Algebraic geometry over groups

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Abstract

Classical commutative algebra provides the underpinnings of classical algebraic geometry. In this paper we will describe, without any proofs, a theory for groups which parallels this commutative algebra and which, in like fashion, is the basis of what we term algebraic geometry over groups.

1 Introduction

For some years now we have attempted to lay the foundations of a theory, which we term algebraic geometry over groups, which bears a surprising similarity to elementary algebraic geometry - hence its name. Here we survey some of the ideas that we will describe in detail elsewhere, focussing our attention on the ideal theory of finitely generated free G–groups (see [BMR2]). In particular we will discuss here group-theoretic counterparts to algebraic sets, coordinate algebras, the Zariski topology and various other notions such as zero-divisors, prime ideals, the Lasker-Noether decomposition of ideals as intersections of prime ideals, the Noetherian condition, irreducibility and the Nullstellensatz. A number of interesting concepts arise, focussing attention on some fascinating new aspects of infinite groups. The impetus for much of this work comes mainly from the study of equations over groups.

2 G-groups

The central notion in this ideal theory is that of a G–group, where G is a group that is fixed throughout. A group H is termed a G–group if it contains a designated copy of G, which we will for the most part identify with G. Notice that we allow for the possibility that $G = 1$ and also that $G = H$; in particular G is itself a G-group. Such G–groups form a category in the obvious way. A morphism from one G–group H to a G–group H', here termed a G-homomorphism, is a homomorphism from H to H' which is the identity on G. The kernels of these morphims are termed *ideals*; they are simply the normal subgroups which meet G in the identity. As noted above, G is itself a G–group and a G-homomorphism from the G–group H to the G–group G can be viewed as a retraction of H onto its subgroup G, i.e., a homomorphism from H into itself which is the identity on G. All of the everyday notions of group theory can be carried over, with a little care, to the category of G-groups, including finitely generated and finitely presented G–groups.

3 Morphisms and homological algebra

Let H be a G-group. Then we denote the group of all G-automorphisms of H by $Aut_G(H)$. In general there is no obvious way to view $Aut_G(H)$ as a G–group unless G is abelian and H has trivial center. At the opposite extreme if G itself has trivial center, the center of the G–group H is an ideal of H as are all the terms of the upper central series of H. The theory of nilpotent groups can then be recast in the category of G-groups, where a G-group H is G-nilpotent if its upper central series terminates after finitely many steps with an ideal I with quotient $H/I \cong G$, where $\cong$ is understood as a G–isomorphism. We will not dwell any more on this side of the theory at this point, leaving it to the reader to think further about the matter. However, before turning our attention to a different aspect of the theory, we would like to indicate how one can introduce some homological notions into the study of G–groups. With this in mind, suppose that A is a given abelian group such that the automorphism group $Aut(A)$ of A contains a designated copy of G, i.e., it comes provided with the structure of a G–group. Suppose furthermore, that H is a G–group. Then we term A a G H–module if H comes equipped with a G–homomorphism into the G-group $Aut(A)$. Armed with this definition one can then develop the homological algebra of G–groups with the corresponding homology and cohomology groups. This is a fascinating aspect of the general theory which we have still to explore. It touches on extension theory and many other related topics. Finally we remark that the theory of groups acting on trees also can be reworked from this point of view, with applications to the struture theory of free G–groups and related groups. Whether the notion of a G–group can help to provide a reasonable description of the subgroups of finitely presented groups is another topic worth considering further (cf., e.g., [HG]).

4 Products

In dealing with various products, it is sometimes useful to let the *coefficient group* G vary. In particular, if H_i is a G_i-group for each i in some index set I, then the unrestricted direct product $\overline{\prod}_{i \in I} H_i$ can be viewed as a $\overline{\prod}_{i \in I} G_i$-group, in the obvious way. If H_i is a G-group for each i, then we will sometimes think of the unrestricted direct product P of the groups H_i as a G-group by taking the designated copy of G in P to be the diagonal subgroup of the unrestricted direct product of all of the copies of G in the various factors. In the case of, say, the standard wreath product, if U is a G-group and if T is a G'-group, then their (standard) wreath product $A \wr T$ can be viewed as a $G \wr G'$-group in the obvious way again.

5 Domains

We need to introduce another central notion in this theory, that of a zero–divisor and thence that of an integral domain.

Let H be a G-group. Then we term a non-trivial element $x \in H$ *a G-zero divisor* if there exists a non-trivial element $y \in H$ such that

$$[x, g^{-1}yg] = 1 \ for\ all\ g \in G.$$

Notice that if $G = 1$ then every non-trivial element of the G-group H is a G-zero divisor. We term a G-group H a *G-domain* if it does not contain any G-zero divisors; in the event that $G = H$ we simply say that H is a domain.

