A Characteristic Subgroup of a p-Stable Group

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In 1964, John Thompson introduced the Thompson subgroup. I shall use the following definition:

Definition

For S a finite p-group, let

$$m = max\{|A| : A \text{ is an abelian subgroup of } S\}.$$

$$\mathcal{A}(S) = \{A \leq S : A \text{ is abelian and } |A| = m\},$$

and

$$J(S) = \langle A : A \in \mathcal{A}(S) \rangle.$$

$J(S)$ plays a crucial role in the classification of finite groups G of local characteristic 2, i.e., those in which $F^*(H) = O_2(H)$ for every 2-local subgroup H of G, in particular through the implications of the Thompson factorization

$$H = C_H(Z(S))N_H(J(S)),$$

or the failure thereof, as well as related factorizations. The investigation of such factorizations already plays an important role in the Odd Order Theorem of Feit and Thompson.

Factorizations are very useful. But even more useful, if available, is a non-identity characteristic subgroup C of S such that $H = N_H(C)$, whenever $S \in Syl_p(H)$ and $F^*(H) = O_p(H)$. In 1968, George

Glauberman published a paper, "A characteristic subgroup of a p-stable group", exhibiting just such a characteristic subgroup, $ZJ(S)$, for p-stable groups H.

Definition

A finite group H is p-stable if $F^*(H) = O_p(H)$ and, whenever P is a normal p-subgroup of H and $g \in H$ with $[P, g, g] = 1$, then $gC_H(P) \in O_p(H/C_H(P))$.

Note: By the Hall-Higman Theorem B, if H is solvable (or even p-solvable) with $F^*(H) = O_p(H)$ and p an odd prime, then H is p-stable, unless $p = 3$ and H has non-abelian Sylow 2-subgroups. [The quintessential non-p-stable group is $ASL(2, p)$, the 2-dimensional affine special linear group.]

Theorem

(Glauberman's ZJ-Theorem) Let H be a finite p-stable group for p an odd prime. If $S \in Syl_p(H)$, then $ZJ(S)$ is a characteristic subgroup of H.

Glauberman's ZJ-Theorem played a key role in Bender's revision of the Feit-Thompson Uniqueness Theorem, and in the work of Gorenstein and Walter on groups with dihedral or abelian Sylow 2-subgroups. Indeed, John Walter goaded George into extending his theorem beyond the p-solvable case.

Although $J(S)$ is almost certainly the "right "subgroup for the general local characteristic p-type analysis, it turns out that $ZJ(S)$ is the "wrong"subgroup in the p-stable case.

Inspired by work of Avinoam Mann, Glauberman recently found the right subgroup.

Definition

Let S be a finite p-group. Let $\mathcal{D}(S)$ denote the set of all abelian subgroups A of S satisfying

$$\text{If } x \in S \text{ and } cl(\langle A, x \rangle) \leq 2, \text{ then } [A, x] = 1.$$

[Equivalently, $\mathcal{D}(S)$ is the set of all subgroups A of S such that A centralizes every A-invariant abelian subgroup of S.]

Definition

$$D^*(S) = \langle A : A \in \mathcal{D}(S) \rangle.$$

Note that $Z(S) \in \mathcal{D}(S)$. Indeed, if S has nilpotence class at most 2, then trivially, $D^*(S) = Z(S)$. In particular, $D^*(S) \neq 1$ whenever $S \neq 1$.

Theorem

(D^* Theorem) Let H be a non-identity p-stable finite group with Sylow p-subgroup S. Then $D^*(S)$ is a non-trivial characteristic subgroup of H.

Note:

Lemma

If $D^*(S) \leq T \leq S$, then $D^*(S) \leq D^*(T)$.

This property fails for $ZJ(S)$. Example: $S = R\langle\sigma\rangle$ with $R \in Syl_2(L_3(4))$ and σ a field automorphism of order 2. Then $R = J(S)$ and $ZJ(S) \cong V_4$. Let $T = ZJ(S)\langle\sigma\rangle \cong D_8$. Then $T = J(T)$. So $ZJ(S) \not\leq ZJ(T)$.

[Similar examples can be constructed for all primes p..]

Theorem

$D^*(S)$ is the unique maximal element of $\mathcal{D}(S)$.

Proof.

Let $D = D^*(S)$, $Y = Z(D)$, $X = Z_2(D)$. If $x \in X$ and $A \in \mathcal{D}(S)$, then $[A, x] \leq Y$ and so $cl(\langle A, x\rangle) \leq 2$. So $[A, x] = 1$. Hence $X = Y = D$. Likewise, if $x \in S$ with $cl(\langle D, x\rangle) \leq 2$, then $cl(\langle A, x\rangle) \leq 2$ for all A. So $[A, x] = 1$ for all A. So $[D, x] = 1$ and $D \in \mathcal{D}(S)$. □

We can now see a clear connection with p-stability.

