When Artin groups are sufficiently large . . .

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0. Introduction

I'll talk about recent work with Derek Holt, some also with Laura Ciobanu; it's moved on a bit since I spoke on it at the webinar 11 months ago, and had some excellent questions from Bob and others.

For a family of Artin groups that I shall call the 'sufficiently large' groups, which includes all large type groups, triangle-free groups, and RAAGs we

- characterise the geodesic and shortlex minimal reps. of elements,
- have effective procedures to rewrite words to these forms, hence solve the word problem in these groups, which we prove shortlex automatic.
- We can apply our knowledge of geodesics to derive the rapid decay property for many of these groups, including all of extra-large type.
- For most of those groups we now deduce that Baum-Connes holds.

1. Notation

Let $G = \langle X \mid R \rangle$ be a finitely generated group.

A word over X is a string over $X^{\pm} := X \cup X^{-1}$, an element of $X^{\pm *}$.
w has string length $|w|$; w is **geodesic** if $u =_G w \Rightarrow |w| \leq |u|$.

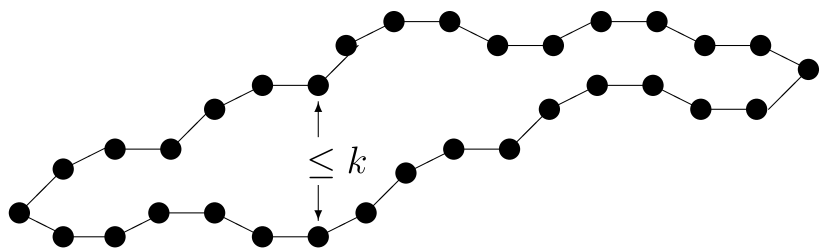
Given an order on $X^{\pm}$, in the **shortlex** word order $u <_{\mathrm{slex}} v$
if either $|u| < |v|$ or $|u| = |v|$ but u precedes v lexicographically.
So, where a $<$ b $<$ c $<$ d $< \cdots <$ y $<$ z,

$$\text{man} <_{\mathrm{slex}} \text{woman} <_{\mathrm{slex}} \text{women}.$$

w is **shortlex geodesic** if $w <_{\mathrm{slex}} u$ for all $u \neq w$ with $u =_G w$.

The set of all geodesics forms a **language** for G, i.e. a subset of $X^{\pm *}$ mapping onto G. The set of shortlex geodesics is a **normal form** for G, i.e. a bijective language.

A group G is **automatic** if it has a language L that is regular (can be recognised by a finite state automaton) and satisfies the following fellow traveller property:
$\exists k$, $v, w \in L$ k-fellow travel ($v \sim_k w$) when $v =_G w$ or $v =_G wx$.
G is **biautomatic** if also $v \sim_k xw$ when $v =_G xw$.



G is **shortlex automatic** if automatic wrt its shortlex geodesic representatives.

In an automatic group, any input word can be rewritten to its representative in L in quadratic time; in particular this solves the word problem.

Defining 'Rapid decay'

A finitely generated group G satisfies **rapid decay** if the operator norm $||.||_*$ for the group algebra $\mathbb{C}G$ is bounded by a constant multiple of the Sobolev norm $||.||_{2,r,\ell}$, for some length function ℓ on G.

We define, for $\phi \in \mathbb{C}G$,

$$||\phi||_* = \sup_{\psi \in \mathbb{C}G} \frac{||\phi * \psi||_2}{||\psi||_2}, \quad \phi * \psi(g) = \sum_{h \in G} \phi(h)\psi(h^{-1}g),$$

$$||\psi||_2 = \sqrt{\sum_{g \in G} |\psi(g)|^2}, \quad ||\phi||_{2,r,\ell} = \sqrt{\sum_{g \in G} |\phi(g)|^2 (1 + \ell(g))^{2r}}.$$

A function $\ell : G \to \mathbb{R}$ is a length function if

$$\ell(1_G) = 0, \quad \ell(g^{-1}) = \ell(g), \quad \ell(gh) \leq \ell(g) + \ell(h), \quad \forall g, h \in G.$$

Why rapid decay?

It relates to the Novikov & Baum-Connes conjectures, was used in Connes-Moscovic's proof of the Novikov conjecture for word hyperbolic groups.

Haagerup identified it as a property of free groups. Jolissaint proved it for word hyperbolic groups, and to be inherited by subgroups, free and direct products etc. It also holds for

- groups with appropriate actions on CAT(0) cube complexes (Chatterji, Ruane), so for Coxeter groups, RAAGs,
- mapping class groups (Behrstock, Minsky), so for braid groups (Artin groups of type A_n),
- groups that are hyperbolic relative to subgroups with RD (Drutu, Sapir) (in fact something weaker than relative hyperbolicity is enough).

