Subdirect products of groups

Groups and computation: in honour of Paul Schupp

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In combinatorial group theory, the constructions of free products, amalgamated free products and HNN- extensions are widely used and very successful versions of subgroup theories have evolved. Constructions dual to these (in the sense of category theory) are direct products, fibre products. These are also useful but a subgroup theory, for instance, has yet to emerge.

In this talk I want to discuss some results about a certain subgroups, called subdirect products, which are in some sense large. Properties of interest for such groups incllude finiteness conditions (finitely generated, finitely presented), homological finiteness, and algorithms (existence and non-existence).

Many of the results I discuss resulted from various projects with collaborators Gilbert Baumslag, Martin Bridson, Jim Howie and Hamish Short who (along wiht me) are sometimes identified by initials BBHMS.

We recall that a **subdirect product** of groups $A_1, \ldots, A_n$ is a subgroup $G \leq A_1 \times \cdots \times A_n$ which projects surjectively onto each factor.

Let $G \leq A_1 \times A_2$ be a subdirect product of two groups.

- Put $L_i = G \cap A_i$.
- G projects onto A_2 with kernel L_1.
- Also A_2 projects onto its quotient A_2/L_2.
- The composition of these maps sends G onto A_2/L_2 with kernel $L_1 \times L_2$.
- By symmetry we have isomorphisms

$$A_1/L_1 \cong G/(L_1 \times L_2) \cong A_2/L_2.$$

If we denote this common quotient by Q, one can easily check that G is a fibre product (= pullback):

Proposition

Let $G \leq A_1 \times A_2$ be a subgroup. Then G is a subdirect product of $A_1 \times A_2$ if and only if there is a group Q and surjections $p_i : A_i \to Q$ such that G is the fibre product of p_1 and p_2, that is

$$G = \{(u, v) \in A_1 \times A_2 \mid p_1(u) =_Q p_2(v)\}.$$

Notice that if A_1 and A_2 are the same group A and $p : A \to Q$ the quotient map, then the fibre product $G = \{(u, v) \in A \times A \mid p(u) = p(v))\}$ is the graph of the equality relaiton in Q.

Suppose that $G \leq A_1 \times \cdots \times A_n$ is subdirect. The following two elementary observations reduce the study of arbitrary subgroups of direct products to studying subdirect products which intersect all the factors.

1. If $A_i \subseteq B_i$ then of course $G \leq B_1 \times \cdots \times B_n$ but is no longer subdirect unless $A_i = B_i$. Indeed $A_1 \times \cdots \times A_n$ is the smallest compatible direct product containing G.

2. If $G \cap A_n = \{1\}$ then projection onto $A_1 \times \cdots \times A_{n-1}$ sends G isomorphically onto $\overline{G}$ which is subdirect in $A_1 \times \cdots \times A_{n-1}$. Thus it is natural to assume that G intersects each of the factors non-trivially.

Subgroups of $F \times F$

We recall some of the difficulties encountered in a direct product of two free groups.

Let $F = \langle a_1, \ldots, a_n \mid \rangle$ be a free group. The direct product $D = F \times F$ of two copies of F has some nice properties. For instance, D has solvable word and conjugacy problems. As F is word hyperbolic, D is bi-automatic and so has quadratic isoperimetric function. The centralizer of an element not lying in a factor is cyclic.

However **the subgroups of $F \times F$ are remarkably complicated**.

A result of Mihailova from 1958 leads to a number of unsolvable algorithmic problems about D.

Let

$$Q = \langle a_1, \ldots, a_n \mid r_1 = 1, \ldots, r_m = 1 \rangle.$$

be a finitely presented group $p : F \to Q$ the natural quotient map.

Consider the **pull-back** or **fibre product** Γ_Q of two copies of the map p. Thus

$$\Gamma_Q = \{(u, v) \in F \times F \mid p(u) =_Q p(v)\}$$

which is the graph of the equality relation among words of Q.

