Two theorems about equationally noetherian groups

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Abstract

An algebraic set over a group G is the set of all solutions of some system $\{f(x_1, \ldots, x_n) = 1 \mid f \in G * \langle x_1, \ldots, x_n \rangle\}$ of equations over G. A group G is equationally noetherian if every algebraic set over G is the set of all solutions of a finite subsystem of the given one. We prove that a virtually equationally noetherian group is equationally noetherian and that the quotient of an equationally noetherian group by a normal subgroup which is a finite union of algebraic sets, is again equationally noetherian. On the other hand, the wreath product $W = U \wr T$ of a non-abelian group U and an infinite group T is not equationally noetherian.

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1 Introduction

Let G be a group, let F_n be the free group, freely generated by $x_1, \ldots, x_n$ and let

$$G[x_1, \ldots, x_n] = G * F_n$$

be the free product of G and F_n. We use functional notation here, denoting an element $f \in G[x_1, \ldots, x_n]$ by

$$f = f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, g_1, \ldots, g_m) \tag{1}$$

thereby expressing the fact that word representing f in $G[x_1, \ldots, x_n]$ involves the *variables* $x_1, \ldots, x_n$ and, as needed, the *constants* $g_1, \ldots, g_m \in G$. We term

$$v = (v_1, \ldots, v_n) \ (v_i \in G) \tag{2}$$

a *root* of f if

$$f(v) = f(v_1, \ldots, v_n, g_1, \ldots, g_m) = 1.$$

If S is a subset of $G[x_1, \ldots, x_n]$ then v is said to be a root of S if it is a root of every $f \in S$. Let S be a subset of $G[x_1, \ldots, x_n]$. The *algebraic set over G defined by S,* or, more simply, the *algebraic set defined by S*, is then, by definition, the set $V(S)$ of all roots of S:

$$V(S) = \{v = (v_1, \ldots, v_n) \mid v_i \in G, f(v) = 1, \ \textit{for all} \ f \in S\}.$$

We term the group G *equationally noetherian* if for every choice of the integer n and every subset S of $G[x_1, \ldots, x_n]$

$$V(S) = V(S_0),$$

where S_0 is a finite subset of S. If there is any ambiguity as to which group G is involved in any algebraic set under consideration, then we use the notation $V_G(S)$ to emphasise the fact that the algebraic set is over G. These notions are introduced and studied in Baumslag, Miasnikov and Remeslennikov [1] (see also [2]).

The object of this note is to prove the following two theorems.

Theorem 1 *Let G be a group and suppose that G contains a subgroup H of finite index which is equationally noetherian. Then G is also equationally noetherian.*

Theorem 1 answers a question raised by Roger Bryant [3].

Theorem 2 *Let G be equationally noetherian and let N be a normal subgroup of G which is a finite union of algebraic sets over G. Then G/N is also equationally noetherian.*

Every singleton of a group G is an algebraic set. So it follows, in particular, that the quotient of an equationally noetherian group by a finite normal subgroup is again equationally noetherian. In addition, it follows also that the quotient of a noetherian group by a normal subgroup which is an algebraic set, is equationally noetherian. Now the center $\zeta(G)$ group G is the set of roots of the system of equations

$$S = \{[g, x] \mid g \in G\}.$$

Consequently $\zeta(G)$ is an algebraic set. So, by Theorem 2, we find

Corollary 1 *If G is equationally noetherian, then so too is $G/\zeta(G)$.*

The converse of Corollary 1 is false. We shall give an appropriate example in section 4. In fact groups which are not equationally noetherian are not hard to come by. A whole family of them is provided by the

Proposition 1 *Let U be any non-abelian group and let T be infinite. Then the wreath product $W = U \wr T$ is not equationally noetherian.*

Some additional examples are described in section 4, one of which settles another question of Roger Bryant [3].

2 The proof of Theorem 1

We recall first from [1] that every subgroup of an equationally noetherian group is equationally noetherian. So, replacing H by the intersection of its finitely many conjugates in G, we can assume that H is normal. Now G embeds in the wreath product $W = H \wr T$, where $T = G/H$ (see, e.g., [4]). W is the semidirect product of T and the direct product of $\mid T \mid$ copies of H. Now the direct product of a finite number of equationally noetherian groups is equationally noetherian [1] and, as noted above, subgroups of equationally noetherian groups are equationally noetherian. Thus it suffices to prove that a group which splits over an equationally normal subgroup of finite index is again equationally noetherian. So we can assume that G is of the form