Why Baum-Connes?

The Baum-Connes conjecture (1982) relates the K-theory of the reduced C^*-algebra $C_r^*(G)$ to the K^G-homology of the classifying space $\underline{E}G$ for proper G-actions, claiming that the assembly maps

$$\mu_i^G : RK_i^G(\underline{E}G) \to K_i(C_r^*(G)), \quad i = 0, 1,$$

are isomorphisms.

It's linked to many other conjectures. In particular when it holds, so does the Kadison-Kaplansky conjecture, i.e. the group ring $\mathbb{Q}G$ contains no non-trivial idempotents.

Baum-Connes is proved for a range of groups using a variety of techniques. In particular it's proved for Coxeter groups (Bozejko et al.,1988), RAAGs by results of Chatterji&Ruane and for braid groups (Schick, 2007), ...

Rapid decay and Baum-Connes

Theorem (Lafforgue, 1998) If G satisfies RD, and acts continuously, isometrically, properly and co-compactly on a CAT(0) metric space, then G satisfies Baum-Connes.

- Using Lafforgue's result Baum-Connes was deduced from RD for co-compact lattices in $SL_3(\mathbb{R})$,$SL_3(\mathbb{C})$; $SL_3(\mathbb{Q}_p)$; $SL_3(\mathbb{H})$ and $E_{6,-26}$ (Lafforgue; Ramagge, Robertson, Steger; Chatterji).
- We'll use Lafforgue's result to deduce Baum-Connes from rapid decay for many Artin groups.

2. Introducing Artin groups

An **Artin group** G is defined by a presentation

$$\langle x_1, x_2, \cdots, x_n \mid \overbrace{x_i x_j x_i \cdots}^{\mathsf{m_{ij}}} = \overbrace{x_j x_i x_j \cdots}^{\mathsf{m_{ij}}}, \quad i \neq j \in \{1, 2, \ldots, n\} \rangle$$
$$\mathsf{m_{ij}} \in \mathbb{N} \cup \infty, \mathsf{m_{ij}} \geq 2,$$

and naturally has a Coxeter group W as a quotient, whose presentation is derived by adding **involutionary relations** $x_i^2 = 1$ to the **braid relations** that define the Artin group.

The group is associated with a Coxeter matrix $(\mathsf{m_{ij}})$, and Coxeter graph Γ, with vertex set $X = \{x_i : i = 1, 2, \ldots, n\}$ (complete graph with $\{x_i, x_j\}$ labelled $\mathsf{m_{ij}}$).

We write $G = G(\Gamma)$, $W = W(\Gamma)$.

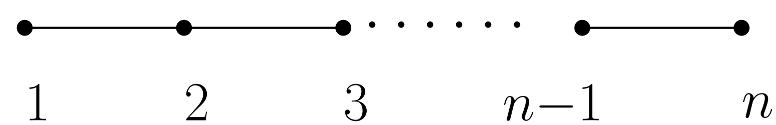
G has

- **dihedral type** if $|X| = 2$,
- **spherical (finite) type** if W is finite,
- **large type** if $m_{ij} \geq 3$, $\forall i, j$,
- **extra-large type** if $m_{ij} \geq 4$, $\forall i, j$,
- **right angled type** (RAAG) if $m_{ij} \in \{2, \infty\}$, $\forall i, j$.
- **is triangle-free** if $\nexists i, j, k$ s.t. $m_{ij}, m_{ik}, m_{jk} < \infty$.

Emil Artin gave his name to the groups, after introducing **braid groups**

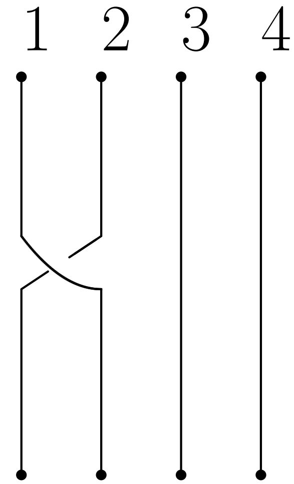
$$\langle x_1, x_2, \cdots, x_n \mid x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}, \quad x_i x_j = x_j x_i, \forall i+1 < j \rangle$$

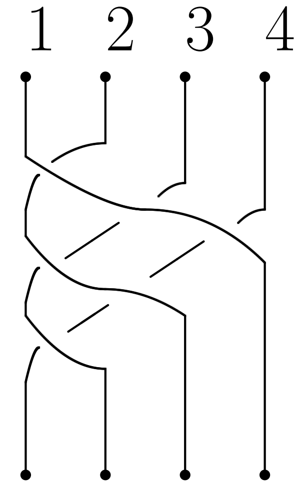
in 1926; these have type A_n.



