Malnormality is decidable in free groups

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Abstract

We prove here that there is an algorithm whereby one can decide whether or not any finitely generated subgroup of a finitely generated free group is malnormal.

1 Introduction

If H is a finitely generated subgroup of a finitely generated free group F then it is well-known that there is an algorithm whereby one can decide whether or not any element of F is contained in H, i.e., the generalized word problem is decidable for finitely generated free groups. There are a number of other problems involving finitely generated subgroups of finitely generated free groups that are also algorithmically solvable; for example, whether a given finitely generated subgroup is of finite index, whether a pair of finitely generated subgroups are conjugate, whether there is an automorphism of the ambient free group which maps onefinitely generated subgroup onto another. There is an algorithms which compute the normalizer of a finitely generated subgroup and the (finitely generated) intersection of two finitely generated subgroups, of a finitely generated free group. Proofs of these results and related references are contained in the book by Lyndon and Schupp [3].

It follows immediately from the fact that the generalised word problem is solvable in a finitely generated free group that there is an algorithm whereby one can decide whether or not any finitely generated subgroup is normal. The purpose of this note is to add one more positive result to this list, which should be compared with the preceding remark, by proving the following

Theorem 1 *There is an algorithm whereby one can decide whether or not a finitely generated subgroup of a free group is malnormal.*

We recall that a subgroup H of a group G is termed *malnormal* if $g^{-1}Hg \cap H = 1$ whenever $g \in G, g \notin H$.

The existence or otherwise of the above algorithm was proposed to us a few years ago by a number of our colleagues and has since been programmed by D.

Serbin in Omsk and added to the algorithms in the software package MAGNUS [4]. In part the interest in malnormality stems from the fact that if

$$G = \{A \star B; H\}$$

is an amalgamated product of two finitely generated free groups A and B with a finitely generated subgroup H amalgamated, then G is hyperbolic if either H is a malnormal subgroup of A or if H is a malnormal subgroup of B [2]. Malnormality also enters into the the study of certain one-relator groups, where the question as to whether such one-relator groups are automatic hinges on the malnormality of certain subgroups of free groups [1].

2 The proof of the theorem

We shall have need of the following definition.

Definition 1 *Let G be a group and let H be a subgroup of G. We term $g \in G$ potentially H-normalizing, or simply pn in the event that H is understood, if*

$$g \notin H \text{ and } g^{-1}Hg \cap H \neq 1$$

We denote the set of all potentially H-normalizing elements by $pn(H)$, which we refer to as the potential normalizer of H.

It follows that H is malnormal in G if $pn(H) = \phi$.

The following simple lemma will be useful in the proof of the theorem.

Lemma 1

$$H.pn(H).H \subseteq pn(H).$$

The proof is straightforward. For suppose that g is pn. Then there exists an element $h \in H, h \neq 1$ such that $g^{-1}hg = h_1 \in H$. So if $u, v \in H$ we have

$$(ugv)^{-1}(uhu^{-1})ugv = v^{-1}h_1v \in H,$$

which completes the proof.

The proof of Theorem 1 will be divided up into four lemmas, the last of which demonstrates that malnormality of finitely generated subgroups of free groups is decidable.

We will need to make use of a so-called Nielsen set of generators of a subgroup of a free group. In order to explain, let F be a free group, freely generated by the set X. As usual, every element $f \in F, f \neq 1$ can be written uniquely as a reduced X-word. We denote the number of elements of $X \cup X^{-1}$ occurring in f by $\ell_X(f)$, which we refer to as the X-length of f. If $g \in F, g \neq 1$, then we write

$$f \circ g$$

in order to express the fact that the X-word fg is reduced as written, i.e., that there is no cancellation on computing the reduced form of the product fg. Now let H be a subgroup of F. Then there exists a set Y of free generators of H, termed a Nielsen set of generators, with the following properties:

1. each $y \in Y$ can be written in the form

$$y = y' \circ \mu(y) \circ y''$$

   where y' and y'' are reduced X-words, $\mu(y)$ is an element of $X \cup X^{-1}$ and

$$\mid \ell_X(y') - \ell_X(y'' \mid\leq 1.$$

2. If y and y_1 are distinct elements of Y, then the letters $\mu(y)^{\pm 1}$ and $\mu(y_1)^{\pm 1}$ do not cancel on computing the reduced form of $y^{\pm 1}y_1^{\pm 1}$.

