residually finite nor automatic.

(OR8) (G.Baumslag) *The same as (OR7), but for a relator of the form* $[u, v]$.

This problem, as well as (OR7)(a), is motivated by the desire to find a non-hopfian one-relator group which is essentially different from any of the Baumslag-Solitar groups [G.Baumslag, D.Solitar, *Some two-generator one-relator non-Hopfian groups.* Bull. Amer. Math. Soc. **68** (1962), 199–201].

(OR9) (D.Moldavanskii) *Are two one-relator groups isomorphic if either of them is a homomorphic image of the other?*

(OR10) *Is every one-relator group without non-abelian metabelian subgroups, automatic?*

Note that hyperbolic groups are automatic, and, in particular, an amalgamated product of two free groups with finitely generated subgroups amalgamated is hyperbolic if at least one of the subgroups is malnormal [O.Kharlampovich and A.Myasnikov, *Hyperbolic groups and free constructions.* Trans. Amer. Math. Soc. **350** (1998), 571–613].

Furthermore, an amalgamated product of two finitely generated abelian groups is automatic [G.Baumslag, S.M.Gersten, M.Shapiro, H.Short, *Automatic groups and amalgams – a survey.* Algortims and Classification in Combinatorial Group Theory (Berkeley, CA, 1989), 179–194, Math. Sci. Res. Inst. Publ., **23.** Springer, New York, 1992].

(OR11) (C.Y.Tang) *Are all one-relator groups with torsion conjugacy separable?*

(OR12) *Are all freely indecomposable one-relator groups with torsion co-hopfian?*

(OR13) **(a)** *Which finitely generated one-relator groups have all generating systems (of minimal cardinality) Nielsen equivalent to each other ?*
(b) *Which finitely generated one-relator groups have only tame automorphisms (i.e., automorphisms induced by automorphisms of the ambient free group)?*

For surveys on Nielsen equivalence in groups, we refer to [G.Rosenberger, *Minimal generating systems for plane discontinuous groups and an equation in free groups.* Groups–Korea 1988 (Pusan, 1988), 170–186, Lecture Notes in Math., **1398**, Springer, Berlin, 1989] and [C.K.Gupta, V.Shpilrain, *Lifting automorphisms: a survey*, Groups '93 Galway/St. Andrews, Vol. 1 (Galway, 1993), 249–263, London Math. Soc. Lecture Note Ser., **211**, Cambridge Univ. Press, Cambridge, 1995].

(OR14) (G.Baumslag, D.Spellman) *Describe one-relator groups which are discriminated by a free group.*

We note that recently, O.Kharlampovich and A.Myasnikov [*Irreducible affine varieties over groups,* J.Algebra **200** (1998), 517–570] proved that every finitely generated group which is discriminated by a free group can be obtained from a free group by applying finitely many free constructions of a very particular type.

(OR15) *If G is a one-relator group with the property that every subgroup of finite index is again a one-relator group, and every subgroup of infinite index is free, must G be a surface group?*

(OR16) *Let $S(n)$ be the orientable surface group of genus n.*

(a) *Are the groups $S(n)$ and $S(m)$ $(m, n \geq 2)$ elementary equivalent? (i.e., $Th(S(m)) = Th(S(n))$?)*
(b) *Is $S(m)$ elementarily equivalent to F_{2m}, the free group of rank $2m$?*

FINITELY PRESENTED GROUPS

Although finitely generated free groups and one-relator groups are finitely presented, we believe they deserve special sections, so you won't find them here.

(FP1) *The triviality problem for groups with a balanced presentation (the number of generators equals the number of relators). See also problem (O1).*

(FP2) *Can a non-trivial finitely presented group be isomorphic to its direct square?*

(FP3) (M.Kervaire, F.Laudenbach) *Let $F_n/R = \langle x_1, ..., x_n | r_1, ..., r_m \rangle$ be a presentation of a non-trivial group. Is it true that a group $\langle x_1, ..., x_n, x_{n+1} | r_1, ..., r_m, s \rangle$ is also non-trivial for any element s from F_{n+1} ?*

For a background to this problem, we refer to [R.Lyndon, *Problems in combinatorial group theory.* Combinatorial group theory and topology (Alta, Utah, 1984), 3–33, Ann. of Math. Stud., **111**, Princeton Univ. Press, 1987].