$G(A_n)$ is faithfully represented as the group of braids on $n+1$ strings,

Generator x_1





Half twist, Δ,
$x_1 x_2 x_3 x_1 x_2 x_1$
$= (x_2 x_1 x_3)^2$

and this braid representation provides a natural solution to the word problem. Artin solved the word problem (1926), Chow (1948) computed the centre.

Artin groups of spherical type

Garside (1969) introduced a normal form for the braid group $G(A_n)$, describing it as the quotient of the braid monoid. Each braid is represented by a product $\Delta^k w$, $k \in \mathbb{Z}$. The word and conjugacy problem are easily soluble.

Garside saw similar behaviour in some other groups, including $G(C_3), G(H_3)$.

Brieskorn-Saito and Deligne (1972) extended Garside's results and more to Artin groups of **spherical** type. The normal forms make Artin groups of spherical type automatic (Thurston(1992) ; Charney (1992)).

For groups of FC type (any full subgraph of Coxeter graph without an ∞ edge has spherical type), Altobelli & Charney (1996,2000) defined normal forms to solve the word problem and prove asynchronous automaticity.

Artin groups: general results

Let $G = G(\Gamma)$ be an Artin group, $W = W(\Gamma)$.

- Each **parabolic** subgroup G_J is an Artin group over the full subgraph of Γ with vertex set J, i.e. **parabolic subgroups embed** (v.d.Lek (1983), Paris (1997)). Hence, since non-free dihedral Artin groups contain $\mathbb{Z}^2$, only free Artin groups are word hyperbolic.
- For $Y_W \subseteq \mathbb{C}^n$ and $X_W = Y_W/W$, $\pi_1(X_W) = G$ and X_W is homotopically equivalent to the **Deligne complex** of cosets of parabolic subgroups of finite type (Lek).
- The Artin monoid embeds (in natural embedding) in the group (Paris,2001).
- $\langle x_i^2 \rangle$ is free, modulo obvious commutation relations, Crisp&Paris (2000); this was Tits' conjecture, arising from Appel and Schupp's work.

Some questions for Artin groups

Let $G = G(\Gamma)$ be an Artin group.

- Does G have soluble word problem, soluble conjugacy problem, a good normal form? Is G automatic?
- What is the centre of G?
- Is G torsion-free?
- Is X_W a $K(\pi; 1)$ for G, that is, a complex with fundamental group G, all higher homotopy groups trivial? - the $K(\pi; 1)$ **conjecture** for G.
- Does G have rapid decay? Does it satisfy Baum-Connes?

The answers are known for various types of Artin groups G, for which either a good normal form or a good action of G is known.

Large and extra-large type

In 1983, 1984, Appel and Schupp defined and studied Artin groups G first of **extra-large**, and then of **large type**. They used small cancellation arguments (exploiting a weak form of relative hyperbolicity) to prove

- the word, generalised word and conjugacy problems are soluble in G,
- parabolic subgroups embed,
- G is torsion-free,
- subgroups $\langle x_i^2, 1 \leq i \leq n \rangle$ are free (this led to Tits' conjecture).

Automaticity of extra-large type groups (Peifer, 1996) and many large type groups (Brady&McCammond, 2000) were proved later using small cancellation arguments.

Beyond large type

An Artin group is

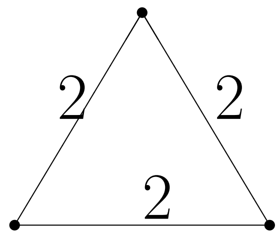
triangle-free if $\nexists$ distinct i, j, k, $\mathsf{m_{ij}}, \mathsf{m_{ik}}, \mathsf{m_{jk}} < \infty$. Pride (Invent. 1986) used small cancellation to solve word and conjugacy problems and verify Tits' conjecture, and that parabolic subgroups embed.

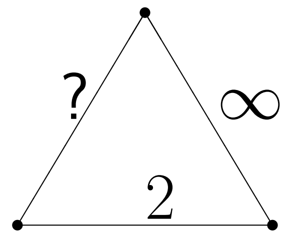
locally non-spherical if no 3-gen. parabolic subgroup has spherical type (includes triangle-free and large type). Chermak (J. Alg 1998) solved word problem in exp. time, proved parabolic subgroups embed.

right angled if $\forall i, j$, $\mathsf{m_{ij}} \in \{2, \infty\}$. There are clear algorithms to solve the word problem, and the groups are biautomatic (Hermiller&Meier 1995).