$$G = TH \ (T \ finite, H \ normal \ in \ G, T \cap H = 1).$$

Notice then that every element $g \in G$ can be uniquely written in the form $g = ta$ $(t \in T, a \in H)$. Suppose that $v \in G^n$. Then v can be expressed as

$$v = (s_1 a_1, \ldots, s_n a_n) \ (s_i \in T, \ a_i \in H) \tag{3}$$

Consider now a word $f \in G[x_1, \ldots, x_n]$ (see (1)). If $g_i = r_i b_i$ $(r_i \in T, b_i \in H \ (i = 1, \ldots, m))$, then

$$f(v) = f(s_1 a_1, \ldots, s_n a_n, r_1 b_1, \ldots, r_m b_m)$$

We re-express $f(v)$ in the form ta by successively moving the various occurrences of the elements of T across to the left of the word, starting with those elements which occur furthest to the left. If we keep track of the words that result from this process, we see first that

$$t = \bar{f}(s_1, \ldots, s_n),$$

where $\bar{f}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, r_1, \ldots, r_m)$ is obtained from $f(x_1, \ldots, x_n)$ by replacing each of the constants $g_i = r_i b_i$ in f by the constants r_i. In particular, $\bar{f}(x_1, \ldots, x_n) \in T[x_1, \ldots, x_n]$. We need to elaborate on the form that we concoct for a. Suppose, first of all, that $T = \{t_1 = 1, \ldots, t_\ell\}$. We then put

$$a_i^j = t_j^{-1} a_i t_j \ (i = 1, \ldots, n, j = 1, \ldots, \ell)$$

and introduce $n\ell$ variables y_i^j $(i = 1, \ldots, n, j = 1, \ldots, \ell)$ in a one-to-one correspondence with the set of a_i^j. Consider now what happens when we start moving the elements of T across to the left hand side of

$$f(s_1 a_1, \ldots, s_n a_n, r_1 b_1, \ldots, r_m a_m).$$

Every time we move an element of $t \in T$ past a b_i we replace it by $t^{-1} b_i t \in H$. The first time we move an element $t_j \in T$ past an a_i, we replace a_i by a_i^j. If we next have to move t_k past a_i^j, then we replace a_i^j by a_i^p, where $t_p = t_j t_k$. At the end of this process, we replace all occurrences of the a_i which have not been changed, by ${a_i}^1$.

The net result of this discussion is that we have defined a word, depending on f and v

$$f'_v \in H[y_1^1, \ldots, y_1^\ell, \ldots, y_n^1, \ldots, y_n^\ell],$$

which has the form

$$f'_v = f'(y_1^1, \ldots, y_1^\ell, \ldots, y_n^1, \ldots, y_n^\ell, c_1, \ldots, c_{m\ell}) \ (c_j \in \ H)$$

such that

$$a = f'_v(a_1^1, \ldots, a_1^\ell, \ldots, a_n^1, \ldots, a_n^\ell, c_1, \ldots, c_{m\ell})$$

(Notice that we have taken advantage of the fact that the conjugates c_k of the b_i that arise from the right-to-left collecting process are contained in the set of all of the conjugates of the elements $b_1, \ldots, b_m$ by *all* of the elements of T).

We have also defined an $n\ell$-tuple

$$v' = (a_1^1, \ldots, a_1^\ell, \ldots, a_n^1, \ldots, a_n^\ell) \in H^{n\ell},$$

which depends only on v. If we think of $v \in G^n$ as given and the elements of $G[x_1, \ldots, x_n]$ as variables, then v gives rise to the function

$$\frac{d}{dv} : \ f \mapsto f'_v \ (f \in G[x_, \ldots, x_n]).$$

Observe that $\frac{d}{dv}$ maps $G[x_1, \ldots, x_n]$ into $H[y_1^1, \ldots, y_1^\ell, \ldots, y_n^1, \ldots, y_n^\ell]$.
For $v = (s_1a_1, \ldots, s_na_n)$, where $s_i \in T, a_i \in H$, we define

$$\lambda(v) = (s_1, \ldots, s_n).$$

Now we can summarise some of the discussion above as

Lemma 1

$$f(v) = \bar{f}(s_1, \ldots, s_n) f'_v(a_1^1, \ldots, a_1^\ell, \ldots, a_n^1, \ldots, a_n^\ell, c_1, \ldots, c_{m\ell})$$

i.e.,

$$f(v) = f(\lambda(v)) f'_v(v').$$

If we review the way in which f'_v is defined, then we find that

Lemma 2 *The word f'_v depends only on $\lambda(v)$.*

We will adopt the notation

$$f'_v = \frac{df}{dv} = \frac{df}{d\lambda(v)}$$

which is unambiguous, in view of Lemma 2. It follows then from Lemma 1 that

Lemma 3 *Let $f \in G[x_1, \ldots, x_n]$ and let $v \in G^n$. Then v is a root of f if and only if $\lambda(v)$ is a root of $\bar{f}$ and v' is a root of $\frac{df}{d\lambda(v)}$.*

