Balanced presentations of the trivial group on two generators and the Andrews-Curtis conjecture

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Abstract

The Andrews-Curtis conjecture states that every balanced presentation of the trivial group can be reduced to the standard one by a sequence of the elementary Nielsen transformations and conjugations. In this paper we describe all balanced presentations of the trivial group on two generators and with the total length of relators ≤ 12. We show that all these presentations satisfy the Andrews-Curtis conjecture.

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1 Introduction

Let $F = F(X)$ be a free group of rank $n \geq 2$ with a basis $X = \{x_1, ..., x_n\}$. Consider the following transformations of an n-tuple $W = (w_1, \ldots, w_n)$ of elements from F:

(AC1) replace w_i by $w_i w_j, \;\; j \neq i$;

(AC2) replace w_i by w_i^{-1};

(AC3) replace w_i by $f w_i f^{-1}$ for some $f \in F$,

and leave w_k fixed for all $k \neq i$.

The transformations (AC1) and (AC2) are usually called elementary Nielsen transformations. We will refer to the transformations (AC1) - (AC3) as the *elementary AC-transformations.*

Let F^n be the cartesian product of n copies of the group F. Two n-tuples V and W from F^n are called *Andrews-Curtis equivalent* (or *AC-equivalent*) if one of them can be obtained from the other by a finite sequence of elementary AC-transformations. In this event, we write $V \sim W$. The relation $\sim$ is an equivalence relation on the set F^n. The following conjecture, which appears to be of interest in topology as well as group theory, was raised by J.J. Andrews and M.L. Curtis in [**AC**].

The Andrews-Curtis conjecture. Elements $w_1, \ldots, w_n \in F(X)$ generate $F(X)$ as a normal subgroup if and only if $(w_1, \ldots, w_n) \sim (x_1, \ldots, x_n)$.

One can formulate this conjecture in terms of presentations. A group presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$ is called *balanced* if $m = n$. The balanced presentation $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$ of the trivial group is called *standard*. We shall say that two presentations with the same generators are AC-equivalent if the tuples of relators in these presentations are AC-equivalent. Plainly, the AC-conjecture is equivalent to the following one: *every balanced presentation of the trivial group is AC-equivalent to the standard one with the same set of generators.*

There is a survey [**BM**] by R.G. Burns and O. Macedonska on the AC-conjecture from group theory viewpoint. For relevant topological results we refer to a survey [**HM**] by C. Hog-Angeloni and W. Metzler. The prevailing opinion seems to be that the AC-conjecture is false. Moreover, several potential counterexamples are known. We say that a balanced presentation of the trivial group $\langle x_1, \ldots, x_n \mid w_1, \ldots, w_n \rangle$ is a *potential counterexample* to the AC-conjecture if, firstly, it is not known to be AC-equivalent to the standard one, and, secondly, no one of the elementary AC-transformations decreases the total length $|w_1| + \ldots + |w_n|$ of the relators. Below we list some of the shortest and the most established potential counterexamples:

(1) $\langle x, y |\ x^{-1}y^2x = y^3,\ y^{-1}x^2y = x^3 \rangle$.

(2) $\langle x, y, z |\ y^{-1}xy = x^2,\ z^{-1}yz = y^2,\ x^{-1}zx = z^2 \rangle$.

(3) $\langle x, y |\ x^3 = y^4,\ xyx = yxy \rangle$.

The first two presentations have been known for almost 20 years; for a discussion we refer to the survey [**BM**]. Example (3) is the second presentation in the series

(4) $\langle x, y |\ x^n = y^{n+1},\ xyx = yxy \rangle,\ n \geq 2$,

which is due to S. Akbulut and R. Kirby [**AK**]. This series has been known for 15 years. Notice that for $n = 2$ the series (4) gives the presentation

(5) $\langle x, y |\ x^2 = y^3,\ xyx = yxy \rangle$,

which, until recently, has also been considered as a potential counterexample. In [**MA**] the first author proved that this presentation satisfies the AC-conjecture. For this purpose he designed a genetic algorithm to search for corresponding sequences of the elementary AC-transformations (see [**MA**] for details). This algorithm is also instrumental for the following theorem which is the main result of this paper.