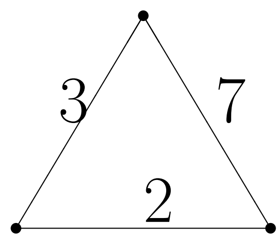
3. Artin groups of sufficiently large type

We say that G has **sufficiently large type** if each rank 3 subdiagram containing an edge with label 2 is one of the two following diagrams:





Large, extra-large, triangle-free and right angled groups all have sufficiently large type. But the left hand example is sufficiently large but not locally non-spherical, while this third diagram defines a group that is locally non-spherical but not sufficiently large:



Rewriting using basic moves

We prove that in sufficiently large Artin groups we can rewrite any input word w to a geodesic or shortlex representative as required using certain sequences of basic (τ-) moves in combination with free cancellation.

Each τ-move is applied to a subword which is maximal

either as a 2-generator subword on a **braid pair**

or as a subword in which all pairs of generators are **free** ($m_{ij} = \infty$) or **commuting** ($m_{ij} = 2$) and the first or last generator commutes with all others.

In the first case, the basic move is a **braid** (β-) move, in the second case a **commuting** (κ-) move.

We can reduce in at worst quadratic time (maybe in linear time?).

The two kinds of basic moves

The β-moves (together with free reduction) are precisely the moves we need to rewrite in the dihedral Artin groups we find as subgroups, the κ-moves in right-angled subgroups.

The sufficiently large condition ensures that combining them in appropriate sequences gives us an effective rewrite system in the full Artin group.

This strategy doesn't work without the sufficiently large condition; e.g. in braid groups.

We need a little more details to describe the two kinds of moves.

Recognising geodesics in dihedral Artin groups

To understand braid moves we look at dihedral Artin groups.

Write ${}_m(a,b)$ for the alternating product $aba\cdots$ of length m, and $(a,b)_m$ for $\cdots bab$. In this notation the dihedral Artin group DA_m has presentation

$$\mathrm{DA}_m = \langle a, b \mid {}_m(a,b) = {}_m(b,a) \rangle.$$

For a word w over a, b,

- we define $p(w)$ to be the minimum of m and the maximal length of a positive alternating subword in w,
- we define $n(w)$ to be the minimum of m and the maximal length of a negative alternating subword in w.

e.g. for $w = ababa^{-1}b^{-1}$, in DA_3, we have $p(w) = 3$, $n(w) = 2$.

Our rewrite strategy is motivated by the following result.

Theorem (Mairesse,Mathéus, 2006)

In a dihedral Artin group DA_m (any $m \in \mathbb{Z}$), a word w is geodesic iff $p(w) + n(w) \leq m$, and is the unique geodesic representative of the element it represents if $p(w) + n(w) < m$.

Rewriting in a dihedral Artin group

The **Garside element** Δ is represented by ${}_m(a,b)$; it's central if m is even, otherwise $a^\Delta = b$. We define the permutation δ of $\{a,b,a^{-1},b^{-1}\}^*$ to be that induced by the permutation $a \mapsto a^\Delta$, $b \mapsto b^\Delta$ of the generators.

