# Groups, Logic, and Computation. GAGTA-2025

Alexander Ushakov

Stevens Institute of Technology

## *Spherical equations in $\mathbb{Z}_2 \wr (F_2 \times F_2)$ are undecidable.*

June 6–13, 2025

*Abstract:*
Recall that a standard form of a spherical equation over a group $G$ is

$$\prod_{j=1}^{k} z_j^{-1} c_j z_j = 1 \quad (k \geq 1),$$

where $c_1, \ldots, c_k \in G$ are constants and $z_1, \ldots, z_k$ are variables. Define a set of spherical equations with $k$ conjugates

$$\mathrm{Sph}_k(G) = \left\{ \prod_{j=1}^{k} z_i^{-1} c_i z_i = 1 \;\middle|\; c_1, \ldots, c_k \in G \right\}$$

that also can be considered as the corresponding Diophantine problem (an algorithmic question to decide if a given equation has a solution or not). A mapping from $\mathrm{Sph}_k(G)$ to $\mathrm{Sph}_{k+1}(G)$ defined by $\prod_{j=1}^{k} z_i^{-1} c_i z_i = 1 \quad \mapsto \quad \prod_{j=1}^{k} z_i^{-1} c_i z_i \cdot z_{k+1}^{-1} 1 z_{k+1} = 1$ is a linear-time reduction. Hence, slightly abusing notation, we write $\mathrm{Sph}_k(G) \subseteq \mathrm{Sph}_{k+1}(G)$. This creates a hierarchy of spherical equations in which

- $\mathrm{Sph}_1(G)$ can be viewed as the set of word-equations;
- $\mathrm{Sph}_2(G)$ can be viewed as the set of conjugacy-equations;
- the set of all spherical equations is the union $\mathrm{Sph}(G) = \bigcup_{i=k}^{\infty} \mathrm{Sph}_k(G)$.

Novikov and Boone constructed a finitely presented group $G$ with undecidable word problem ($\mathrm{Sph}_1(G)$). Miller constructed a finitely presented group $G$ with decidable word problem ($\mathrm{Sph}_1(G)$) and undecidable conjugacy problem ($\mathrm{Sph}_2(G)$). In this work we show that in $\mathbb{Z}_2 \wr (F_2 \times F_2)$ the word problem and the conjugacy problem are decidable, but there exists $k \geq 3$ for which $\mathrm{Sph}_k(G)$ is undecidable. Furthermore, the following holds.

**Theorem.** $G = \langle x, y \mid r_1, \ldots, r_m \rangle$ has undecidable WP $\Rightarrow$ $\mathrm{Sph}_{m+3}$ is undecidable in $\mathbb{Z}_2 \wr (F_2 \times F_2)$.