Theorem 1.1 *Every balanced presentation of the trivial group on two generators with the total length of the relators at most 12 satisfies the Andrews-Curtis conjecture.*

The idea of the proof of this theorem is very simple: we list all balanced presentations of the trivial group on two generators where the total length of relators is at most 12:

$$\mathcal{P} = \{P_1, P_2, P_3, \ldots\}$$

and then apply the genetic algorithm to each of the listed presentations. However, there are two issues to address here. The first one concerns the listing of the trivial presentations. Indeed, it is known that there is no algorithm to decide whether a group given by a finite presentation is trivial or not [**Adjan**], [**Rabin**]. Whether such an algorithm exists for balanced presentations is an open and difficult problem [**BMS**]. In the particular case when the total length of relators in the presentations is at most 12, the following result enables one to list the set $\mathcal{P}$.

Theorem 1.2 *Let G be a group defined by a presentation*

(6) $\langle\, x, y \mid r(x,y),\ s(x,y)\,\rangle$, *where* $|r(x,y)| + |s(x,y)| \leq 12$.

If the abelianization of G is trivial then G is either the trivial group or G is isomorphic to the following finite group of order 120

$$\langle x, y \mid yxy = x^2,\ xyx = y^4\ \rangle.$$

The proof of this theorem is based on computer computations with the software package Magnus. We discuss the proof in Section 2.

The second issue is related to the real time required to carry out the computations with the genetic algorithm. It turns out that there are about 10^6 presentations in the set $\mathcal{P}$. To run the genetic algorithm on each of them would take too much time. So one has to exploit tricks and shortcuts (pribambases) to decrease the time. We discuss this in Section 2. However, it is worthwhile to note here that the presentations from $\mathcal{P}$ with total length up to 10 are relatively easy to reduce to the standard one by AC-transformations. So, the minimal total length of relators in non-trivial examples from $\mathcal{P}$ appears to be 11. Surprisingly enough, all the "difficult" presentations from $\mathcal{P}$ with total length 11 (which cannot be easily reduced to the standard one) are readily seen to be AC-equivalent to the presentation **(5)**: $\langle x, y |\ x^2 = y^3,\ xyx = yxy\rangle$. The most difficult examples to "crack" are those with total length 12, which cannot be easily reduced neither to the standard one nor to the presentation **(5)** (of length 11). We list these most interesting presentations in the corollary below.

Corollary 1.3 *The following presentations of the trivial group are AC-equivalent to $\langle x, y |\ x, y\rangle$:*

$$\langle x, y |\ x^{-1}y^2x = y^3,\ x^2 = y^\epsilon x y^\delta\rangle,$$

where $\epsilon, \delta \in \{1, -1\}$.

One can find in [**MA**] the corresponding sequences of AC-transformations that reduce the presentations above to the standard presentation of the trivial group.

It follows now from Theorem 1.1 that the minimal total length of relators in potential counterexamples, that still stand, is 13. This allows us to formulate the following

Corollary 1.4 *The presentation* **(3)***:*

$$\langle x, y |\ x^3 = y^4,\ xyx = yxy\rangle$$

is, at the present time, a minimal potential counterexample to the Andrews-Curtis conjecture.

2 Description of the algorithms

We start with few known algorithms, which play an important part in our proofs.

In 1936 J.H.C. Whitehead [**Whit**] gave an algorithm which for given m-tuples $U = (u_1, ..., u_m)$ and $V = (v_1, ..., v_m)$ from F^m decides whether there exists an automorphism $\phi \in Aut(F)$ such that $\phi(u_i) = v_i, i = 1, \ldots, m$. Furthermore, if such an automorphism exists, then the algorithm finds one. In general, the time-complexity of the Whitehead algorithm is exponential. However, in the particular case when one needs to check whether a given element $f \in F$ can be mapped by an automorphism of F to the element x_1, the Whitehead method is polynomial in time with respect to the length of f (for a given fixed F). Recall that an element $f \in F$ is called *primitive* if $\phi(f) = x_1$ for some $\phi \in Aut(F)$. It follows that we can recognize primitive elements in a free group of rank two quite effectively.