This letter $\mu(y)$ is called the *middle* or, more usually, the *central letter of* $y \in Y \cup Y^{-1}$.

We begin with the proof of

Lemma 2 *Let f be an element in F, which is pn and of minimal length in HfH. If*

$$fhf^{-1} = h_1 \ (h, h_1 \in H, h \neq 1),$$

then at least one of the letters of h appears in the reduced form of fhf^{-1}, i.e., in h_1.

Proof. Suppose that all of h cancels in fhf^{-1}. Since, f is of minimal length in $pn(H)$, f cannot end in h^{-1}. Consequently, h cancels partly with f and also with f^{-1}. Hence, h can be written in the form $h = u \circ v \circ u^{-1}$, $f = f_1u^{-1}$ with u of maximal possible length. Then $fhf^{-1} = f_1vf_1^{-1}$ and v cancels completely with either f_1 or f_1^{-1}. Suppose, $f_1 = f_2v^{-1}$. Then $f = f_1u^{-1} = f_2v^{-1}u^{-1}$. Now $\ell_X(v^{-1}u^{-1}) > \frac{1}{2}\ell_X(h)$, which implies that $\ell_X(fh) < \ell_X(f)$, contradicting the choice of f at the outset.

We will adopt the notation introduced in Lemma 2 throughout the rest of this note without further mention.

Next we prove the

Lemma 3 *Suppose f is of minimal length in fH and that $h = a_1 \ldots a_n$ ($a_i \in Y \cup Y^{-1}$) is a reduced Y-product. Then either the central letter of a_1 remains in*

$$fa_1 \ldots a_j \ldots a_n \tag{1}$$

and in this case the cancellation with f in fh is exactly that of f in fa_1; or the central letter $\mu(a_1)$ cancels in the product 1, and in this case a_1 is of even length and exactly half of a_1 cancels with f.

The proof follows immediately from the fact that f is minimal in fH and from the properties of a Nielsen set of generators.

We come next to the the third lemma needed to prove Theorem 1.

Lemma 4 *Let m be the number of elements in Y. Suppose that f is of minimal length in the coset fH and that $a_1 \dots a_n$ $(a_i \in Y \cup Y^{-1})$ is a reduced Y-product. Furthermore, suppose that the central letter of a_j cancels in the product*

$$fa_1 \dots a_j \dots a_n \ (j < n) \tag{2}$$

but that the central letter of a_{j+1} does not. Then

1. *$j \leq m$;*
2. *for any $k = 1, \dots, j$ the length $\ell_X(a_k)$ is even and $\mu(a_k)$ cancels in 2;*
3. *for any $k = 1, \dots, j-1$ exactly the right half a_k'' of a_k cancels completely in $a_k a_{k+1}$;*
4. *$\ell_X(a_1) < \dots < \ell_X(a_j)$;*
5. *if the right half of a_j does not cancel completely with a_{j+1}, then*

   $$a_1 \dots a_{j-1} a_j' \mu(a_j)$$

   is precisely the part of $a_1 \dots a_n$ that cancels with f and

   $$\ell_X(a_1 \dots a_{j-1} a_j' \mu(a_j)) = \frac{1}{2} \ell_X(a_j);$$

6. *if the right half of a_j does cancel with a_{j+1}, then $a_1 \dots a_j p$, where p is an initial segment of a_{j+1} of length at most half that of a_{j+1} and at least half of that of a_j, is precisely the part of $a_1 \dots a_n$ that cancels with f and $\ell_X(a_1 \dots a_j p) \leq \frac{1}{2} \ell_X(a_{j+1})$.*