We recall here that a subgroup M of a group H is *malnormal* if whenever $h \in H,\ h \notin M$, then $h^{-1}Mh \cap M = 1$. A group H is termed a *CSA-group* if every maximal abelian subgroup M of H is malnormal. If H is such a CSA-group and G is a non-abelian subgroup of H, then H, viewed as a G-group, is a G-domain. Notice that every torsion-free hyperbolic group is a CSA-group. This demonstrates, together with the Theorems A1, A2 and A3 below, that there is a plentiful supply of G-domains.

Theorem A1 *If U is a G-domain and if T is a G'-domain, then the wreath product $U \wr T$ is a $G \wr G'$-domain.*

Further domains can be constructed using amalgamated products.

Theorem A2 *Let A and B be domains. Suppose that C is a subgroup of both A and B satisfying the following condition:*

$(*)$ $\qquad$ *if* $c \in C, c \neq 1,$ *then either* $[c, A] \nsubseteq C$ *or* $[c, B] \nsubseteq C.$

Then the amalgamated free product $H = A *_C B$ is a domain.

Theorem A3 *The free product, in the category of G-groups, of two G-domains is a G-domain whenever G is a malnormal subgroup of each of the factors.*

6 Free G-groups

It is not hard to identify the finitely generated free G–groups. They take the form

$$G[X] = G[x_1, \ldots, x_n] = G* < x_1, \ldots, x_n >,$$

the free product of G and the free group $F(X)$ freely generated by $X = \{x_1, ..., x_n\}$. These G-groups can be likened to algebras over a unitary commutative ring, more specially a field, with G playing the role of the coefficient ring. We view $G[X]$ as a non–commutative analogue of a polynomial algebra over a unitary commutative ring in finitely many commuting variables, and the elements $f \in G[X]$ as polynomials in the non-commuting *variables* $x_1, \ldots, x_n$, with *coefficients* in G. We use functional notation here,

$$f = f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, g_1, \ldots, g_m) \tag{1}$$

thereby expressing the fact that the word representing f in $G[X]$ involves the *variables* $x_1, \ldots, x_n$ and, as needed, the *constants* $g_1, \ldots, g_m \in G$.

7 G-equationally Noetherian groups

Let H be a G–group and let $f \in G[X]$. We term

$$v = (a_1, \ldots, a_n) \ \in H^n \tag{2}$$

a *root* of f if

$$f(v) = f(a_1, \ldots, a_n, g_1, \ldots, g_m) = 1.$$

We sometimes say that f *vanishes at* v. If S is a subset of $G[X]$ then v is said to be a *root of* S if it is a root of every $f \in S$, i.e., S *vanishes at* v. In this event we also say that v is an *H-point* of S. We denote the set of all roots of S by $V_H(S)$. So

$$V_H(S) = \{v \in H^n \mid f(v) = 1 \ for\ all\ f \in S\}.$$

Then a G-group H is called G-equationally Noetherian if for every $n > 0$ and every subset S of $G[x_1, \ldots, x_n]$ there exists a finite subset S_0 of S such that

$$V_H(S) = V_H(S_0).$$

In the event that $G = H$ we simply say that G is equationally Noetherian, instead of G is G-equationally Noetherian. These G-equationally Noetherian groups play an important part in the theory that we are developing.

The class of all G-equationally Noetherian groups is fairly extensive. This follows from the two theorems below.

Theorem B1 *Let a G-group H be linear over a commutative, Noetherian, unitary ring, e.g., a field. Then H is G-equationally Noetherian.*

A special case of this theorem was first proved by Roger Bryant [BR] in 1977 and another special case, that of free groups, by Victor Guba [GV] in 1986. It follows, in particular, that G-groups which are either polycyclic (see [AL]) or finitely generated and metabelian (see [RV1]) or free nilpotent or free metabelian (seeBW), are G-equationally Noetherian. Not all G-equationally Noetherian G-groups are linear.

Theorem B2 *Let $\mathcal{E}_G$ be the class of all G-equationally Noetherian groups. Then the following hold:*

1. *$\mathcal{E}_G$ is closed under G-subgroups, finite direct products and ultrapowers;*

2. *$\mathcal{E}_G$ is closed under G-universal (G-existential) equivalence, i.e., if $H \in \mathcal{E}_G$ and H' is G-universally equivalent (G-existentially equivalent) to H, then $H' \in \mathcal{E}_G$;*

3. *$\mathcal{E}_G$ is closed under G-separation, i.e., if $H \in \mathcal{E}_G$ and H' is G-separated by H, then $H' \in \mathcal{E}_G$.*

(We briefly defer the definition of G–separation until **8**.) Here two groups are said to be G-universally equivalent if they satisfy the same G-universal sentences. These are formulas of the type

$$\forall x_1 \ldots \forall x_n (\bigvee_{j=1}^{s} \bigwedge_{i=1}^{t} (u_{ji}(\bar{x}, \bar{g}_{ij}) = 1 \ \& \ w_{ij}(\bar{x}, \bar{f}_{ij}) \neq 1))$$

where $\bar{x} = (x_1, \ldots, x_n)$ is an n-tuple of variables, $\bar{g}_{ij}$ and $\bar{f}_{ij}$ are arbitrary tuples of elements (constants) from G.