***(FP4)** (R.Bieri, R.Strebel) *Is it true that if the relation module of a group G is finitely generated, then G is finitely presented ?*

A paper by M.Bestvina and N.Brady with a negative solution of this problem has been published recently [*Morse theory and finiteness properties of groups.* Invent. Math. **129** (1997), 445–470]. A somewhat simpler method was subsequently used by W.Dicks and I. Leary [*Presentations for subgroups of Artin groups*, Proc. Amer. Math. Soc., to appear]. The latter paper is available at *http://manwe.mat.uab.es/dicks/pub.html*

(FP5) (J.Stallings) *If a finitely presented group is trivial, is it always possible to replace one of the defining relators by a primitive element without changing the group?*

(FP6) **(a)** (S.I.Adian) *Is it true that a finitely presented group has either polynomial or exponential growth?*

(b) *(R.I.Grigorchuk) Is it true that every finitely presented group contains either a free 2-generator semigroup, or a nilpotent subgroup of finite index ?*

The point here is that there are examples of groups of intermediate growth (between polynomial and exponential), but all these groups are infinitely presented – see [R.I.Grigorchuk, *On the Milnor problem of group growth* (Russian). Dokl. Akad. Nauk SSSR **271** (1983), 30–33]; [R.I.Grigorchuk, *Construction of p-groups of intermediate growth that have a continuum of factor-groups* (Russian). Algebra i Logika **23** (1984), 383–394, 478]; [R.I.Grigorchuk, *Degrees of growth of p-groups and torsion-free groups* (Russian). Mat. Sb. **168** (1985), 194–214, 286]; [R.I.Grigorchuk, A.Maki, *On a group of intermediate growth that acts on a line by homeomorphisms* (Russian). Math. Notes USSR **53** (1993), 146–157].

(FP7) (C.Y.Tang) *Is there a non-free non-cyclic finitely presented group all of whose proper subgroups are free?*

(FP8)*Is every knot group virtually free-by-cyclic?*

For various properties of knot groups, we refer to the book [L.P.Neuwirth, *Knot groups.* Annals of Mathematics Studies **56**, Princeton University Press, Princeton, N.J. 1965].

***(FP9)** (G.Baumslag) *Is every finitely generated group discriminated by a free group, finitely presented?*

It is – see [O.Kharlampovich, A.Myasnikov, *Irreducible affine varieties over groups*, J.Algebra **200** (1998), 517–570].

*(**FP10**) (G.Baumslag) *Is a finitely generated free-by-cyclic group finitely presented?*

It is; see the preprint [M.Feighn, M.Handel, *Mapping tori of free group automorphisms are coherent*] at *http://andromeda.rutgers.edu/~feighn/*

(FP11) (G.Baumslag, F.B.Cannonito, C.F.Miller) *Is every countable locally linear group embeddable in a finitely presented group?*

(FP12) (A.Olshanskii) *If a relatively free group is finitely presented, is it virtually nilpotent?*

(FP13) (S.Ivanov) *Is every finitely presented Noetherian group virtually polycyclic?*

(FP14) (M.I.Kargapolov) *Is every residually finite Noetherian group virtually polycyclic?*

(FP15) (J.Wiegold) *Is every finitely generated perfect group G (i.e., $[G, G] = G$) the normal closure of a single element?*

(FP16) (P.Scott) *Let p, q, r be distinct prime numbers. Is the free product $Z_p * Z_q * Z_r$ the normal closure of a single element?*

(FP17) (D.Anosov) *Is there a non-cyclic finitely presented group each element of which is a conjugate of some power of a single element?*

(FP18) (V.N.Remeslennikov) *Is every countable abelian group embeddable in the centre of some finitely presented group?*

*(**FP19**) (R.Hirshon) *Let G be a finitely generated residually finite group, and ϕ an endomorphism of G. Is it true that $\phi^{k+1}(G)$ is isomorphic to $\phi^k(G)$ for some k?*

R.Hirshon himself [*Some properties of endomorphisms in residually finite groups*, J. Austral. Math. Soc. Ser. A **24** (1977), 117–120] proved the assertion in the case where $\phi(G)$ has finite index in G. However, the answer is negative in general [D.Wise, *A continually descending endomorphism of a finitely generated residually finite group*, Bull. London Math. Soc., to appear]. A preprint is available at *http://zebra.sci.ccny.cuny.edu/web/abstracts/1997/97-09-23A.html*