We call a geodesic word v **critical** if it has the form

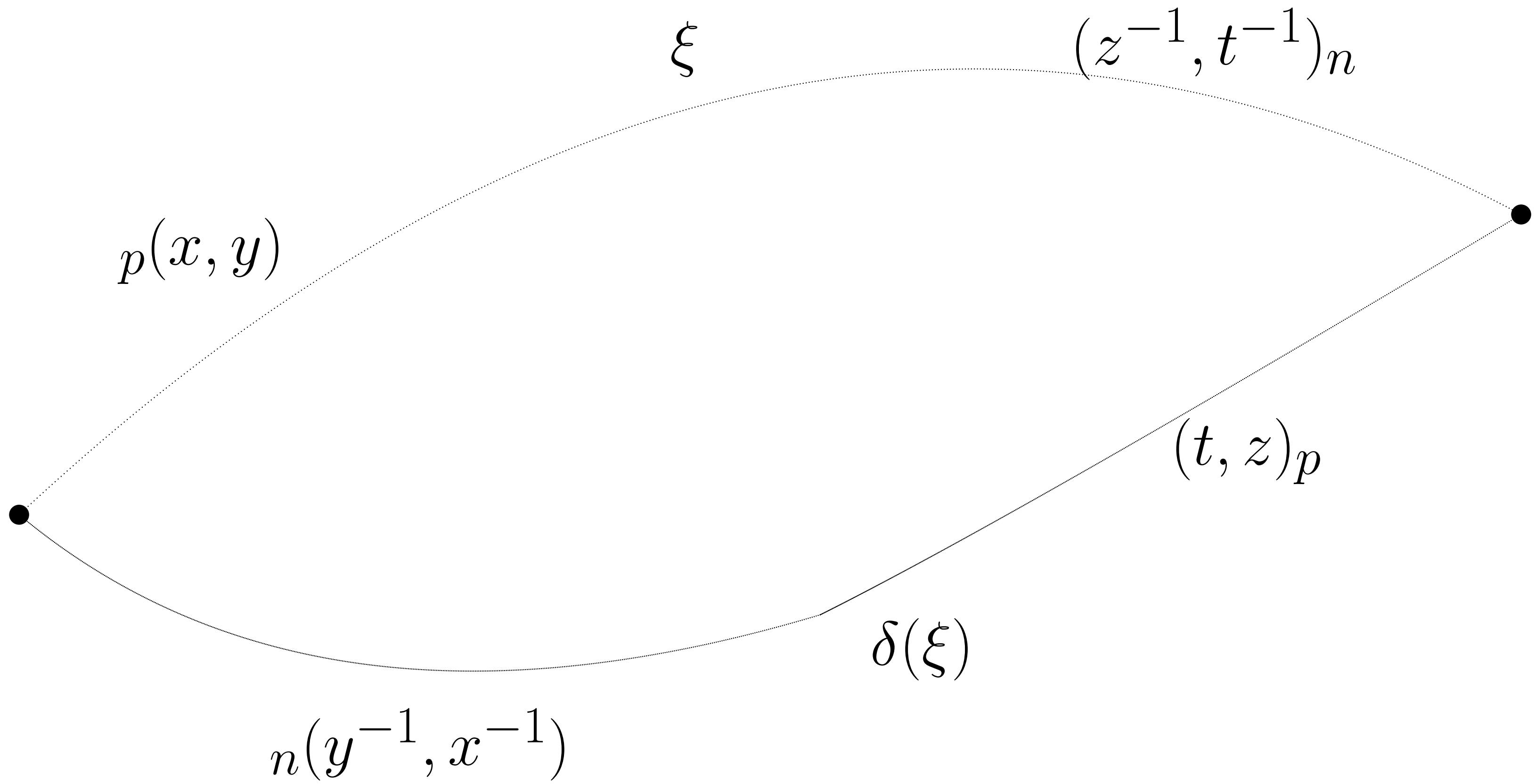
$$w = {}_p(x,y)\xi(z^{-1},t^{-1})_n \quad \text{or} \quad \tau(w) = {}_n(y^{-1},x^{-1})\delta(\xi)(t,z)_p,$$

where $\{x,y\} = \{z,t\} = \{a,b\}$, $p = p(v)$, $n = n(v)$, $p+n = m$.
(We add an extra condition when p or n is zero.)

Such words fall into pairs of words related by the involution τ, representing the same element, beginning and ending with different generators.

We call application of τ to a critical subword of w a **β-move** on w.

Two critical words related by a β-move.



Example:

In $G = \mathrm{DA}_3$, aba^{-1} is critical, and $\tau(aba^{-1}) = b^{-1}ab$.

Applying that β-move to the critical subword aba^{-1} in the non-geodesic word $w = ababa^{-1}b^{-1}$, we see that

$$w = ab(aba^{-1})b^{-1} \to ab(b^{-1}ab)b^{-1},$$

and the final word freely reduces to aa, which is geodesic.

It is straightforward to derive the following from Mathéus and Mairesse' criterion for geodesics.

Theorem (Holt, Rees, PLMS 2012)

If w is freely reduced over $\{a, b\}$ then w is shortlex minimal in DA_m unless it can be written as $w_1w_2w_3$ where w_2 is critical, and $w' = w_1\tau(w_2)w_3$ is either less than w lexicographically or not freely reduced.

Definining κ-moves to deal with commuting pairs.

In our 2012 paper, we proved that we could rewrite in large type Artin groups using sequences of β-moves applied to 2-generator subwords. But β-moves alone aren't enough to achieve all the reduction we need when some pairs of generators commute; we need κ-moves to deal with those pairs.