It is not hard to construct examples of G-groups that are not G-equationally Noetherian (see [BMRO]). We note also here two theorems proved in [BMRO], namely that if a G–group H contains a G–equationally Noetherian G–subgroup of finite index, then H itself is G–equationally Noetherian; and if Q is a finite ideal of a G–equationally Noetherian G–group H, then H/Q is G–equationally Noetherian. Finally, we remark, taking for granted here the notions to be introduced in **13**, that if G is an equationally Noetherian torsion-free hyperbolic group and if A is an unitary associative ring with the appropriate properties, then the completion G^A is G-equationally Noetherian.

8 Separation and discrimination

We concern ourselves next with specific approximation techniques in groups and rings. Two notions play an important part here, namely that of separation and discrimination.

Let H be a G-group. Then we say that a family

$$\mathcal{D} = \{D_i \ \mid \ i \in I\}$$

of G-groups *G-separates* the G-group H, if for each non-trivial $h \in H$ there exists a group $D_i \in \mathcal{D}$ and a G-homomorphism $\phi : H \longrightarrow D_i$ such that $\phi(h) \neq 1$.

Similarly, we say that $\mathcal{D}$ *G-discriminates* H, if for each finite subset $\{h_1, \ldots, h_n\}$ of non-trivial elements of H there exists a $D_i \in \mathcal{D}$ and a G-homomorphism $\phi : H \longrightarrow D_i$ such that $\phi(h_j) \neq 1, \ j = 1, \ldots, n$.

If $\mathcal{D}$ consists of the singleton D, then we say that D *separates* H in the first instance and that D *discriminates* H in the second. If G is the trivial group, then the notions of separation and discrimination are often expressed in the group-theoretical literature by saying, respectively, that H is *residually* $\mathcal{D}$ and that H is *fully residually* $\mathcal{D}$ or H is ω*-residually* $\mathcal{D}$.

The following two theorems are often useful.

Theorem C1 [BMR2] *Let G be a domain. Then a G-group H is G-discriminated by G if and only if H is a G-domain and H is G-separated by G.*

Benjamin Baumslag introduced and exploited this idea in the case of free groups [BB].

We say that the G-group H is *locally G-discriminated* by the G-group H' if every finitely generated G-subgroup of H is G-discriminated by H'.

Theorem C2 *Let H and H' be G-groups and suppose that at least one of them is G-equationally Noetherian. Then H is G-universally equivalent to H' if and only if H is locally G-discriminated by H' and H' is locally G-discriminated by H.*

The idea to tie discrimination to universal equivalence is due to V. Remeslennikov [RV1], who formulated and proved a version of Theorem C2 in the case of free groups.

9 Ideals

As usual, the notion of a domain leads one to the notion of a prime ideal. An ideal P of the G-group H is said to be a *prime ideal* if H/P is a G-domain. Prime ideals are especially useful in describing the ideal structure of an arbitrary G-equationally Noetherian G-domain H.

An ideal Q of the G-group H is termed *irreducible* if $Q = Q_1 \cap Q_2$ implies that either $Q = Q_1$ or $Q = Q_2$, for any choice of the ideals Q_1 and Q_2 of H. Irreducibility is important in dealing with ideals of a free G-group $G[X]$. In the theory that we are developing here, we define, by analogy with the classical case, the *Jacobson G-radical* $J_G(H)$ of the G-group H to be the intersection of all maximal ideals of H with quotient G-isomorphic to G; if no such ideals exist, we define $J_G(H) = H$. Similarly, we define the *G-radical* $Rad_G(Q)$ of an ideal Q of a G-group H to be the pre-image in H of the Jacobson G-radical of H/Q, i.e., the intersection of all the maximal ideals of H containing Q with quotient G-isomorphic to G.

More generally, if K is any G-group, then we define the *Jacobson K-radical* $J_K(H)$ of the G-group H to be the intersection of all ideals of H with quotient G-embeddable into K; similarly we define the K-radical $Rad_K(Q)$ of an ideal Q of H to be the pre-image in H of the Jacobson K-radical of H/Q. Finally, an ideal of a G-group is said to be a *K-radical ideal* if it coincides with its K-radical.

A finitely generated G-group H is called a *K-affine* G-group if $J_K(H) = 1$. The K-affine groups will play an important role in the abstract characterization of coordinate groups defined over K.