HYPERBOLIC AND AUTOMATIC GROUPS

(H1) (a) *Are hyperbolic groups residually finite?*
(b) *Does every hyperbolic group have a proper subgroup of finite index?*

I.Kapovich and D.Wise [*The equivalence of some residual properties of word-hyperbolic groups*, preprint] proved the equivalence of (a) and (b). The preprint is available at *http://math.cornell.edu/~daniwise/papers.html*

(H2) *Are hyperbolic groups linear?*

(H3) Do hyperbolic groups with torsion have solvable isomorphism problem?

We note that Z.Sela [*The isomorphism problem for hyperbolic groups.* Ann. of Math. (2) **141** (1995), 217–283] has solved the isomorphism problem for torsion-free hyperbolic groups that do not split (as an amalgamated product or an HNN extension) over the trivial or the infinite cyclic group.

(H4) (A.Myasnikov) *Given a finite presentation of a hyperbolic group (which is not necessarily a Dehn presentation), is it possible to find a Dehn presentation for this group in polynomial time?*

Note that every hyperbolic group has a Dehn presentation – see [I.G.Lysenok, *Some algorithmic properties of hyperbolic groups.* Math. USSR Izv. **35** (1990), 145–163].

(H5) (A.Myasnikov) *Given a finite presentation of an automatic group, can one decide if this group is hyperbolic?*

Papasoglu [*An algorithm detecting hyperbolicity.* Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994), 193–200, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., **25**, Amer. Math. Soc., Providence, RI, 1996] gave a partial algorithm to recognize hyperbolic groups. Given a finite presentation $\langle S, R\rangle$, the algorithm terminates if the group $G = \langle S, R\rangle$ is hyperbolic and gives an estimate of the hyperbolicity constant δ.

(H6) (S.Gersten) *Are all automatic groups biautomatic?*

For a background on problems (H6) through (H10) we refer to [S.M.Gersten, *Problems on automatic groups.* Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), 225–232, Math. Sci. Res. Inst. Publ., **23**, Springer, New York, 1992].

(H7) (S.Gersten) *Does every automatic group have a solvable conjugacy problem?*

(H8) (S.Gersten) *Is every biautomatic group which does not contain any $Z \times Z$ subgroups, hyperbolic?*

(H9) (S.Gersten) *Can the group $< x, y; yxy^{-1} = x^2 >$ be a subgroup of an automatic group?*

(H10) (S.Gersten) *Is a retract of an automatic group automatic?*

(H11) *Does every hyperbolic group act properly discontinuously and co-compactly by isometries on a $CAT(k)$ space, where $k < 0$?*

(H12) (G.Baumslag, A.Myasnikov, V.Remeslennikov) *Is every hyperbolic group equationally noetherian? (A group is called equationally noetherian if every system of equations in finitely many variables in this group is equivalent to a finite subsystem).*

For a background on equationally noetherian groups, we refer to [G.Baumslag, A.Myasnikov, V.Roman'kov, *Two theorems about equationally Noetherian groups*, J. Algebra **194** (1997), 654–664]. Here we just mention that free groups are equationally noetherian; this is due to R.Bryant [*The verbal topology of a group.* J.Algebra **48** (1977), 340–346] and [V.S.Guba, *Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems.* Math. Notes USSR **40** (1986), 688–690].

For a general discussion on this and related problems, we refer to [G.Baumslag, A.Myasnikov, V.Remeslennikov, Algebraic geometry over groups I: Algebraic sets and ideal theory, preprint].

(H13) *Are combable groups automatic?*

(H14) (A.Myasnikov) *We call a subgroup H of a group G malnormal if for any element g in G, but not in H, one has $H^g \cap H = \{1\}$. Is this property algorithmically decidable for finitely generated subgroups of a hyperbolic group?*

We note that malnormality is decidable in *free* groups – see [G.Baumslag, A.Myasnikov, V.Remeslennikov, *Malnormality is decidable in free groups*, to appear]. Another idea on how to check malnormality is essentially contained in [J.Stallings, *Topology of finite graphs.* Invent. Math. **71** (1983), 551–565].