Suppose that

- u is a word over generators that pairwise commute,
- a is a generator commuting with all letters in u,
- neither a nor a^{-1} is in u,

then we call ua and au **right** and **left κ-critical**.

A κ-move exchanges κ-critical subwords ua and au within a word.

Applying sequences of moves in sufficiently large Artin groups.

When we have more than 2 generators, we reduce to shortlex minimal form using sequences of β-moves and κ-moves, each applied to a subword either on a braid pair of generators or on a set of generators any two forming either a commuting or a free pair.

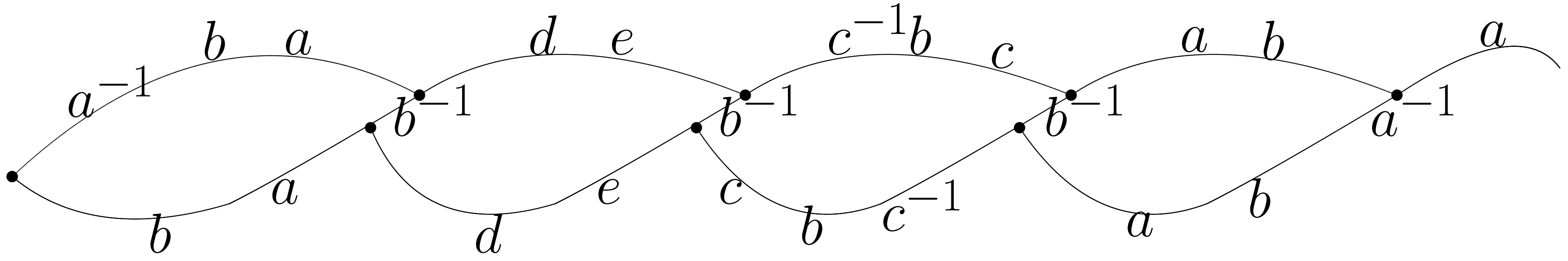
We use sequences that move either to the right or to the left.

Example:

$$G = \langle a, b, c, d, e \mid aba = bab,\ aca = cac,\ bcbc = cbcb,\ bd = db,\ be = eb \rangle$$

First consider $w = a^{-1}badec^{-1}bcaba$. The 2 generator subwords are all geodesic in the dihedral Artin subgroups (in fact also in G). The two maximal a, b subwords are β-critical in DA_3. Applying a β-move to the leftmost critical subword creates a κ-critical subword bde. When bde is transformed to deb, we see a new β-subword.

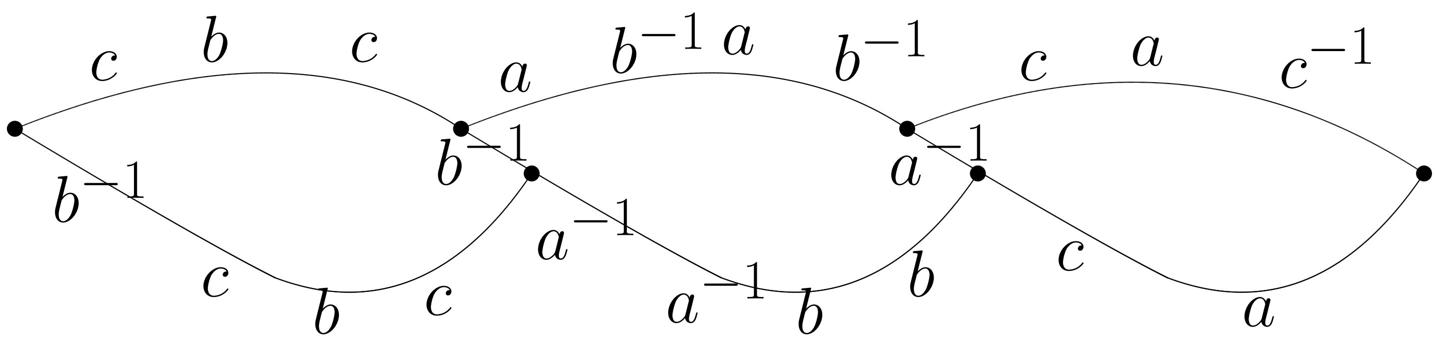
In fact, a sequence of 3 more β-moves transforms w to a word that is not freely reduced. The free reduction is then $badedcbc^{-1}ab$,



Reducing $a^{-1}badec^{-1}bcaba$.