10 The affine geometry of G-groups

Let

$$H^n = \{(a_1, \ldots, a_n) \mid \ a_i \in H\}$$

be *affine n-space over the G-group H* and let S be a subset of $G[X]$. Then we define, as in **7**,

$$V_H(S) = \{v \in H^n \mid f(v) = 1, \ for \ all \ f \in S\}$$

and term it the *(affine) algebraic set* over H defined by S.

We sometimes denote $V_H(\{s_1, s_2, \ldots\})$ by $V_H(s_1, s_2, \ldots)$.

The union of two algebraic sets in H^n is not necessarily an algebraic set. We define a topology on H^n by taking as a sub-basis for the closed sets of this topology, the algebraic sets in H^n. We term this topology the *Zariski topology*. If H is a G-domain, then the union of two algebraic sets is again algebraic and so in this case the closed sets in the Zariski topology consist entirely of algebraic sets.

Given a G-group H, the algebraic sets over H can be viewed as the objects of a category, where morphisms are defined by polynomial maps, i.e., if $Y \subseteq H^n$, and $Z \subseteq H^p$ are algebraic sets then a map $\phi : Y \to Z$ is a *morphism* in this category (or a *polynomial map*) if there exist $f_1, \ldots, f_p \in G[x_1, \ldots, x_n]$ such that for any $(a_1, \ldots, a_n) \in Y$

$$\phi(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_p(a_1, \ldots, a_n)).$$

It turns out that this category is isomorphic to a sub-category of the category of all G-groups. In order to explain more precisely what this sub-category consists of we need to turn our attention to the *ideals* of algebraic sets.

11 Ideals of algebraic sets

Let H be a G-group, let n be a positive integer, let H^n affine n-space over H, $G[X] = G[x_1, \ldots, x_n]$ and let $Y \subseteq H^n$. Then we define

$$I_H(Y) = \{f \in G[X] \mid f(v) = 1 \ for \ all \ \ v \in Y\}.$$

Suppose now that S is a non-empty subset of $G[X]$ and that $Y = V(S)$. Every point $y = (y_1, \ldots, y_n) \in H^n$ defines a G-homomorphism ϕ_y of $G[X]$ into H, via evaluation, i.e., by definition, if $f \in G[X]$, then $\phi_y(f) = f(y)$. It follows that

$$I_H(Y) = \bigcap_{y \in Y} ker \ \phi_y.$$

Hence $I_H(Y)$ is an ideal of $G[X]$ provided only that Y is non-empty. If $Y = \emptyset$ and $G \neq 1$, then $I(Y) = G[X]$ is not an ideal. We shall, notwithstanding the inaccuracy, term $I(Y)$ *the ideal of* Y under all circumstances.

In the event that Y is an algebraic set in H^n, then we define the *coordinate group* $\Gamma(Y)$ of Y to be the G-group of all polynomial functions on Y. These are the functions from Y into H which take the form

$$y \mapsto f(y) \ (y \in Y),$$

where f is a fixed element of $G[x_1, \ldots, x_n]$. It is easy to see that

$$\Gamma(Y) \cong G[X]/I(Y).$$

The ideals $I_H(Y)$ completely characterize the algebraic sets Y over H, i.e., for any algebraic sets Y and Y' over H we have:

$$Y = Y' \Longleftrightarrow I_H(Y) = I_H(Y').$$

Similarly, the algebraic sets Y are characterized by their coordinate groups $\Gamma(Y)$:

$$Y \cong Y' \Longleftrightarrow \Gamma(Y) \cong \Gamma(Y');$$

here $\cong$ represents isomorphism in the appropriate category.

Amplifying the remark above, it turns out that if H is a G-group, then the category of all algebraic sets over H is equivalent to the category of all coordinate groups defined over H, which is exactly the category of all finitely generated G-groups that are G-separated by H. The latter result comes from the abstract description of coordinate groups. We need another notion from commutative algebra in order to explain how this comes about.

The ideal Q of $G[X]$ is called H-closed if $Q = I_H(Y)$ for a suitable choice of the subset Y of H^n. It is then not hard to see that the H-closed ideals of $G[X]$ are precisely the H-radical ideals of $G[X]$. Therefore, a finitely generated G-group Γ is a coordinate group of an algebraic set $Y \subseteq H^n$ (for a suitable n) if and only if $J_H(\Gamma) = 1$, which is equivalent to G-separation of Γ in H.

An elaboration of this approach yields some analogues of the Lasker-Noether theorem, which we will describe in **18**.

12 The Zariski topology of equationally Noetherian groups.

In the event that the G-group H is G-equationally Noetherian, it turns out (irrespective of the choice of n) that the Zariski topology satisfies the descending chain condition on closed subsets of H^n, i.e., every properly descending chain of closed subsets of H^n is finite. Indeed, we have the following important theorem.