We come now to the proof of Theorem 1. To this end, suppose that S is a subset of $G[x_1, \ldots, x_n]$ and $V(S)$ is the variety defined by S.

According to Lemma 3 for any $v \in G^n$ we have

$$v \in V_G(S) \Longleftrightarrow \lambda(v) \in V_T(\bar{S}) \;\;\&\;\; v' \in V_H(\frac{dS}{d\lambda(v)}), \tag{4}$$

where here

$$\bar{S} = \{\bar{f} \mid f \in S\} \subseteq T[x_1, \ldots, x_n]\},$$
$$\frac{dS}{d\lambda(v)} = \{\frac{df}{d\lambda(v)} \mid f \in S\} \subseteq H[x_1, \ldots, x_{n\ell}]\}.$$

A finite group T is equationally noetherian [1]; hence $V_T(\bar{S}) = V_T(\bar{S_0})$ for some finite subset $S_0 \subseteq S$. The group H is also equationally noetherian, so for every $\lambda \in V_T(\bar{S})$ there exists a finite subset $R_\lambda \subseteq S$ such that

$$V_H(\frac{dR_\lambda}{d\lambda}) = V_H(\frac{dS}{d\lambda}). \tag{5}$$

Put

$$S_1 = \bigcup\{R_\lambda \mid \lambda \in V_T(\bar{S})\}.$$

S_1 is a finite subset of S because $V_T(\bar{S})$ is finite. We claim that $V_G(S) = V_G(S_0 \cup S_1)$. Indeed, if $v \in V_G(S_0 \cup S_1)$, then by Lemma 3

$$\lambda(v) \in V_T(\bar{S_0}) = V_T(\bar{S}), \tag{6}$$

and hence $v \in V_G(R_{\lambda(v)})$ by the choice of S_1. It follows, again by Lemma 3, that

$$v' \in V_H(\frac{dR_{\lambda(v)}}{d\lambda(v)}) = V_H(\frac{dS}{d\lambda(v)}). \tag{7}$$

Now, from (6), (7) and (5) we see that $v \in V_G(S)$. The reverse inclusion $V_G(S_0 \cup S_1) \subseteq V_G(S)$ is obvious. This completes the proof of Theorem 1.

3 The proof of Theorem 2

Our objective now is to prove that if G is an equationally noetherian group and if N is a normal subgroup of G that is a finite union of algebraic sets in G, then $H = G/N$ is equationally noetherian. Suppose that H is not equationally noetherian. Then there exists an integer n and a subset S of $H[x_1, \ldots, x_n]$ such that

$$V_H(S) \neq V_H(S_0)$$

for every finite subset S_0 of S. Let f_0 be any arbitrarily chosen element of S. Then

$$V_H(S) \neq V_H(f_0).$$

So there exists $v_1 \in V_H(f_0)$ and an element $f_1 \in S$ such that $f_1(v_1) \neq 1$. Similarly,

$$V_H(S) \neq V_H(f_0, f_1).$$

Hence there exists $v_2 \in V_H(f_0, f_1)$ and an element $f_2 \in S$, such that $f_2(v_2) \neq 1$. In this way we concoct an infinite sequence $f_0, f_1, f_2, \ldots$ of elements of S and an infinite sequence $v_1, v_2, \ldots$ of elements of H^n, such that

$$v_i \in V_H(f_1, \ldots, f_{i-1}) \textit{ and } f_i(v_i) \neq 1 \textit{ in } H. \tag{8}$$

Notice that, in particular, $f_0(v_i) = 1$ for all i. Now let ϕ be the homomorphism of $G[x_1, \ldots, x_n]$ onto $H[x_1, \ldots, x_n]$ which maps G canonically onto H and each x_i to itself and let θ be the map of G^n onto H^n which extends the canonical homomorphism of G onto H. Choose, for each m, a pre-image $\bar{f}_m$ of f_m under ϕ and a pre-image $\bar{v}_m$ in G^m of v_m.