On the other hand, there is no uniform algorithm that determines for *any* hyperbolic group G and its arbitrary finitely generated subgroup H whether H is malnormal in G or not. The following argument was communicated to us by D.Wise.

For any finitely presented group Q, Rips' construction provides a short exact sequence

$$1 \to H \to G \to Q \to 1,$$

where H is finitely generated and non-trivial and G is hyperbolic.

Note that H is malnormal if and only if $H = G$. Consequently, H is malnormal if and only if Q is trivial. Hence, a uniform algorithm for detecting malnormality would provide an effective procedure to determine whether or not a finitely presented group Q is trivial. But this is known to be algorithmically undecidable.

Still, it is conceivable that for each particular hyperbolic group an algorithm for detecting malnormality of its finitely generated subgroups might exist. We note that there exists an algorithm (due to D.Holt) which decides whether or not a given finitely generated *quasiconvex* subgroup of a hyperbolic group is malnormal.

(H15) (A.Myasnikov) *If a finitely generated subgroup H of a hyperbolic group is malnormal (see above), does it follow that H is quasiconvex?*

(H16) *Is every finitely presented metabelian automatic group virtually abelian ?*

Note that a finitely generated nilpotent group is automatic if and only if it is virtually abelian – see [D.Epstein, J.Cannon, D.Holt, S.Levy, M.Paterson, W.Thurston, *Word processing in groups.* Jones and Bartlett Publishers, Boston, 1992].

BRAID GROUPS

See also problem (O10).

(B1) *Are braid groups linear?*

There are two canonical representations of braid groups by matrices over Laurent polynomial rings – the Burau and Gassner representations (the latter is actually a representation of the pure braid group which is a subgroup of finite index in the whole braid group). Both of these representations are faithful for $n = 2, 3$ (a general reference here is [J.S.Birman, *Braids, links and mapping class groups*, Ann. Math. Studies **82**, Princeton Univ. Press, 1974]). A proof of the Gassner representation being faithful for every n (which implies braid groups being linear) was recently claimed in [S.Bachmuth, *Braid groups are linear groups*, Adv. Math. **121** (1996), 50–61]. However, there is a controversy around this paper since several people believe they have found essential gaps in the proof (see J.S.Birman's review article 98h:20061 in Math. Reviews). This makes us consider Problems (B1), (B2) open.

(B2) *Is the Gassner representation of the pure braid group P_n faithful for every n ?*

See the background to (B1).

(B3) *Is the Burau representation of the braid group B_n faithful for $n = 4, 5$?*

The Burau representation was shown to be non-faithful for $n \geq 10$ in [J.Moody, *The faithfulness question for the Burau representation*, Proc. Amer. Math. Soc. **119** (1993), 671–679], and then for $n \geq 6$ in [D.Long and M.Paton, *The Burau representation is not faithful for $n \geq 6$*, Topology **32** (1993), 439–447].

(B4) (J.Birman) *Let $F = F_n$ be the free group of rank n generated by $a_1, ..., a_n$. Is there a solution of the equation $y_1 a_1 y_1^{-1} ... y_n a_n y_n^{-1} = a_1 ... a_n$ with all y_i from the second commutator subgroup F'' ?*

The answer is "no" if and only if the Gassner representation of the pure braid group P_n is faithful – cf. problem (B2).

(B5) (J.Birman) *Give necessary and sufficient conditions for a square matrix over Laurent polynomial ring to be the Burau matrix of some braid.*

For a background, see [J.S.Birman, *Braids, links and mapping class groups*, Ann. Math. Studies **82**, Princeton Univ. Press, 1974].

(B6) (V.Lin) *Let $n \geq 4$.*
(a) *Does the braid group B_n have a non-trivial non-injective endomorphism?*

(b) *Is it true that every non-trivial endomorphism of the commutator subgroup* $[B_n, B_n]$ *is an automorphism ?*

For a background and discussion on the problems (B6)–(B8), we refer to a recent preprint by V.Lin [*Braids, permutations, polynomials.I*] which can be either found on the Max Planck Institut für Mathematik electronic preprint server, or requested from the author at *vlin@techunix.technion.ac.il.* Here we only note that automorphisms of braid groups were described in [J.Dyer, E.Grossman, *The automorphism groups of the braid groups,* Amer. J. Math. **103** (1981), 1151–1169].