Mihailova observed that

Lemma

Γ_Q is finitely generated by the pairs $(a_1, a_1), \ldots, (a_n, a_n)$ and $(r_1, 1), \ldots, (r_m, 1)$.

Since $(u, v) \in \Gamma_Q \;\Leftrightarrow\; u =_Q v$ this shows the following:

Theorem (Mihailova)

If Q has unsolvable word problem, then the problem of deciding whether an arbitrary pair $(u, v) \in F \times F$ lies in the finitely generated subgroup Γ_Q is recursively unsolvable. That is, the membership problem for Γ_Q in $F \times F$ is unsolvable.

Combining this with the Adian-Rabin construction for triviality and other observations, one can show have the following:

Theorem (CFM)

- *The problem of determining whether a finite set of elements generates* $F \times F$ *(for* $n \geq 2$*) is recursively unsolvable.*
- *The isomorphism problem for finitely generated subgroups of* $F \times F$ *is unsolvable.*
- *Certain of the finitely generated subgroups* Γ_Q *have an unsolvable conjugacy problem.*

We next ask **how to find a presentation for** Γ_Q.

Let $N = \ker\ p : F \to Q$ so that $Q = F/N$.

We observe that

- $\Gamma_Q \cap F \times 1 = N \times 1 := N_1$.
- So the kernel of the projection of Γ_Q onto the second factor $F_2 = 1 \times F$ is a copy of N
- and the sequence $1 \to N_1 \to \Gamma_Q \to F_2 \to 1$ is exact.
- But $\Gamma_Q \cap 1 \times F = 1 \times N = N_2$.

So to present Γ_Q we have to add relations saying that N_1 and N_2 commute. Now if Q is infinite and thus N is not finitely generated, this requires infinitely many relations.

Grunewald made this precise by proving the following:

Theorem (Grunewald)

If Q is infinite, then Γ_Q is **not** *finitely presented.*

More generally Baumslag and Roseblade (1983) showed that only the "obvious" subgroups of $F \times F$ are finitely presented. Before explaining this more carefully we make some rather general observations.

More general properties

It is easy to see **when subdirect products are normal subgroups**.

Proposition

Let $G \leq A_1 \times \cdots \times A_n = D$ *be a subdirect product of the groups* $A_1, \ldots, A_n$ *and let* $L_i = G \cap A_i$*. Then the following are equivalent:*

- *-* G *is normal in* D*;*
- *- each* A_i / L_i *is abelian;*
- *-* G *contains the derived group* $[D, D]$*.*

We next ask **when is a subdirect product $G \leq A \times B$ is finitely generated ?** Of course we should assume that A and B are finitely generated since they are quotients of G.

Proposition

Let A and B be finitely generated and suppose $G \leq A \times B$ is subdirect.

- *If G is finitely generated and B is finitely presented, then $G \cap A$ is finitely normally generated.*
- *If G is finitely presented, then B is finitely presented if and only if $G \cap A$ is finitely normally generated.*

In the other direction we have:

Proposition

Suppose that $G \leq A \times B$ *is a subdirect product of two finitely generated groups* A *and* B*. If either* $G \cap A$ *or* $G \cap B$ *is finitely normally generated, then* G *is finitely generated.*

Combining these we conclude:

Corollary

Suppose that $G \leq A_1 \times A_2$ *is the subdirect product of two finitely presented groups* A_1 *and* A_2. *Let* $L_i = G \cap A_i$ *Then* G *is finitely generated if and only if one (and hence both) of* A_1/L_1 *and* A_2/L_2 *are finitely presented.*

Finite generation is a 1-dimensional property and finite presentation is a 1-and-2-dimensional property. So here a 1-dimensional property of of a fibre product G is related to a 1-and-2-dimensional property of the quotient Q.

As we will see later, two dimensional properties of a fibre product are related to 1, 2 and 3 dimensional properties of groups in the construction.