Now N is a finite union of algebraic sets, say

$$N = V_G(S_1) \cup \ldots \cup V_G(S_p).$$

The group G is noetherian, therefore we can assume that all the sets $S_i \subseteq G[x]$ are finite. Note that words in S_i have at most one variable.

Consider now the set $\bar{f}_0(\bar{v}_1), \bar{f}_0(\bar{v}_2), \ldots, \bar{f}_0(\bar{v}_m), \ldots$ of values under $\bar{f}_0$ of the elements $\bar{v}_m$. Then all of these elements lie in N. Hence we can find an infinite subsequence $i_1(0) < i_2(0) < \ldots$ of the sequence $1, 2, \ldots$ such that the elements $\bar{f}_0(\bar{v}_{i_m(o)})$ all lie in the same algebraic set, say $V_G(S_{p_0})$ (here $p_0 \in \{1, \ldots, p\}$). Thus the values of $\bar{f}_0$ on these elements are roots of the finite set S_{p_0}. Therefore, for $m = 1, 2, \ldots$

$$\bar{v}_{i_m(0)} \in V_G(S_{p_0}(\bar{f}_0))$$

where $S_{p_0}(\bar{f}_0)$ is the finite subset of $G[x_1, \ldots, x_n]$ defined as follows

$$S_{p_0}(\bar{f}_0) = \{s(\bar{f}_0) \mid s \in S_{p_0}\}.$$

Consider now $\bar{f}_1$. Since f_1 takes on the value 1 at each of $v_{i_m(0)} > 1$, all of the images of this elements $\bar{v}_{i_m(0)}$ under $\bar{f}_1$ lie in N. Therefore there is an infinite subsequence $i_1(1) < \ldots < i_m(1) \ldots$ of $\{i_m(0)\}$ such that $\bar{f}_1$ takes on values from one and the same algebraic set, say $V_G(S_{p_1})$, at each element $\bar{v}_{i_m(1)}$. Again, if we define

$$S_{p_1}(\bar{f}_1) = \{s(\bar{f}_1) \mid s \in S_{p_1}\},$$

then for any m

$$\bar{v}_{i_m(1)} \in V_G(S_{p_1}(\bar{f}_1)).$$

Continuing in this way we arrive, at the $k-th$ stage at an infinite subsequence $i_1(k) < i_2(k) < \ldots$ of the previously defined sequence $\{i_m(k-1)\}$ such that $\bar{f}_k$ takes on values from an algebraic set $V_G(S_{p_k})$, for some fixed $p_k \in \{1, \ldots, p\}$, at every $\bar{v}_{i_m(k)}$. We then define

$$S_{p_k}(\bar{f}_k) = \{s(\bar{f}_k) \mid s \in S_{p_k}\}.$$

and observe that for any m

$$\bar{v}_{i_m(k)} \in V_G(S_{p_k}(\bar{f}_k)).$$

Consider now the set

$$T = \bigcup\{S_{p_k}(\bar{f}_k) \mid \ k = 0, 1, \ldots\}$$

of elements of $G[x_1, \ldots, x_n]$. By our construction for any $v \in G^n$

$$v \in V_G(T) \Longrightarrow f_k(v) = 1 \ in \ H \quad (k = 0, 1, 2 \ldots). \tag{9}$$

Since G is equationally noetherian, $V_G(T) = V_G(T_0)$, where T_0 is a finite subset of T. We can assume that

$$T_0 = S_{p_0}(\bar{f}_0) \cup \ldots \cup S_{p_\ell}(\bar{f}_\ell).$$

for some finite ℓ. Notice that

$$\bar{v}_{i_m(k)} \in V_G(S_{p_0}(\bar{f}_0) \cup \ldots \cup S_{p_\ell}(\bar{f}_\ell)).$$

whenever $k \geq \ell$. But then for $\mu_0 = i_1(\ell)$ we have $\bar{v}_{\mu_0} \in V_G(T_0)$. This implies that $\bar{v}_{\mu_0} \in V_G(T)$. Therefore by (9)

$$f_{\mu_0}(v_{\mu_0}) = 1,$$

which contradicts the choice of the element v_{μ_0} (see (8)). Thus we have arrived at a contradiction which completes the proof of the theorem.