(B7) (V.Lin) *Let* $n \geq 4$.
(a) *Does the braid group* B_n *have a proper torsion-free non-abelian factor group?*
(b) *Does the commutator subgroup* $[B_n, B_n]$ *have a proper torsion-free factor group?*

(B8) (V.Lin) *Let* $n \geq 4$.
(a) *Is it true that every automorphism of the commutator subgroup* $[B_n, B_n]$ *can be extended to an automorphism of the whole group* B_n *?*
(b) *Is it true that every non-trivial endomorphism of* $[B_n, B_n]$ *can be extended to an endomorphism of* B_n *?*

NILPOTENT GROUPS

(N1) (A.Myasnikov) *Let* G *be a free nilpotent group of finite rank. Suppose an element* $g \in G$ *is fixed by every automorphism of* G*. Is it true that* $g = 1$ *?*

V.Bludov has communicated the following example of a non-trivial element g of a free nilpotent group of rank 2 and nilpotency class $k \geq 8$, which is fixed by every automorphism: $g = [a, [a, b], [a, b, b], [a, b], ..., [a, b]]$, where there are $(2k-3)$ occurrences of $[a, b]$ after $[a, b, b]$. (Here a and b are generators of the free nilpotent group). However, the problem for free nilpotent groups of bigger rank remains open.

(N2) *Let* G *be a finitely generated nilpotent group. Is the isoperimetric function of* G *equivalent to a polynomial?*

(N3) (B.I.Plotkin) *Is it true that every locally nilpotent group is a homomorphic image of a torsion-free locally nilpotent group?*

(N4) (G.Baumslag) *Let* G *be a finitely generated torsion-free nilpotent group. Is it true that there are only finitely many non-isomorphic groups in the sequence* $Aut(G)$, $Aut(Aut(G))$, ... *?*

Hamkins [*Every group has a terminating transfinite automorphism tower.* Proc. Amer. Math. Soc. **126** (1998), 3223–3226] established the property in the title.

(N5) (G.Baumslag) *Is the property of being directly indecomposable decidable for finitely generated nilpotent groups?*

(N6) (A.Myasnikov) *Describe all finitely generated nilpotent groups of class 2 which have genus 1. (We say that a group G has genus 1 if every group with the same set of finite homomorphic images as G, is isomorphic to G).*

See the background to the problem (F14).

METABELIAN GROUPS

Some of the problems about free groups (particularly (F1), (F3)) are also of interest when asked about free metabelian groups.

(M1) *The isomorphism problem for finitely presented metabelian groups.*

There is an algorithm to determine whether or not a given finitely generated metabelian group is free metabelian – see [J.R.J.Groves, C.F.Miller, III, *Recognizing free metabelian groups.* Illinois J. Math. **30** (1986), 246–254] and the paper by Noskov cited in the background to the problem (F14).

We also note that "most" algorithmic problems about finitely presented metabelian groups are solvable – see [G.Baumslag, F.Cannonito, D.Robinson, *The algorithmic theory of finitely generated metabelian groups*, Trans. Amer. Math. Soc. **344** (1994), 629–648] and references thereto.

(M2) *Is the automorphism group of a free metabelian group of rank > 3 finitely presented ?*

The automorphism group of a free metabelian group of finite rank is known to be finitely generated unless the rank equals 3 – see [S.Bachmuth, H.Mochizuki, $\mathrm{Aut}(F) \to \mathrm{Aut}(F/F")$ *is surjective for free group F of rank ≥ 4,* Trans. Amer. Math. Soc. **292** (1985), 81–101] and [S.Bachmuth, H.Mochizuki, *The nonfinite generation of* $\mathrm{Aut}(G)$*, G free metabelian of rank* 3, Trans. Amer. Math. Soc. **270** (1982), 693–700].

(M3) (F.B.Cannonito) *Is there an algorithm which decides whether or not a given finitely presented solvable group is metabelian?*

(M4) (P.Hall) *Are projective groups of infinite countable rank in the class of metabelian groups free metabelian?*

For groups of finite rank, the answer is affirmative – see [V.A.Artamonov, *Projective metabelian groups and Lie algebras* (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 226–236, 469].

(M5) (G.Baumslag) *What can one say about the integral homology of a finitely generated metabelian group?*

For a survey on homological properties of metabelian groups, we refer to [Yu.V.Kuz'-min, *Homology theory of free abelianized extensions.* Comm. Algebra **16** (1988), 2447–2533].