Theorem D1 *Let H be a G-group. Then for each integer $n > 0$, the Zariski topology on H^n is Noetherian, i.e., satisfies the descending chain condition on closed sets, if and only if H is G-equationally Noetherian.*

This implies, in particular, that every closed subset of H^n is a finite union of algebraic sets. As is the custom in topology, a closed set Y is termed *irreducible* if $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed sets, implies that either $Y = Y_1$ or $Y = Y_2$. So, by the remark above, every closed subset Y of H^n can be expressed as a finite union of irreducible algebraic sets:

$$Y = Y_1 \cup \ldots \cup Y_n.$$

These sets are usually referred to as the *irreducible components* of Y, which turn out to be unique. This shifts the study of algebraic sets to their irreducible components. It turns out that the irreducible ideals are the algebraic counterpart to the irreducible algebraic sets. Indeed, let H be a G-domain; then a closed subset $Y \subseteq H^n$ is irreducible in the Zariski topology on H^n if and only if the ideal $I_H(Y)$ is an irreducible ideal in $G[X]$.

An elaboration of this approach yields an important characterization of irreducible algebraic sets in terms of their coordinate groups.

Theorem D2 *Let H be a G-equationally Noetherian G-domain and let Y be an algebraic set in H^n. Then the following conditions are equivalent:*

1. *Y is irreducible;*
2. *$I_H(Y)$ is a prime ideal in $G[X]$;*
3. *$\Gamma(Y)$ is a G-equationally Noetherian G-domain;*
4. *$\Gamma(Y)$ is G-discriminated by H.*

In the event that $G = H$ we can add one more equivalent condition to Theorem D2, which establishes a surprising relationship between coordinate groups of irreducible algebraic sets over G and finitely generated models of the universal theory of the group G.

Theorem D3 *Let G be an equationally Noetherian domain and let Y be an algebraic set in G^n. Then the following conditions are equivalent:*

1. *Y is irreducible;*
2. *$\Gamma(Y)$ is G-universally equivalent to G.*

Moreover, any finitely generated G-group which is G-universally equivalent to G is the coordinate group of some irreducible algebraic set over G.

13 Exponential groups

As we noted in the discussion above, any G-group which is G-separated by a G-equationally Noetherian group is again G-equationally Noetherian. A rich collection of such groups arises for various choices of G from certain torsion-free abelian groups. In order to explain, let A be an additively written abelian group. We term A *unitary* if it comes equipped with a distinguished non-zero element, which we denote by 1. Such unitary abelian groups can be likened to topological spaces with base points. Distinguishing a non-zero element in an additively written abelian group amounts simply to the specification of one of its cyclic subgroups, with a specific choice of generator. A typical example of a unitary abelian group is the additive group A^+ of a unitary ring A with the ring identity as the distinguished element. Unitary abelian groups form a category where morphisms are the unitary homomorphisms (i.e., homomorphisms which map 1 to 1) and the subobjects are unitary subgroups (subgroups containing the distinguished element 1). A group H is termed an A-group if it comes equipped with a function $H \times A \to H$:

$$(h, \alpha) \mapsto h^\alpha$$

satisfying the following conditions for arbitrary $g, h \in H$ and $\alpha, \beta \in A$:

$$h^1 = h, \quad h^{\alpha+\beta} = h^\alpha h^\beta, \; g^{-1}h^\alpha g = (g^{-1}hg)^\alpha,$$

and if g and h commute,

$$(gh)^\alpha = g^\alpha h^\alpha.$$

In the event that A is a unitary ring we assume also that

$$h^{\alpha\beta} = (h^\alpha)^\beta$$

We shall simply refer to such A-groups, in all of their incarnations, as *exponential* groups. The most important of these exponential groups for our purposes here are those where A is the ring of polynomials $Z[x]$ of integral polynomials in the variable x, introduced by R.C. Lyndon in [LRC1], [LRC2], in a successful attempt to parametrise the set of all solutions of equations in a single variable over a free group. It turns out, in particular, that if A satisfies some separation conditions and if, for example, G is a torsion-free hyperbolic group, then G^A is G–separated by G [BMR2]. This holds true also in the special case where $A = Z[x]$ and $G = F$, a free group. Since a free group F is F-equationally Noetherian, it follows that $F^{Z[x]}$ is F-equationally Noetherian.

$F^{Z[x]}$ can be constructed from F by means of iterated amalgamated products, starting with F and using free abelian groups repeatedly as a second factor. Consequently the structure of the finitely generated subgroups of $F^{Z[x]}$ can be extracted by using the Bass-Serre theory of groups acting on trees. However it seems likely that a more precise understanding of these subgroups will have a bearing on a deeper understanding of the elementary theory of non-abelian finitely generated free groups.