Proof. Since the central letter of a_1 cancels, it follows from Lemma 3, that the length of a_1 is even, that the first half of a_1 cancels with f and hence $f_1 = fa_1$ has the same length as f and ends with a_1''. In particular, f_1 is of minimal length in $f_1 H (= fH)$. If the right half of a_1 does not cancel completely with a_2, then the cancellation with f is exactly that of a_1 with f, which as we noted before, is a_1' - the left half of a_1. In this event $j = 1$, (1), (2), (3) hold, (4) does not apply. Suppose next that $j > 1$ and consider f_1 in the place of f. Then it follows, by induction on j, that

$$\ell_X(a_2) < \ell_X(a_3) < \dots \ell_X(a_j)$$

.

Now f_1 ends with a_1'' and by Lemma 3, not more then half of a_2 cancels with f_1. Since a_1'' must cancel in $f_1 a_2$, $\ell_X(a_1) < \ell_X(a_2)$. So, (2) holds. (3) and (4) follows again immediately by induction on j. Finally, (1) follows immediately from (2). This completes the proof of Lemma 4.

Finally we prove

Lemma 5 *Suppose that*

$$c_1 \dots c_m f a_1 \dots a_n f^{-1} = 1,$$

where $c_i, a_j \in Y \cup Y^{-1}$, and f is of minimal length in HfH. Then there exists $f' \in HfH$ of the same length as f which is a product of two pieces of generators from $Y \cup Y^{-1}$.

We have already noted that not all of $a_1 \dots a_m$ cancels on forming $fa_1 \dots a_m f^{-1}$. Therefore, f must cancel completely in the product

$$c_1 \dots c_m f a_1 \dots a_n$$

.

1. f cancels completely in $fa_1 \dots a_m$. By Lemma 3, there exists $j \leq n$ such that $f' = fa_1 \dots a_{j-1}$ has the same length as f and f' cancels completely into $a_j' \mu(a_j)$ which must be of even length. So f' is a piece of a_j and satisfies the derived conclusion.

2. Suppose f does not cancel completely in $fa_1 \dots a_n$. Then choosing j as before, put

$$f_1 = f a_1 \dots a_{j-1}.$$

   Then f_1 cancels with not more then half of a_j, say, b_j. This f_1 ends with a piece of a_j . So

$$f_1 f_2 \circ p$$

   and f_2 cancels completely on forming $c_1 \dots c_m f_1$. There exists an i such that $\mu(c_i)$ cancels but $\mu(c_{i-1}$ does not. Then $f_1' = c_i c_{i+1} \dots c_m f_1$ has the same length as f_1. Then f_1' cancels completely on forming

$$c_{i-1} f_1' a_j$$

   But by Lemma 4 no more than half of c_{i-1} nor more than half of a_j cancel. Therefore f_1' is a product of a piece of c_{i-1} and a piece of a_j and both pieces are respectively of length at most one half of c_{i-1} and a_j.

This completes of Lemma 5.

As noted earlier, Theorem 1 is a consequence of Lemma 5.

3 Examples

We illustrate Theorem 1 by giving here two examples. The first of these is

Example 1 *The subgroup H of the free group F on a, b generated by*

$$[a, b], [a^2, b^2], [a^3, b^3], [a^4, b^4], [a^5, b^5], [a^6, b^6], [a^7, b^7], [a^8, b^8], [a^9, b^9], [a^10, b^10]$$

is malnormal in F.

The determined reader can check the truth of this assertion by following out the steps in the proof of the theorem.

By way of contrast, we have analogously, the

Example 2 *The subgroup of F generated by*

$$ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^10b^10$$

is not malnormal.

Again, the determined reader can verify the truth of this assertion by hand.

There is an easier method for verifying both of the above assertions, namely to make use fo the software package MAGNUS [4] that is under development at the Mathematics Department of the City College of the City University of New York.

References

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[4] MAGNUS, *A software package for computing and experimenting in and with infinite groups*, Available by anonymous ftp from

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