Lemma

Let H be a p-stable group and let $S \in Syl_p(H)$ and $T = O_p(H)$. Then $D^*(S) \leq D^*(T) \triangleleft H$. In particular, if W is the normal closure of $D^*(S)$ in H, then W is an abelian normal subgroup of H.

Proof.

Let $D = D^*(S)$. Then D is an abelian normal subgroup of S and so $[T, D, D] = 1$. Hence, by p-stability, since $C_H(T) = Z(T)$, we have $DZ(T)/Z(T) \leq O_p(H/Z(T)) = T/Z(T)$. Thus $D \leq T$, whence $D \leq D^*(T)$, and then $W \leq D^*(T)$. $\square$

Theorem

(D^* Theorem) If H is p-stable and $S \in Syl_p(H)$, then $D^*(S) \lhd H$.

Proof.

As before, let $D = D^*(S)$ and $W = \langle D^H \rangle$. We wish to show $W \in \mathcal{D}(S)$. Choose $x \in S$ with $[W, x, x] = 1$. We must show that $[W, x] = 1$. Let $g \in H$. Then $[D^g, x, x] = 1$ and we wish to conclude that $[D^g, x] = 1$. Let $C = C_H(W) \lhd H$. Then p-stability implies that $x \in C_1$, the pre-image of $O_p(H/C)$. As $C_1 \lhd H$, $C_1 = C(S^g \cap C_1)$. Write $x = cx_1$ with $c \in C$, $x_1 \in S^g$. Then $[W, x_1, x_1] = 1$ and so $[D^g, x_1, x_1] = 1$. As $x_1 \in S^g$, $[D^g, x_1] = 1$. So $[D^g, x] = 1$ for all $g \in G$. Hence $[W, x] = 1$, as desired. $\square$

Recall: If S_1 has nilpotence class at most 2, then $D^*(S_1) = Z(S_1)$. Moreover, if also $J(S_1)$ is abelian and S_1 is of class exactly 2, then

$$D^*(S_1) = Z(S_1) < ZJ(S_1) = J(S_1).$$

On the other hand, if $S_2 = A\langle x\rangle$ with A abelian, $x^p \in A$, and $[A, x, x] \neq 1$, then

$$Z(S_2) < ZJ(S_2) = J(S_2) = A = D^*(S_2).$$

Finally, if S_1 and S_2 are as above, and $S = S_1 \times S_2$, then $D^*(S) = Z(S_1) \times ZJ(S_2)$ and $Z(S) \neq D^*(S) \neq ZJ(S)$.

With some additional effort it is possible to prove that always

$$Z(S) \leq D^*(S) \leq ZJ(S).$$

The following weak Replacement Theorem is a refinement by Mann of a theorem of J. D. Gillam.

Theorem

Let S be a finite metabelian p-group and let $A \in \mathcal{A}(S)$. Then $\langle A^S \rangle$ contains an abelian subgroup B normal in S, with $|B| = |A|$.

Theorem

Every maximal normal abelian subgroup of S contains $D^*(S)$ and $D^*(S) \leqslant ZJ(S)$.

Proof.

Let $D = D^*(S)$. If A is a normal abelian subgroup of S, then $[D, A, A] = 1$ and so $[D, A] = 1$, whence $D \leqslant A$ if A is a maximal normal abelian subgroup of S. Now let $A \in \mathcal{A}(S)$ and set $Q = DA$. Then Q is metabelian. So by the Gillam-Mann Theorem, there exists $B \in \mathcal{A}(\langle A^Q \rangle) \subseteq \mathcal{A}(Q)$ with $B \lhd Q$. Then $D \leq B$ and so $Q = BA \leq \langle A^Q \rangle$. Thus, $A = Q$, whence $D \leq A$. It follows that $D \leq ZJ(S)$. $\square$

Final notes: Helmut Bender has observed that it is possible to use $D^*(S)$ to provide a recursive definition of a characteristic subgroup $D^{**}(S)$ with properties analogous to Glauberman's subgroup $K^\infty(S)$. Namely, let $D_1 = D^*(S)$, $D_2 = D^*(C_S(D_1)\ mod\ D_1)$, $D_3 = D^*(S_2\ mod\ D_2)$, with S_2 the stabilizer of the chain $1 < D_1 < D_2$. Continue ad infinitum. Then $D^{**}(S)$ is the union of all the D_i. [$K^\infty(S)$ has been used effectively by Paul Flavell, and he is considering possible applications of $D^{**}(S)$, or variants thereof.]

Also, Glauberman has raised the question whether a subgroup analogous to $D^*(S)$ can serve as a replacement for the 2-subgroup used by Stellmacher in his work on S_4-free groups.