The Z[x]-structure of $F^{Z[x]}$ is also interesting in its own right. Whether the Z[x]-subgroups which are closed under centralisers are free as Z[x]-groups is an intriguing question - this is true of the 2-generator Z[x]-subgroups.

14 Dimension

There are two natural notions of dimension of algebraic sets, the usual topological one and another that comes out of the Noether normalisation theorem, involving transcendence degree. The notion of transcendence degree can be carried over to this algebraic geometry over groups. Here we find that there is an analogue for G–groups of the Grushko-Neumann theorem [GIA], [NBH] on the minimal number of generators of a free product as well as a theorem of Wagner [WDH], which is concerned with free quotients of free G–groups. Both of them play a role here.

15 Generic points, local rings and completions

Most of the elementary notions associated with algebraic geometry seem to have analogues in algebraic geometry over groups, in particular generic points, local rings and completions. Derivations into group algebras and the analogue of a tangent space are topics that we hope to pursue in due course. We refer the reader to the book by R. Hartshorne [HR] for a general discussion of algebraic geometry and the terms used here.

16 Curves and one-relator groups

Let now J be a group defined by a single defining relation

$$J =< x_1, \ldots , x_n; r = 1 >,$$

where r is cyclically reduced and involves all of the generators listed. Let G be the subgroup generated by any proper subset of the given generators of J, which is free by Magnus' Freiheitssatz (see e.g., [MKS]). Then we can think of J as a one-relator G–group and $Hom_G(J, T)$ can be thought of as a curve in T^n, irrespective of how we choose the G–group T. It is a fascinating idea to explore one-relator groups from this perspective. The geometric properties of these curves and the relevance to the structure of the groups involved are yet another interesting area of study. Whether this point of view will have an impact on the isomorphism problem for one-relator groups, in particular, is still to be determined.

17 Lie and associative rings

The theory that we have developed for groups carries over also to lie and associative rings. The equation theoretic aspects of lie rings are much less developed than the corresponding theory for groups. The nature of lie rings which are discriminated by a free lie ring is unresolved, whether free lie rings are equationally noetherian is undecided and so on. The general theory is itself intriguing and worthy of development. And the focus on tangent spaces alluded to above brings in to play associative rings in the form of the integral group ring of free and related groups.

18 Decomposition theorems

The categorical equivalence, described above, between algebraic sets over the G-group H and the finitely generated G-groups which are G-separated by H, leads to an analogue of a theorem often attributed to Lasker and Noether.

Theorem E1 *Let H be a G-equationally Noetherian G-domain. Then each H-closed ideal in $G[X]$ is the intersection of finitely many prime H-closed ideals, none of which is contained in any one of the others, and this representaion is unique up to order. Conversely distinct irredundant intersections of prime H-closed ideals define distinct H-closed ideals.*

Theorem E1 has a counterpart for ideals of arbitrary finitely generated G-groups.

Theorem E2 *Let H be a finitely generated G-group and let K be a G-equationally Noetherian G-domain. Then each K-radical ideal in H is a finite irredundant intersection of prime K-radical ideals. Moreover, this representaion is unique up to order. Furthermore, distinct irredundant intersections of prime K-radical ideals define distinct K-radical ideals.*

Theorems E1 and E2 lead to several interesting consequences.

Theorem F1 *Let H be a G-equationally Noetherian G-domain. If Y is any algebraic set in H^n, then the coordinate group $\Gamma(Y)$ is a subgroup of a direct product of finitely many G-groups, each of which is G-discriminated by H.*

Theorem F2 *Let H be a non-abelian equationally Noetherian torsion-free hyperbolic group (here $G = 1$). Then every finitely generated group E which is separated by H is a subdirect product of finitely many finitely generated groups, each of which is discriminated by H.*

Theorem F2 plays a very useful part in the proof that finitely generated groups which are discriminated by a free group are finitely related [KM1], [KM2].

19 The Nullstellensatz

We need next a variation of the notion of an algebraically closed group, which is due to W.R. Scott [SWR]. Here a non-trivial G-group H is termed *G-algebraically closed* if every finite set of equations and inequations of the form

$$f = 1 \;\; \textit{and} \; f \neq 1 \qquad (f \in G[x_1, \ldots, x_n])$$

that can be satisfied in some G-group containing H, can also be satisfied in H. This class of G-groups will play a part in the discussion that follows. Notice, in the event that $G = H$ we have the standard notion of algebraically closed group due to W.R. Scott.

Hilbert's classical Nullstellensatz is often formulated for ideals of polynomial algebras over algebraically closed fields. One such formulation asserts that every proper ideal in the polynomial ring $K[x_1, \dots, x_n]$ over an algebraically closed field K, has a root in K. It is easy to prove an analogous result for G-groups (notice that this includes the case where $G = H$, below).

