Verifying Biautomaticity

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Example: $G = \langle a, b, t \mid [a, t] = [b, t] = 1 \rangle$

$$
\begin{tikzcd}
\cdot \arrow[r, "a"] \arrow[d, "t"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "a"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "t"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "b"'] & \cdot
\end{tikzcd}
$$

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\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "t"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "b"'] & \cdot
\end{tikzcd}
\quad \longrightarrow \quad
\begin{tikzcd}
\cdot \arrow[r, "a"] \arrow[d, "t"'] \arrow[dr, "x"] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "a"'] & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "t"'] & \cdot \arrow[d, "t"] \arrow[dl, "y"'] \\
\cdot \arrow[r, "b"'] & \cdot
\end{tikzcd}
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\end{tikzcd}
$$

Labels of triangles determine a partial multiplication.

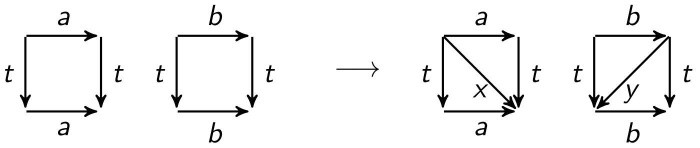
Example: $G = \langle a, b, t \mid [a,t] = [b,t] = 1 \rangle$

$$\begin{tikzcd} {} \arrow[r, "a"] \arrow[d, "t"'] & {} \arrow[d, "t"] \\ {} \arrow[r, "a"'] & {} \end{tikzcd} \quad \begin{tikzcd} {} \arrow[r, "b"] \arrow[d, "t"'] & {} \arrow[d, "t"] \\ {} \arrow[r, "b"'] & {} \end{tikzcd} \quad \longrightarrow \quad \begin{tikzcd} {} \arrow[r, "a"] \arrow[d, "t"'] \arrow[dr, "x"] & {} \arrow[d, "t"] \\ {} \arrow[r, "a"'] & {} \end{tikzcd} \quad \begin{tikzcd} {} \arrow[r, "b"] \arrow[d, "t"'] & {} \arrow[d, "t"] \arrow[dl, "y"] \\ {} \arrow[r, "b"'] & {} \end{tikzcd}$$

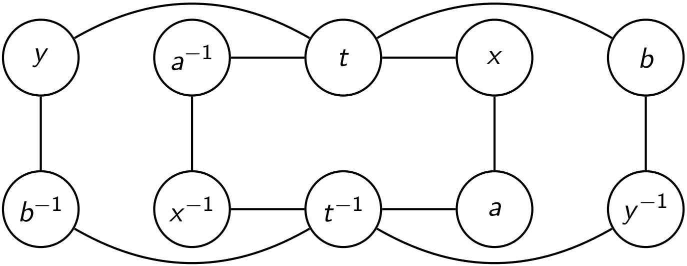
Labels of triangles determine a partial multiplication.

	a	a^{-1}	b	b^{-1}	t	t^{-1}	x	x^{-1}	y	y^{-1}
a		1			x			t^{-1}		
a^{-1}	1					x^{-1}	t			
b				1		y^{-1}			t	
b^{-1}			1		y					t^{-1}
t	x			y		1		a^{-1}		b
t^{-1}		x^{-1}	y^{-1}		1		a		b^{-1}	
x		t				a		1		
x^{-1}	t^{-1}				a^{-1}		1			
y			t			b^{-1}				1
y^{-1}				t^{-1}	b				1	

Example: $G = \langle a, b, t \mid [a,t] = [b,t] = 1 \rangle$



G is biautomatic if there are no cycles of length < 6 in the Cayley diagram of the partial multiplication table.

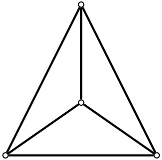


Prees

Definition

A pree is a set with a partial multiplication affording an identity, inverses, and a partial associative law. Its universal group has generators P and relations corresponding to all defined products.

The meaning of the partial associative law is that all triangles in the Cayley diagram correspond to products defined in the pree.

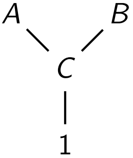


Equivalently if three triangles fit around a common vertex, the label of the perimeter yields no new products. (If it does, they can be added to the pree.)

Universal groups of prees

The universal group $U(P)$ of a pree P has generators P and relators corresponding to the products defined in P.

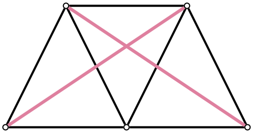
An amalgamated free product of groups is the universal group of an amalgam of groups.



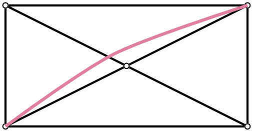
Every finitely presented group is the universal group of a finite pree.

Prees and their universal groups have been studied by Baer, Gaglione, Hoare, Kushner, Lipshutz, Rimlinger, Spellman, Stallings,

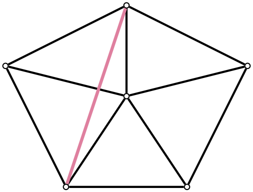
Axioms for prees



Axiom P5. One of two products must be defined. Prees satisfying this axiom are called pregroups.



Axiom A4. One of two diagonals must be defined.



Axiom A5. At least one chord must be defined.

Theorem (RG 2014)

Let P satisfy A4 and A5.