4 Some examples.

Our first objective now is to prove Proposition 1, which was formulated in section 1. Suppose then that $W = U \wr T$ is the wreath product of a non-abelian group U by an infinite group T. Let B be the normal closure of U in W and let S be the following subset of $W[x]$

$$S = \{[a, b^x] \mid a, b \in B\}.$$

Since U is non-abelian and T is infinite, $V(S)$ is empty. To see this, observe that if x is a root of S, then $x = tc$, where $t \in T$ and $c \in B$. Now choose u and v to be two elements of U which do not commute. Then

$$[u, (tc)^{-1}tcv(tc)^{-1}tc] \neq 1$$

contradicting the fact that $x = tc$ is a root of S. On the other hand if S_0 is a finite subset of S, then there are only finitely many elements of B and T that arise in S_0. Since T is infinite and all of the elements a and b are contained in B, there exists an element $t \in T$ such that the supports of all of the conjugates b^t of all of the elements b that occur in these finitely many equations, are disjoint from the supports of each a, which implies that every such a commutes with every b^t. This completes the proof of Proposition 1.

In particular, we have

Example 1 *The wreath product of the symmetric group of degree three S_3 by an infinite cyclic group C is not noetherian.*

Since any direct power of a finite group is equationally noetherian [1], it follows that even a split extension $S_3 \wr C$ of an equationally noetherian group $\oplus_{i=1}^{\infty} S_3$ by an equationally noetherian group, indeed an infinite cyclic group C, is not always equationally noetherian, in contrast with Theorem 1.

Now Roger Bryant [3] has proved that a finitely generated abelian-by-nilpotent group is equationally noetherian. Notice that, by Proposition 1, the wreath product of the quaternion group of order 8 by an infinite cyclic group is not noetherian, i.e., there exist finitely generated nilpotent-by-abelian groups which are not equationally noetherian. This answers another one of Bryant's questions.

Now we will construct an example of a group G, with the center $\zeta(G)$ of order p, such that G is not equationally noetherian but $G/\zeta(G)$ is equationally noetherian.

Example 2 *Let p be any given prime and let G be the group presented as follows:*

$$G =< t, a, b, c; [t^{-m}at^m, t^{-m}bt^m] = c, [t, c] = [a, c] = [b, c] = 1, c^p = 1$$

$$[t^{-m}at^m, t^{-n}at^n] = [t^{-m}bt^m, t^{-n}bt^n] = 1 \ (\textit{all integer } m, n), \ [t^{-m}at^m, t^{-n}bt^n] = 1 \ (m \neq n) > .$$

Here $C = gp(c)$ is the center of G and it has order p. Notice that $G/C \simeq Z^2 \wr Z$. Consequently, G/C is equationally noetherian because it is linear [1]. To see that G is not equationally noetherian, consider the subset

$$S = \{[x, y^{z^n}] \mid \ (n = 0, 1, \ldots)\}$$

of $G[x, y, z]$. Suppose that

$$V(S) = V(S_0)$$

where S_0 is a finite subset of S. Then we can assume without any loss of generality, that

$$S_0 = \{[x, y^{z^n}] \mid \ (n = 0, 1, \ldots, k)\}.$$

But $x = a$, $y = t^{-k-1}bt^{k+1}$ and $z = t$ is a root of S_0 but not a root of S. So G is not equationally noetherian, as claimed.

In conclusion we will construct an example of a (restricted) direct product P of finite (hence equationally noetherian) groups which is not equationally noetherian. We prepare the way by first observing that the following lemma holds.

Lemma 4 *Let*

$$S = \{[x, y^{z^n}] \mid \ (n = 0, 1, \ldots)\}$$

and let $G = S_3 \wr C_n$, where C_n is a cyclic group of order n and let $H = S_3 \wr C$. Then

1. *the algebraic set $V_G(S)$ cannot be defined by a subset of S with fewer than n elements;*

2. *the variety $V_H(S)$ cannot be defined by any finite subset of S.*

The proof of Lemma 4 is analogous to the proof in Example 2 and will therefore be omitted.

Finally we have the following

Example 3 *Let P be the restricted direct product of the groups $S_3 \wr C_n$,*

$$P = \prod_{n=1}^{\infty} S_3 \wr C_n,$$

and let

$$S = \{[x, y^{z^n}] \mid \ (n = 0, 1, \ldots)\}.$$

Then the algebraic set $V_P(S)$ can not be defined by any finite subset of S and so, in particular P is not equationally noetherian.

It suffices here to note, that if $V_P(S) = V_P(S_0)$ for some finite $S_0 \subseteq S$, then $V_{G_n}(S) = V_{G_n}(S_0)$ for any direct factor $G_n = S_3 \wr C_n$ of P. This, however, is impossible by Lemma 4.

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