Subgroups of a direct product with a free group

A few years ago I found a remarkably simple proof of the (yet to be stated) theorem of Baumslag and Roseblade. Thinking of a direct product $A \times F$ as an HNN-extension of A, one can prove:

Theorem (CFM)

Let $A \times F$ be the direct product of a group A with a free group F. Suppose that $G \leq A \times F$ is a subgroup which intersects F non-trivially.

- *If G is finitely generated, then G has a subgroup G_0 of finite index which is an HNN-extension of the form*

  $$G_0 = \langle C, t \mid t^{-1}bt = b, \ \forall b \in L \rangle$$

  where C is finitely generated and $L = G \cap A \leq C$.
- *If G is finitely presented, then $L = G \cap A$ is finitely generated.*

The proof uses a theorem of Marshall Hall which implies that an element $1 \neq t \in F$ generates a free factor of a subgroup of finite index in F, that is, t is a primitive element in a subgroup of finite index. We remark that if $G \cap F$ contains a primitive element, then passing to a subgroup of finite index is unnecessary.

Since finitely generated normal subgroups of free groups have finite index, from the above we can easily deduce

Theorem (Baumslag and Roseblade)

Let $F_1 \times F_2$ be the direct product of two free groups F_1 and F_2. Suppose that $G \leq F_1 \times F_2$ is a subgroup and define $L_i = G \cap F_i$.

- *If either $L_i = 1$ then G is free.*
- *If both L_i are non-trivial and one of them is finitely generated, then $L_1 \times L_2$ has finite index in G.*
- *Otherwise, G is not finitely presented.*

By a **surface group** we mean the fundamental group of a connected 2-manifold. Such a group is either free (of finite or countably infinite rank) or else has a subgroup of index at most two which is trivial or has a presentation of the form

$$\mathcal{P}_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle.$$

Like free groups, the groups of closed, orientable surfaces of genus $g > 1$ have the property that a non-trivial finitely generated normal subgroup has finite index. So the above arguments carry over to the product of such a surface group and a free group. This proves the following for genus $g > 1$:

Theorem

Suppose that A is the group of a closed, orientable surface and that F is free. If $G \leq A \times F$ is a finitely presented subgroup of their direct product, then G is either a surface group or virtually a direct product of surface groups.

Baumslag and Roseblade's proved their theorem using a spectral sequence argument to show that when A is free $H_1(L,\mathbb{Z})$ is a section (quotient of a subgroup) of $H_2(G_0,\mathbb{Z})$. If L is not finitely generated, it follows that $H_2(G_0,\mathbb{Z})$ is not finitely generated and so G_0 is not finitely presented.

In the context of our Theorem on direct products with a free group, the exact Mayer-Vietoris sequence for the HNN-extension is

$$\cdots \to H_{n+1}(G_0,\mathbb{Z}) \to H_n(L,\mathbb{Z}) \xrightarrow{0} H_n(C,\mathbb{Z}) \to \cdots$$

where the map $H_n(L,\mathbb{Z}) \to H_n(C,\mathbb{Z})$ is the difference of the induced maps on associated subgroups. But this is the zero map since t commutes with L. This proves a homological version of our Theorem:

Theorem (CFM)

Let $A \times F$ be the direct product of a group A with a free group F. Suppose that $G \leq A \times F$ is a subgroup which intersects F non-trivially. Let $L = G \cap A$. Then G has a subgroup G_0 of finite index with $L \leq G_0$ such that $H_n(L,\mathbb{Z})$ is a quotient of $H_{n+1}(G_0,\mathbb{Z})$ for $n \geq 0$.