(M6) (R.Goebel) *Is there a group which is NOT isomorphic to the outer automorphism group of any metabelian group with a trivial centre?*

SOLVABLE GROUPS

(S1) (A.I.Mal'cev) *Describe the automorphism group of a free solvable group of finite rank. In particular, is this group finitely generated?*

The automorphism group of a free solvable group of derived length > 2 and *rank* > 2 cannot be generated by elementary Nielsen automorphisms – see [C.K.Gupta, F.Levin, *Tame range of automorphism groups of free polynilpotent groups*, Comm. Algebra **19** (1991), 2497–2500] and [V.Shpilrain, *Automorphisms of F/R' groups*, Internat. J. Algebra Comput. **1** (1991), 177–184]. Moreover, every free solvable group of derived length $d > 2$ and rank $r > 2$ has automorphisms that cannot be lifted to automorphisms of the free solvable group of derived length $d+1$ and the same rank r – see [V.Shpilrain, *Non-commutative determinants and automorphisms of groups*, Comm. Algebra **25** (1997), 559–574]. It is not known however whether or not those automorphism groups are finitely generated.

(S2) (M.I.Kargapolov) *The word problem for groups admitting a single defining relation in the variety of all solvable groups of a given derived length.*

We note that the word problem for groups admitting *finitely many* defining relations in the variety of all solvable groups of a given derived length > 2, is, in general, unsolvable – see [O.G.Kharlampovich, *A finitely presented solvable group with unsolvable word problem* (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 852–873].

(S3) (M.I.Kargapolov) *Is it true that every group of rank* > 2 *admitting a single defining relation in the variety of all solvable groups of a given derived length, has trivial centre?*

E.Timoshenko [*Center of a group with one defining relation in the variety of* 2-*solvable groups* (Russian). Sibirsk. Mat. Z. **14** (1973), 1351–1355, 1368] settled this problem in the affirmative for metabelian groups. C.K.Gupta and V.Shpilrain [*The centre of a one-relator solvable group*, Internat. J. Algebra Comput. **3** (1993), 51–55] settled the problem (also in the affirmative) for solvable groups of arbitrary derived length, under an additional assumption that the relator is not a proper power modulo any term of the derived series.

(S4) (M.I.Kargapolov) *Is there a number $N = N(k, d)$ so that every element of the commutator subgroup of a free solvable group of rank k and derived length d, is a product of N commutators?*

The answer is "yes" for free metabelian groups – see [Kh.S.Allambergenov, V.A.Romankov, *Products of commutators in groups* (Russian). Dokl. Akad. Nauk UzSSR 1984, 14–15] and for free solvable groups of derived length 3 – see [A.H.Rhemtulla, *Commutators of certain finitely generated soluble groups.* Canad. J. Math. **21** (1969), 1160–1164].

(S5) (P.M.Neumann) *Is it true that if A, B are finitely generated solvable Hopfian groups, then $A \times B$ is Hopfian?*

(S6) (V.N.Remeslennikov) *The conjugacy problem for finitely generated abelian-by-polycyclic groups.*

(S7) (D.Robinson) *Is there a finitely presented solvable group satisfying the maximum condition on normal subgroups, with unsolvable word problem?*

(S8) (G.Baumslag, V.Remeslennikov) *Is a finitely generated free solvable group of derived length 3 embeddable in a finitely presented solvable group?*

(S9) (B.Fine, V.Shpilrain) *Let u be an element of a group G. We call u a test element if, whenever $\phi(u) = u$ for some endomorphism ϕ of the group G, this ϕ is actually an automorphism of G. Does the free solvable group of rank 2 and derived length $d > 2$ have any test elements?*

The most obvious candidate for a test element in a group generated by x and y would be $u = [x, y]$. This however is *not* a test element in a free solvable group of derived length $d > 2$ – see [N.Gupta, V.Shpilrain, *Nielsen's commutator test for two-generator groups*, Math. Proc. Cambridge Philos. Soc. **114** (1993), 295–301].

Department of Mathematics, The City College of New York, New York, NY 10031

e-mail addresses: gilbert@rio.sci.ccny.cuny.edu, alexei@rio.sci.ccny.cuny.edu, shpil@groups.sci.ccny.cuny.edu