We call a sequence of basic moves like this a **rightward length reducing sequence.**

Now consider $w = cbcab^{-1}ab^{-1}cac^{-1}$, in which cac^{-1} is critical. Applying a β-move to this critical subword creates a new critical subword, $ab^{-1}ab^{-1}a^{-1}$, to which we can then apply a further β-move. After one more β-move, w is transformed to the word $w' = b^{-1}cbca^{-1}a^{-1}bbca$, of the same length as w but preceding w lexicographically.



We call a sequence like this a **leftward lex reducing sequence.**

Proposition (Holt, Rees)

Let $G = \langle X \rangle$ be a sufficiently large Artin group, $a \in X^{\pm 1}$.

(i) If w is geodesic but wa is not, then w admits a rightward length reducing sequence to a geodesic.

(i) If w is shortlex geodesic but wa is not, then w admits either a rightward length reducing sequence or a leftward lex reducing sequence, for which the resultant word is shortlex geodesic.

(iii) If v, w are distinct geodesics representing the same element, ending with distinct letters a, b, then $a, \neq b^{-1}$, all other geodesics representing the same element end in either a or b, and a single rightward sequence of basic moves transforms v to a word ending in b.

These results form the basis of the proof of the following theorem.

Theorem A (Holt, Rees) PLMS 2012 + preprint 2012

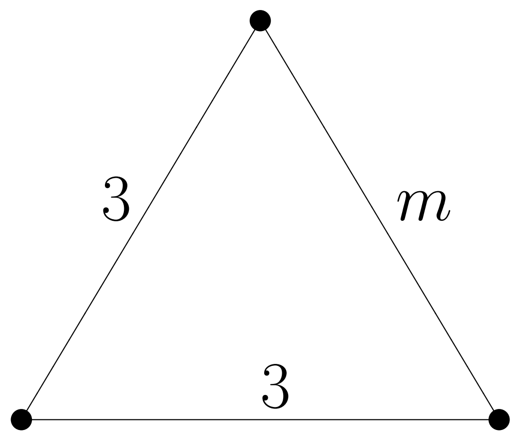
Sufficiently large Artin groups are shortlex automatic, with regular geodesics.

We identify the set of shortlex geodesics as the set of words admitting neither a rightward length reducing nor a leftward lex reducing reduction.

The regularity of geodesics is proved by verifying Neumann&Shapiro's 'Falsification by fellow traveller' condition; this follows immediately from part (iii) of the proposition.

Theorem B (Ciobanu, Holt, Rees, preprint 2012)

Let $G(\Gamma)$ be an Artin group of large type for which the unlabelled graph FΓ formed by omitting ∞ edges from Γ contains no subgraph



with $m \in \mathbb{N}$. Then G satisfies rapid decay.

In particular all Artin groups of extra-large type satisfy rapid decay.

Verifying rapid decay

We verify rapid decay by verifying the property

$$\forall \phi, \psi \in \mathbb{C}G, k, l, m \in \mathbb{N},$$
$$|k-l| \leq m \leq k+l, \Rightarrow ||(\phi_k * \psi_l)_m||_2 \leq P(k)||\phi_k||_2||\psi_l||_2.$$

where ϕ_k means the restriction of ϕ to elements of word length k, etc.

We do this by analysing the factorisations of elements of length m as products of elements g_1, g_2 of length k, l; both geodesic $(m = k+l)$ and non-geodesic $(m < k+l)$ factorisations.

It would be enough to have

- a polynomial bound (in k) on the number of geodesic factorisations,
- a further condition connecting non-geodesic and geodesic factorisations.

But we don't have that!

Even in a dihedral Artin group DA_m, the powers Δ^n of $\Delta = {}_m(a, b)$ have too many divisors. Every positive word of length n is a left divisor of Δ^n; for $m \geq 3$ these words represent exponentially many elements.

So we have to try a bit harder.