A very similar phenomenon happens with respect the integral homology of the groups involved. Using spectral sequences or mostly the usual 5-term sequence for an extension, we show:

Theorem

Let A and B be groups with both $H_1(-,\mathbb{Z})$ and $H_2(-,\mathbb{Z})$ finitely generated. Suppose that $G \leq A \times B$ is a subdirect product of A and B. Then $H_1(G,\mathbb{Z})$ is finitely generated if and only if one (and hence both) of $H_2(A/(G \cap A),\mathbb{Z})$ and $H_2(B/(G \cap B),\mathbb{Z})$ is finitely generated.

In the special case of free factors we conclude the following:

Corollary

Suppose that $G \leq F_1 \times F_2$ is a subdirect product of two free groups F_1 and F_2. Let $L_i = G \cap F_i$. Then

$$H_1(G, \mathbb{Z}) \cong H_1(F_2, \mathbb{Z}) \oplus H_2(F_1/L_1, \mathbb{Z}) \oplus C$$

where C is a subgroup of $H_1(F_1, \mathbb{Z})$ and hence is free abelian of rank at most the rank of F_1.

Since it is known how to construct two generator groups with prescribed countable $H_2(-, \mathbb{Z})$, one can apply the pull back construction to conclude:

Corollary (Baumslag and Roseblade)

Let F_1 and F_2 be non-abelian free groups. Then there are continuously many subdirect products $G \leq F_1 \times F_2$ having non-isomorphic $H_1(G, \mathbb{Z})$.

As another application we note the following example.

- Let F be a finitely generated free group and suppose that F/L has finitely generated $H_2(F/L, \mathbb{Z})$ but is not finitely presented.
- Let $G \leq F \times F$ be the pullback or fibre product corresponding to this presentation.
- Then $H_1(G, \mathbb{Z})$ is finitely generated by the previous theorem, but G is not finitely generated our earlier characterization.

We record this as the following:

Corollary

There is a subdirect product $G \leq F \times F$ of two finitely generated free groups such that $H_1(G, \mathbb{Z})$ is finitely generated, but G is not finitely generated.

The Stallings-Bieri examples

These examples are constructed as follows:

- Let $F_1 = \langle a_1, b_1 \mid \rangle$, …, $F_n = \langle a_n, b_n \mid \rangle$ be free groups of rank 2.
- Let $Q = \langle c \mid \rangle$ be an infinite cyclic group.
- Let ψ_n be the map from the direct product $F_1 \times \cdots \times F_n$ to Q defined by $a_i \mapsto c$ and $b_i \mapsto c$.
- Define $SB_n = \ker \psi_n$.

It is easy to check that SB_n is a subdirect product of the F_i and that (for $n > 1$) SB_n is finitely generated by the elements $a_i b_i^{-1}$, $a_i a_j^{-1}$ and $b_i b_j^{-1}$.

Theorem (Stallings-Bieri)

- *For* $n > 2$ *the group* SB_n *is finitely presented;*
- *for* $n > 1$ *the group* SB_n *is of type* FP_{n-1} *(or even better of type* F_{n-1}*); but*
- $H_n(SB_n.\mathbb{Z})$ *is not finitely generated, so* SB_n *but not of type* FP_n*.*

Observe that projection onto the first $n-1$ factors maps SB_n surjectively onto $F_1 \times \cdots \times F_{n-1}$ with kernel L_n which is the normal closure in F_n of $a_n b_n^{-1}$ which is a primitive element.

Also one can check

$$SB_{n-1} = SB_n \cap (F_1 \times \cdots \times F_{n-1}).$$

The fact that $H_n(SB_n.\mathbb{Z})$ is not finitely generated follows inductively from our homological version of the Theorem on direct products with a free group.

Later we will give some results which give more information on when finitely generated subdirect products of $F_1 \times \cdots \times F_n$ are finitely presented.

Definitions and basic properties of limit groups

Recall that a group G is said to be *residually free* if for every element $1 \neq g \in G$ there is a homomorphism $\phi : G \to F$ from G onto a free group F such that $\phi(g) \neq 1$.

A group is residually free if and only if it is isomorphic to a subgroup of an unrestricted direct product of free groups.