The other algorithm that we used in our proofs is the Todd-Coxeter algorithm [**TC**], [**Jon**]. This is a systematic procedure for enumerating cosets of a given finitely generated subgroup of a given finitely presented group. In particular, if the group given by a finite presentation is finite, then the Todd-Coxeter algorithm will eventually recognize this, and it will give a multiplication table of the group. At the present time, the Todd-Coxeter algorithm (and its variations) are among the most powerful application of computers to group theory (see [**Havas**] for details).

The third algorithm we want to mention here allows one to check whether the abelianization $G/[G, G]$ of the group G given by a finite presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$ is trivial or not.

This algorithm is relatively fast (at least in our case); it calculates the canonical invariants of the abelian group $G/[G, G]$. One can find a complete description of the algorithm in the book [**Sims**] by C.Sims.

Now we explain how these methods can be used in proving Theorems 1.1 and 1.2. We combine both proofs into a single procedure that was carried out by a computer. This procedure consists of the following steps.

1. Generate a list L_1 of balanced presentations on two generators and with the total length of relators ≤ 12. There are about $6 \cdot 10^6$ of such presentations.

2. For each presentation P in the list L_1, check to see whether the abelianization of the group defined by P is trivial or not. Delete, one by one, all presentations from L_1 with a non-trivial abelianization. Denote by L_2 the

resulting list. Plainly, all the trivial groups from L_1 are also in L_2. There are about 10^6 presentations in L_2.

3. Apply the Whitehead algorithm to each of the relators in every presentation P in L_2 to check whether the relator is a primitive element in F or not. If the relator is primitive then the group defined by P is trivial. Moreover, it is not hard to see that, in this event, the presentation P is AC-equivalent to the standard one **[MA]**. Delete, one by one, all the presentations from L_2 with primitive elements among its relators. All the deleted presentations satisfy the AC-conjecture. Denote by L_3 the resulting list. There are 122240 presentations in L_3.

4. Observe that a cyclic permutation of a relator is a particular case of the transformation $(AC3)$. We can use this to reduce the number of presentations in L_3 which belong to the same AC-equivalence class. Compare presentations in L_3, by cyclically permuting their relations, and leave only one presentation from each equivalence class. Denote the resulting list by L_4. Only 1648 presentations are left in the list L_4.

5. At this point we want to sort out the presentations in L_4 which define the trivial group. We apply the Todd-Coxeter algorithm to each presentation P from L_4 to compute the order of the group defined by P. Notice that the Todd-Coxeter algorithm is time-consuming, so we want to apply it to as few presentations as possible. Luckily, all the groups in the list L_4 happened to be finite, so the Todd-Coxeter algorithm eventually stopped and gave the answer. It turns out that all the groups in L_4 are trivial, except 16 groups of order 120. It follows now that all the groups from L_2 are either trivial or of order 120. To finish the proof of Theorem 1.2 it suffices to notice that all these groups are isomorphic to each other, and hence they are isomorphic to this particular one, given by the presentation

$$\langle x, y \mid yxy = x^2,\ xyx = y^4 \ \rangle.$$

   Indeed, every group of order 120 with trivial abelianization is a central extension of a cyclic group of order 2 by the simple group A_5. All such groups are isomorphic. Denote by L_5 the list of all presentations from L_4 which define the trivial group. There are 1632 presentations in L_5.

6. This is the last and the most time-consuming step. We apply the genetic algorithm from **[MA]** (and some of its variations) to check whether the presentations from L_5 satisfy the AC-conjecture. We show that every presentation from L_5 is AC-equivalent either to the standard presentation of the trivial group, or to the presentation **(5)**, or to one of the presentations in Corollary 1.3, which are already known to satisfy the AC-conjecture (see **[MA]**).

   At this step, all the presentations of the trivial group on two generators and with total length at most 12 have been shown to satisfy the AC-conjecture. This proves Theorem 1.1.

All routines and procedures which we used here are available via the Internet at *www.grouptheory.org* as a part of the software package Magnus.

Notice that in order to show that two presentations are AC-equivalent we use a modification of the genetic algorithm from **[MA]** in which the fitness function is replaced by a new one. Namely, in this case we used the sum of the Hamming distances between the relators as cyclic words. In most of the occasions the genetic algorithm with this new fitness function worked very fast. We refer to **[Hol]** and **[Mit]** for a general discussion on genetic algorithms.

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