Theorem G1 *Let H be a G-algebraically closed G-group. Then every ideal in $G[X]$, which can be generated as a normal subgroup by finitely many elements, has a root in H.*

Another form of the Nullstellensatz for polynomial rings can be expressed as follows. Suppose that S is a finite set of polynomials in $K[x_1, \dots, x_n]$ and that a polynomial f vanishes at all of the zeroes of S; then some power of f lies in the ideal generated by S. With this in mind, we introduce the following definition.

Definition 1 *Let H be a G-group and let S be a subset of $G[x_1, \dots, x_n]$. Then we say that S satisfies the Nullstellensatz over H if*

$$I(V_H(S)) = gp_{G[X]}(S),$$

where here $gp_{G[X]}(S)$ denotes the normal closure in $G[X]$ of S.

It follows, as in the classical case, that an ideal in $G[X]$ satisfies the Nullstellensatz if and only if it is H-radical. Notice that in the event that $V_H(S)$ is non-empty, then $gp_{G[X]}(S)$ is actually the ideal of the G-group $G[X]$ generated by S, i.e., the smallest ideal of $G[X]$ containing S.

The following version of the Nullstellensatz then holds.

Theorem G2 *Let H be a G-group and suppose that H is G-algebraically closed. Then every finite subset S of $G[x_1, \dots, x_n]$ with $V_H(S) \neq \emptyset$, satisfies the Nullstellensatz; indeed, $I(V_H(S)) = gp_{G[X]}(S)$.*

There is a simple criterion for determining whether a given set satisfies the Nullstellensatz. Indeed, suppose that H is a G-group and that $V_H(S) \neq \emptyset$, where S is a subset of $G[x_1, \dots, x_n]$. Then S satisfies the Nullstellensatz over H if and only if $G[X]/gp_{G[X]}(S)$ is G-separated by H. Notice, that if the group H is torsion-free and a set $S \subseteq G[X]$ satisfies the Nullstellensatz over H, then the ideal $Q = gp_{G[X]}(S)$ is isolated in $G[X]$, i.e., $f^n \in Q$ implies $f \in Q$ (for any $f \in G[X]$).

It is not easy to determine which systems of equations, e.g., over a free group, satisfy the Nullstellensatz.

20 Connections with representation theory

The set $Hom(J, T)$ of all homomorphisms of a finitely generated group J into a group T has long been of interest in group theory. If J is a finite group and T is the group of all invertible $n \times n$ matrices over the field $\mathcal{C}$ of complex numbers, then the study of $Hom(J, T)$ turns into the representation theory of finite groups. If T is an algebraic group over $\mathcal{C}$, then $Hom(J, T)$ is an affine algebraic set, the geometric nature of which lends itself to an application of the Bass-Serre theory of groups acting on trees, with deep implications on the structure of the fundamental groups of three dimensional manifolds.

The algebraic geometry over groups that we develop here can be viewed also as a contribution to the general representation theory of finitely generated groups. In order to explain, we work now in the category of G-groups, where G is a fixed group; notice that in the event that $G = 1$, this is simply the category of all groups.

Now let J be a finitely generated G–group, equipped with a finite generating set $\{x_1, \ldots, x_n\}$, and let T be an arbitrarily chosen G–group. Express J as a G–quotient group of the finitely generated free G-group $F = G[x_1, \ldots, x_n]$:

$$J \cong F/Q.$$

Then we have seen that the set $Hom_G(J, T)$ of G-homomorphims from J into T can be parametrized by the roots of Q in T^n and hence carries with it the Zariski topology. Observe that if the Nullstellensatz applies to Q, then the coordinate group of this space can be identified with J, which explains its importance here. It is not hard to see that $Hom_G(J, T)$ is independent of the choice of generating set, i.e., the algebraic sets obtained are isomorphic in the sense that we have already discussed. So $Hom_G(J, T)$ is a topological invariant of the G-group J, which we refer to as the space of all representations of J in T. The group $Inn_G(T)$ of G-inner automorphisms of T, i.e., those inner automorphisms of T which commute element-wise with G, induce homeomorphisms of $Hom_G(J, T)$ and so we can form the quotient space of Hom(J, T) by $Inn_G(T)$, which we term the space of inequivalent representations of J in T. This is a finer invariant than $Hom(J, T)$, akin to the space of inequivalent representations or characters of representations of a finite group. In the event that J and T are algorithmically tractable and satisfy various finiteness conditions it is interesting to ask whether these spaces can be described in finite terms and, if so, whether they are computable and how they can be used to provide information about J, assuming complete knowledge of T. The obvious questions involving the various algebro-geometric properties of $Hom_G(J, T)$ arise, in particular for finitely generated metabelian groups. Whether this touches on the isomorphism problem for such groups remains to be seen.