1. *P embeds in its universal group and has the induced multiplication.*
2. *If P is finite, its universal group is biautomatic.*

Generalizes [Gersten and Short 1990] for C(3)-T(6) small cancellation presentations with all pieces of length 1.

Finite C(3)-T(6) groups are cyclic, but all finite groups are universal groups of A4-A5 prees (namely themselves).

Normal Forms

[Diekert, Duncan, Miasnikov 2010] extends the Knuth-Bendix procedure to a procedure for geodesically perfect rewriting system.

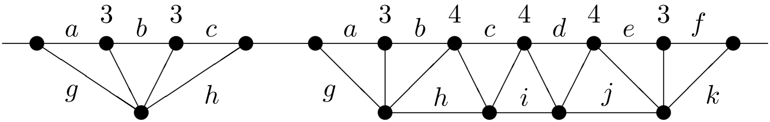
Definition

A geodesically perfect rewriting system contains

1. Length-reducing reductions which rewrite any word to an equivalent geodesic word;
2. Length-preserving reductions which rewrite any two equivalent geodesic words to each other.

Theorem (RG 2015)

Let P satisfy A4 and A5. The universal group of P has a regular geodesically perfect rewriting system.



Length reducing reductions for universal groups of A4-A5 prees.

Questions

1. Does every hyperbolic group have a finite geodesically perfect rewriting system?
2. Does every automatic group have a geodesically perfect rewriting system?

One-relator groups

Question

Is every one relator group whose relator is a commutator automatic?

Theorem (RG)

Suppose G is presented with a single rectangular relator:

$$a_1 \cdots a_m b_1 \cdots b_n c_1 \cdots c_m \; d_1 \cdots d_n$$

and further suppose that the the corners $a_m b_1, b_n c_1, c_m d_1, d_n a_1$ are pairwise distinct and do not occur elsewhere in the relator. Then G is automatic.

Example: $G = \langle\, a, b \mid [\, a^2, b^3 \,] \,\rangle$

It suffices to show that for some free group F, $G * F$ is automatic (Baumslag, Gersten, Shapiro, Short 1991 Theorem F)

$$
\begin{tikzcd}
\circ \arrow[r, "b"] \arrow[d, "a"'] & \circ \arrow[r, "b"] & \circ \arrow[r, "b"] & \circ \arrow[d, "a"] \\
\circ \arrow[d, "a"'] & & & \circ \arrow[d, "a"] \\
\circ \arrow[r, "b"'] & \circ \arrow[r, "b"'] & \circ \arrow[r, "b"'] & \circ
\end{tikzcd}
$$

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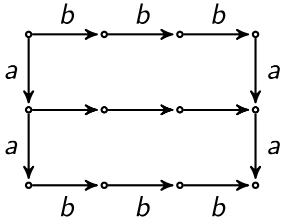

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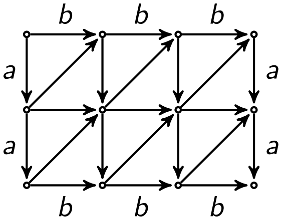

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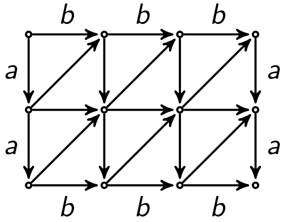
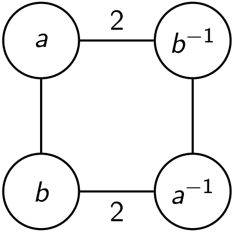
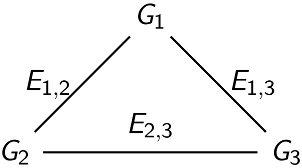




Triangles of groups



Triangles of groups [Gersten, Stallings 1990]. Non-positively curved triangles embed in their universal groups. Negatively curved triangles with finite vertex groups have hyperbolic universal groups.

Non-positively curved triangles with finite vertex groups have biautomatic universal groups [Floyd and Parry 1995].

Slightly positively curved triangles [Chermak 1995]

Positively curved triangles with vertex groups $BS(1,2)$ [Allcock 2012]

Triangles of prees

Theorem (RG)

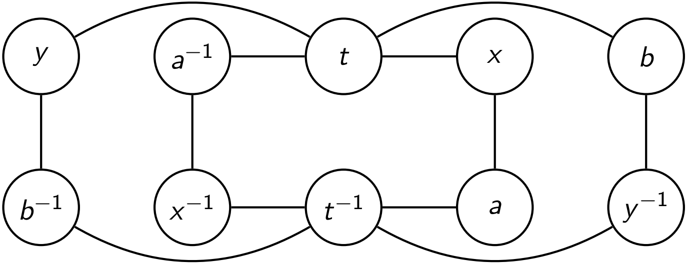
Let T be a triangle of prees satisfying A4 - A5 and such that the edge prees have the induced multiplication and satisfy the following conditions.

1. *The distance between any two nontrivial elements of an edge pree is ≥ 3 in each of its vertex groups.*
2. *The distance between any two nontrivial elements in distinct edge prees of a vertex group is ≥ 2.*

Then T also satisfies A4-A5.

Example

Vertex prees:



Edge prees: $\{1, a, a^{-1}\}$ and $\{1, b, b^{-1}\}$.

The triangle of universal groups of edge and vertex prees has all angles 0.
The universal group of the triangle is biautomatic but not hyperbolic.