Examples of residually free groups include free abelian groups $\mathbb{Z}^n$ and direct products of free groups of finite rank and surface groups. Since being free is a hereditary property, the subgroups of these are residually free as well.

A group G is a *fully residually free* if for every finite subset $X \subset G$ there is a homomorphism $\phi : G \to F$ from G to a free group F such that $\phi|_X : X \to F$ is injective, that is, the images of the elements of X are all different. Clearly fully residually free groups are residually free. A finitely generated, fully residually free group is called a *limit group*.

Limit groups play a crucial role in the work of Kharlampovich and Miasnikov and the work of Sela on the Tarski problem.

Examples of limit groups are free abelian groups $\mathbb{Z}^n$, free groups of finite rank and surface groups. If G is a limit group and C is a maximal abelian subgroup of G then $G *_C (C \times \mathbb{Z}^n)$ is a limit group.

Proposition (Some properties of limit groups)

Let Γ be a limit group. Then

- Γ *is torsion free, finitely presented, CAT(0) and has a finite* $K(\Gamma, 1)$.
- *Finitely generated subgroups of* Γ *are limit groups.*
- *For* $a, b, c \in \Gamma \setminus \{1\}$ *commutativity is transitive, that is,* $[a, b] = [b, c] = 1$ *implies* $[a, c] = 1$.
- Γ *has the same universal theory as a free group.*
- *If* S *is a subgroup of* Γ *with* $H_1(S, \mathbb{Q})$ *finite dimensional, then* S *is finitely generated (and hence is a limit group).*
- *Limit groups have good algorithms.*

Motivated by the Baumslag-Roseblade Theorem and the Stallings-Bieri examples. BHMS proved the following first for surface groups and later for limit groups.

Theorem (BHMS)

Let $\Gamma_1, \ldots, \Gamma_n$ *be non–abelian limit groups and let* $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ *be a finitely generated subdirect product which intersects each factor non-trivially. Then either :*

1. S *is of finite index;* or
2. S *is of infinite index and has a finite-index subgroup* $S_0 < S$ *such that* $H_j(S_0; \mathbb{Q})$ *has infinite* $\mathbb{Q}$*-dimension for some* $j \leq n$*.*

Here is one consequence of the above result.

Theorem (BHMS)

If $\Gamma_1, \ldots, \Gamma_n$ *are limit groups and* $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ *is a subgroup of type* $FP_n(\mathbb{Q})$*, then* S *is virtually a direct product of* n *or fewer limit groups.*

A key fact about finitely generated residually free groups is a result from algebraic geometry over groups:

Theorem (Baumslag-Miasnikov-Remeslennikov)

A finitely generated residually free group can be embedded into a direct product of finitely many limit groups.

For finitely presented residually free groups, Kharlampovich and Miasnikov give an algorithm for finding such an embedding. BHMS later gave another algorithm for finding such an embedding and obtained additional algroithmic results.

VSP and the 1-2-3 Theorem

Recall that a subdirect product is *full* if it intersects each of the direct factors non-trivially. A subgroup $S < G_1 \times \cdots \times G_n$ is said to be *virtually surjective on pairs* (VSP) if for all $i \neq j \in \{1, \ldots, n\}$, the projection $p_{ij}(S) \subset G_i \times G_j$ has finite index. (We implicitly assume that $n \geq 2$.)

A subgroup $S < \Gamma$ is termed *separable* if for every $\gamma \in \Gamma \setminus S$ there exists a normal subgroup $K \lhd \Gamma$ of finite index such that $\gamma \notin SK$.

Proposition (BHMS)

Let $G_1, \ldots, G_n$ be finitely generated groups and let $S \subset G_1 \times \cdots \times G_n$ be a subgroup. If $p_{ij}(S) \subset G_i \times G_j$ is of finite index for all $i, j \in \{1, \ldots, n\}$, then there exist finite-index subgroups $G_i^0 \subset G_i$ such that $\gamma_{n-1}(G_i^0) \subset S$.