21 Some unanswered questions

We record, finally, some unanswered questions, which seem interesting to us.

1. Let the G-group H be hyperbolic. Is H G-equationally Noetherian?
2. If G is G-equationally Noetherian, is $G[x_1, \dots, x_n]$ also G-equationally Noetherian?
3. Are there some natural conditions which ensure that an extension of one equationally Noetherian group by another equationally group is equationally Noetherian (here $G = 1$)?
4. Are parafree groups (i.e., residually nilpotent groups with the same lower central quotients as some fixed free group) equationally noetherian?

Question 3 is an analogue of the corresponding well-known theorem concerned with extensions of algebraic groups. One can formulate, in the context of algebraic geometry over groups, analogous notions including the analogue of a lie group.

22 Related work

We refer the reader to the paper [BMR1] for references to related work and to a full account of the subject matter being reported on here.

References

[AL] Auslander L. *On a problem of Philip Hall.* Annals of Math., **1967** (1986), pp. 112-126.

[BH] Bass H. *Groups acting on non–archimedian trees.* Arboreal group theory, 1991, p. 69–130.

[BB] Baumslag B. *Residually free groups.* Proc.London Math. Soc., 1967, v.17, 3, p.402–418.

[BG] Baumslag G., *On generalized free products.* Math. Zeitschr., 1962, v.7, 8, p. 423–438.

[BMR1] Baumslag, G., Myasnikov A., and Remeslennikov V. N. *Algebraic geometry over groups. I. Ideal theory.* To appear J. algebra.

[BMR2] Baumslag, G., Myasnikov A., and Remeslennikov V. *Discriminating completions of hyperbolic groups.* To be submitted for publication.

[BMRO] Baumslag G., Myasnikov A. and Roman'kov V. *Two theorems about equationally Noetherian groups.* Journal of Algebra **194** (1997) pp. 654–664.

[BR] Bryant R. *The verbal topology of a group.* Journal of Algebra **48** (1977), pp. 340–346.

[GIA] Grushko, I. A. *Ueber die Basen einem freien Produkten von Gruppen*, Mat. Sb. **8** (1940), pp. 169–182.

[GV] Guba V. *Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems.* Mat. Zametki **40** 3 (1986), pp. 321–324.

[HR] Hartshorne R. *Algebraic geometry.* Springer-Verlag, New York (1977).

[HG] Higman G. *Subgroups of finitely presented groups*, Proc. Royal Soc. London Ser. A **262** (1961), pp. 455-475.

[KM1] Kharlampovich O. and Myasnikov A. *Irreducible affine varieties over a free group. I: irreducibility of quadratic equations and Nullstellensatz.* J. of Algebra **200** 2 (1998), pp. 472–516.

[KM2] Kharlampovich O. and Myasnikov A. *Irreducible affine varieties over a free group. II: Systems in triangular quasi-quadratic form and description of residually free groups.* J. of Algebra, 1998,v. 200, n. 2, p. 517-570.

[LRC1] Lyndon R.C. *Equations in groups.* Bol. Soc. Bras. Mat. **11** (1980), pp. 79–102.

[LRC2] Lyndon R.C. *Groups with parametric exponents.* Trans. American Math. Soc., **9** 6 (1960), pp. 518-'-533.

[MKS] Magnus, W., Karrass A. and Solitar D. *Combinatorial group theory: Presentations of groups in terms of generators and relations.* Interscience (John Wiley and Sons), New York, London, Sydney (1966).

[MR1] Myasnikov A.G., Remeslennikov V.N. *Exponential groups I: foundations of the theory and tensor completion.* Siberian Math. J., **5** (1994).

[MR2] Myasnikov A.G., Remeslennikov V.N., *Exponential groups II: extensions of centralizers and tensor completion of CSA-groups,* International Journal of Algebra and Computation **6**(6) (1996), pp. 687–711.

[NBH] Neumann, B. H. *On the number of generators of a free product.* J. London Math. Soc. **18** (1943), pp. 12–20.

[RV1] Remeslennikov V.N. *E–free groups.* Siberian Math. J. **30** 6 (1989), pp. 153–157.

[RV2] Remeslennikov V.N. *Matrix representation of finitely generated metabelian groups*, Algebra i logika, **8** (1969), pp. 72–76.

[SWR] Scott W.R. *Algebraically closed groups.* Proc. American Math. Soc. **2**, (1951), pp. 118–121.

[SJ] Stallings J., *Finiteness properties of matrix representations.* Annals of Math. **124** (1986), pp. 337–346.

[WDH] Wagner, D. H. *On free products of groups*, Trans. American Math. Soc. **84** (1957), pp. 352–378.

[WB] B. A. F. Wehrfritz, *Infinite linear groups.* Springer-Verlag, New York, Heidelberg, Berlin (1973).