In case the projections of S are surjective onto pairs one can see this as follows. Consider an $(n-1)$-fold commutator in G_1, say $([x_1, \ldots, x_{n-1}], 1 \ldots, 1)$. By surjective on pairs, there are elements $(x_1, 1, *, \ldots, *)$, $(x_2, *, 1, *, \ldots, *)$, and so on in S where the $*$ entries are unnamed elements. Then their $(n-1)$-fold commutator is $([x_1, \ldots, x_{n-1}], 1 \ldots, 1)$ which therefore lies in S.

Theorem (The VSP Criterion)

Let $S < G_1 \times \cdots \times G_n$ be a subgroup of a direct product of finitely presented groups. If S is virtually surjective on pairs (VSP), then it is finitely presented and separable.

Note that we do not assume,*a priori*, that the subgroup S is finitely generated. The converse of this theorem is false in general; even finitely presented full subdirect products need not satisfy VSP. For example, if N is a finitely-generated torsion-free nilpotent group that is not cyclic, and if $\phi : N \times N \to \mathbb{Z}$ is a homomorphism whose restriction to each factor is non-trivial, then the kernel of ϕ is a finitely presented, separable full subdirect product without VSP.

An essential ingredient in the proof of VSP Criterionis the following asymmetric version of the 1-2-3 Theorem.

Theorem (Asymmetric 1-2-3 Theorem)

Let $f_1 : \Gamma_1 \to Q$ *and* $f_2 : \Gamma_2 \to Q$ *be surjective group homomorphisms. Suppose that* Γ_1 *and* Γ_2 *are finitely presented, that* Q *is of type* F_3*, and that at least one of* $\ker f_1$ *and* $\ker f_2$ *is finitely generated. Then the fibre product of* f_1 *and* f_2*,*

$$P = \{(g, h) \mid f_1(g) = f_2(h)\} \subset \Gamma_1 \times \Gamma_2,$$

is finitely presented.

Indeed we prove an *effective* version of this result which yields an explicit finite presentation for P.

We remark that the 1-2-3 Theorem originally arose in an earlier project concerning subdirect products of hyperbolic group. BBMS used this together with a **Rips construction**.

Theorem (Rips)

Any finitely presented group Q is a quotient of a torsion-free small cancellation (hence word hyperbolic) group E_Q by a finitely generated normal subgroup N_Q, so $Q \cong E_Q/N_Q$.

Here are two results proved by combining Rips and 1-2-3 with various tactics.

Theorem (BBMS)

The direct product $A \times B$ *of two word hyperbolic (which necessarily has a quadratic isoperimetric function) can have a finitely presented subgroup* G *with an exponential isoperimetric function.*

Theorem (Bridson and CFM)

Let F *be a free group of rank 2. There is a (finitely presented) torsion-free word hyperbolic group* Γ *such that the group* $D = \Gamma \times \Gamma \times F$ *has a recursive set of finitely presented subgroups* H_i $(i \geq 0)$ *such that the problem of determining whether or not* $H_i \cong H_0$ *is recursively unsolvable.*

Subdirect products of limit groups

We next turn to the question of determining which subgroups of a direct product of limit groups are finitely presented.

In order to state our next theorem concisely we introduce the following temporary definition: an embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ of a residually free group S as a full subdirect product of limit groups is said to be *neat* if Γ_0 is abelian (possibly trivial), $S \cap \Gamma_0$ is of finite index in Γ_0, and Γ_i is non-abelian for $i = 1, \ldots, n$.

Theorem (BHMS)

Let S be a finitely generated residually free group. Then the following conditions are equivalent:

1. *S is finitely presentable;*
2. *S is of type $\mathrm{FP}_2(\mathbb{Q})$;*
3. *$\dim H_2(S_0;\mathbb{Q}) < \infty$ for all subgroups $S_0 \subset S$ of finite index;*
4. *there exists a neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ such that the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $1 \leq i < j \leq n$;*
5. *for every neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$, the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $1 \leq i < j \leq n$.*

Corollary

For all $n \in \mathbb{N}$, a residually free group S is of type $\mathrm{F_n}$ if and only if it is of type $\mathrm{FP_n}(\mathbb{Q})$.

We now turn to some novel examples of finitely presented residually free groups. Let Φ_i be the free group with basis $\{a_i, b_i\}$. Then for any finite subset $E \subset \mathbb{Z}$ and $c > 1$ we define certain finitely generated subgroups $S(E, c)$ of the direct product of $|E|$ copies of Φ_i.

As a concrete example we have $S = S(\{1,2,3,4\},3)$ is the subgroup of $\Phi_1 \times \Phi_2 \times \Phi_3 \times \Phi_4$ generated by the following 12 elements: the four images of the generators of Γ

$$(a_1, a_2, a_3, a_4)\ ,\ (b_1, b_2, b_3, b_4)$$

$$(a_1, a_2^2, a_3^3, a_4^4)\ ,\ (b_1, b_2^2, b_3^3, b_4^4)$$

together with the eight elements

$$([[a_1, b_1], a_1], 1, 1, 1)\ ,\ ([[a_1, b_1], b_1], 1, 1, 1)\ ,\ (1, [[a_2, b_2], a_2], 1, 1)\ ,\ \ldots$$

$$\ldots\ ,\ (1, 1, 1, [[a_4, b_4], a_4])\ ,\ (1, 1, 1, [[a_4, b_4], b_4])$$

which are normal generators for the subgroups $\gamma_3(\Phi_i)$ for $1 \leq i \leq 4$.

Observe that $(1, a_2, a_3^2, a_4^3)$ and $(a_1^3, a_2^2, a_3, 1)$ are in S.

Theorem (BHMS)

For any positive integer c*, and any finite subset* $E \subset \mathbb{Z}$ *of cardinality at least* $c+1$*, the group* $S(E,c)$ *is a finitely presentable subdirect product of the non-abelian free groups* Φ_n *(*$n \in E$*) and* $S(E,c) \cap \Phi_n = \gamma_c(\Phi_n)$ *for each* $n \in E$*.*

Motivated by these examples, we construct a collection of examples of finitely presentable residually free groups which is complete up to commensurability.

Definition

Let $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_n\}$ be a finite collection of 2 or more limit groups, let $c \geq 2$ be an integer, and let $\underline{g} = \{(g_{k,1}, \ldots, g_{k,n}),\ 1 \leq k \leq m\}$ be a finite subset of $\Gamma := \Gamma_1 \times \cdots \times \Gamma_n$. Define $T = T(\mathcal{G}, \underline{g}, c)$ to be the subgroup of Γ generated by $\underline{g}$ together with the c-th term $\gamma_c(\Gamma)$ of the lower central series of Γ.

Theorem

Let $T(\mathcal{G}, \underline{g}, c)$ be defined as above.

1. *If, for all $1 \leq i < j \leq n$, the images in $H_1\Gamma_i \times H_1\Gamma_j$ of the ordered pairs $(g_{k,i}, g_{k,j})$ generate a subgroup of finite index, then the residually free group $T(\mathcal{G}, \underline{g}, c)$ is finitely presentable.*
2. *Every finitely presentable residually free group is either a limit group or else is commensurable with one of the groups $T(\mathcal{G}, \underline{g}, c)$.*

Our next result provides the advertised algorithm for embedding a finitely presented residually free group as a subdirect product of limit groups. The embedding constructed is quite canonical.

Theorem (BHMS)

There is an algorithm that, given a finite presentation of a residually free group S, will construct an embedding $\iota : S \hookrightarrow \exists\mathrm{Env}(S)$, so that

1. $\exists\mathrm{Env}(S) = \Gamma_{\mathrm{ab}} \times \exists\mathrm{Env}_0(S)$ *where* $\Gamma_{\mathrm{ab}} = H_1(S, \mathbb{Z})/(\mathrm{torsion})$ *and* $\exists\mathrm{Env}_0(S) = \Gamma_1 \times \cdots \times \Gamma_n$ *is a direct product of non-abelian limit groups* Γ_i*. The intersection of S with the kernel of the projection* $\rho : \exists\mathrm{Env}(S) \to \exists\mathrm{Env}_0(S)$ *is the centre $Z(S)$ of S, and $\rho(S)$ is a full subdirect product.*
2. *Each* $L_i := \Gamma_i \cap S$ *contains a term of the lower central series of a subgroup of finite index in Γ_i and so* $\exists\mathrm{Env}(S)/(L_1 \times \cdots \times L_n)$ *is virtually nilpotent.*
3. [Universal Property] *For every map* $\phi : S \to D = \Lambda_1 \times \cdots \times \Lambda_m$*, with $\phi(S)$ subdirect and Λ_i non-abelian limit groups, there exists a unique homomorphism* $\hat{\phi} : \exists\mathrm{Env}_0(S) \to D$ *with* $\hat{\phi} \circ \rho|_S = \phi$*;*
4. [Uniqueness] *If* $\phi : S \hookrightarrow D$ *embeds S as a full subdirect product, then* $\hat{\phi} : \exists\mathrm{Env}_0(S) \to D$ *is an isomorphism respecting direct sums.*

The group $\exists\mathrm{Env}(S)$ in the previous theorem is called the *existential envelope* of S and the associated factor $\exists\mathrm{Env}_0(S)$ is the *reduced existential envelope*. The projection ρ embeds $S/Z(S)$ in $\exists\mathrm{Env}_0(S)$, and $\rho(S) \subset \exists\mathrm{Env}_0(S)$ is always a full subdirect product. The subgroup $S \subset \exists\mathrm{Env}(S)$ is always a subdirect product but it is full if and only if S has a non-trivial centre.

We now turn briefly to algorithmic properties.

Theorem (BHMS)

Every finitely presented residually free group has a solvable conjugacy problem.

Theorem (BHMS)

If G is a finitely presented residually free group and $H \subset G$ is a finitely presentable subgroup, then there is an algorithm that, given a word in the generators of G, will determine whether or not the element of G it defines belongs to H.

Theorem (BHMS)

The class of finitely presented residually free groups is recursively enumerable.

The isomorphism problem for finitely presented residually free groups remains open.

Our canonical embedding algorithm may be helpful in this regard. In the special case of those finitely presented groups whose envelop has at most two non-abelian factors, there is such an algorithm.

Direct products of finitely generated groups can be really strange. Perhaps the most dramatic example is the following:

Theorem (Mary Tyer-Jones 1971)

There exists a non-trivial finitely generate group G which is isomorphic to its direct square, that is $G \cong G \times G$.

This is rather dramatic. The group is clearly non-hopfian. Moreover, it is perfecte and the rank (minimum number of generators) of the finite direct powers G^n is constant. Also, for finitely generated groups, there is no cancellation law, that is, $A \times B \cong A \times C$ does **not** imply $B \cong C$.

One conjectures there exists a similarly exotic finitely presented groups but this is still open.

However, for finitely presented groups, Gilbert Baumslag and I proved the following along these lines:

Theorem (GB+CFM 1988)

There is a non-trivial finitely presented group G which maps onto its direct square. So there is an epimorphism $\phi : G \twoheadrightarrow G \times G$.

Theorem (GB+CFM 1988)

There is a non-trivial finitely presented group H whose commutator subgroup $K = [H, H]$ is finitely generated, non-trivial and isomorphic to its direct square, that is, $K \cong K \times K$