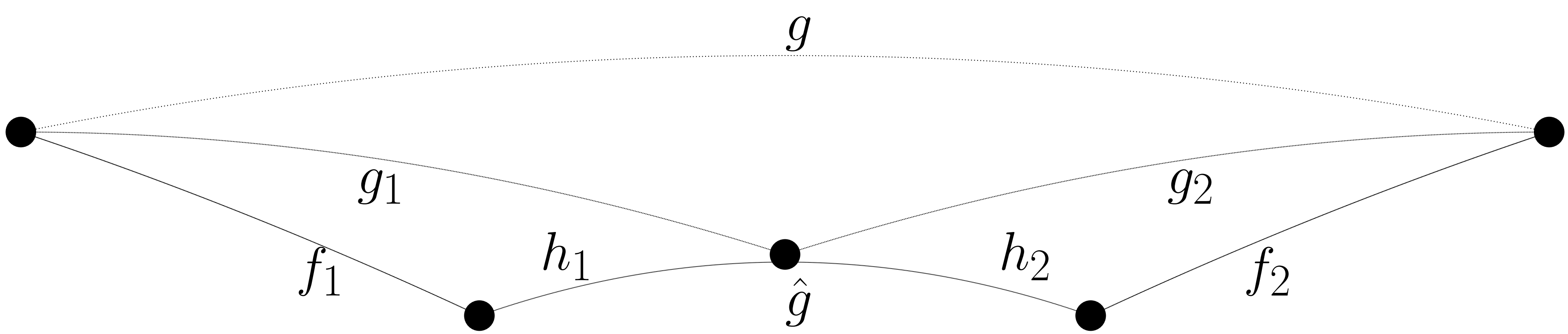
We achieve what we need by relating each geodesic or non-geodesic factorisation $g_1 g_2$ of g to another factorisation, using a **merging process**.

We can control that process and certain associated sets of factorisations using polynomial bounds.

The merging process

Given a geodesic or non-geodesic factorisation $g_1 g_2$ of g, the merging process derives a particular factorisation $f_1 \hat{g} f_2$, for which the factorisations of g_1 as $f_1 h_1$ and g_2 as $h_2 f_2$ are **permissible**. Then $\hat{g} = h_1 h_2$.

Essentially (with some modification) our set $\mathcal{P}$ of permissible factorisations to the set of geodesic factorisations that don't split the Garside Δ_{ij} element of any dihedral subgroup; and $\hat{g} = \Delta_{ij}^r$, for some i, j, r.



The conditions D1, D2 on $\mathcal{P}$ that we need for rapid decay

For $\mathcal{P}_{k,l}(g) := \{(g_1, g_2) : |g_1| = k, |g_2| = l, (g_1, g_2) \in \mathcal{P}, g_1 g_2 =_G g\}$,

D1: $\exists$ poly. $P_1(x)$: $\sup_{|g|=k+l} |\mathcal{P}_{k,l}(g)| \leq P_1(\bar{k} := \min(k, l))$.

D2: $\exists$ sets $S(g, k, l) \subset G^3$ ($\forall g, k, l$), and polys. $P_2(x), P_3(x)$, $K > 0$:

If $g = g_1 g_2$ with $|g_1| = k, |g_2| = l$,
then $\exists (f_1, \hat{g}, f_2) \in S(g, k, l)$, s.t. $g = f_1 \hat{g} f_2$ and $\hat{g} = h_1 h_2$,
where $(f_1, h_1) \in \mathcal{P}_{k-p_1, p_1}(g_1)$, $(h_2, f_2) \in \mathcal{P}_{p_2, l-p_2}(g_2)$, $p_1, p_2 \leq K\bar{k}$.

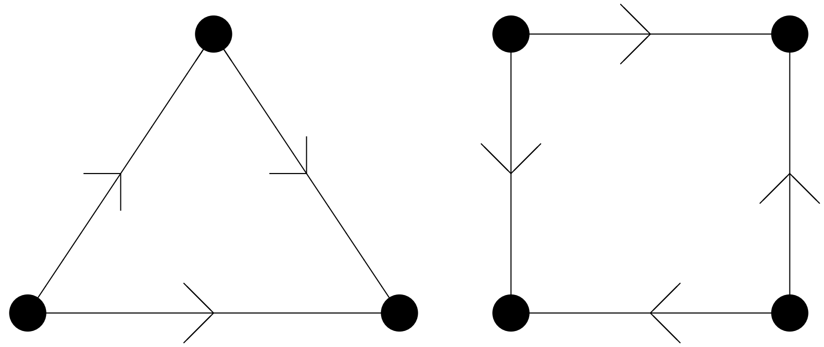
(a) for all g, k, l, $|S(g, k, l)| \leq P_2(\bar{k})$,

(b) $|\{\hat{g} : \exists g, \ (f_1, \hat{g}, f_2) \in S(g, k, l)\}| \leq P_3(\bar{k})$.

Theorem C (Ciobanu, Holt, Rees, preprint 2012)

An Artin group $G = G(\Gamma)$ satisfying the hypotheses of Theorem B must satisfy the Baum-Connes conjecture if one of the following holds.

1. G is 3-generated.
2. FΓ is triangle-free.
3. FΓ can be oriented to exclude both the following as induced subgraphs.



In particular, any large type Artin group $G(\Gamma)$ for which FΓ is triangle-free satisfies the Baum-Connes conjecture.

Finding an appropriate CAT(0) action

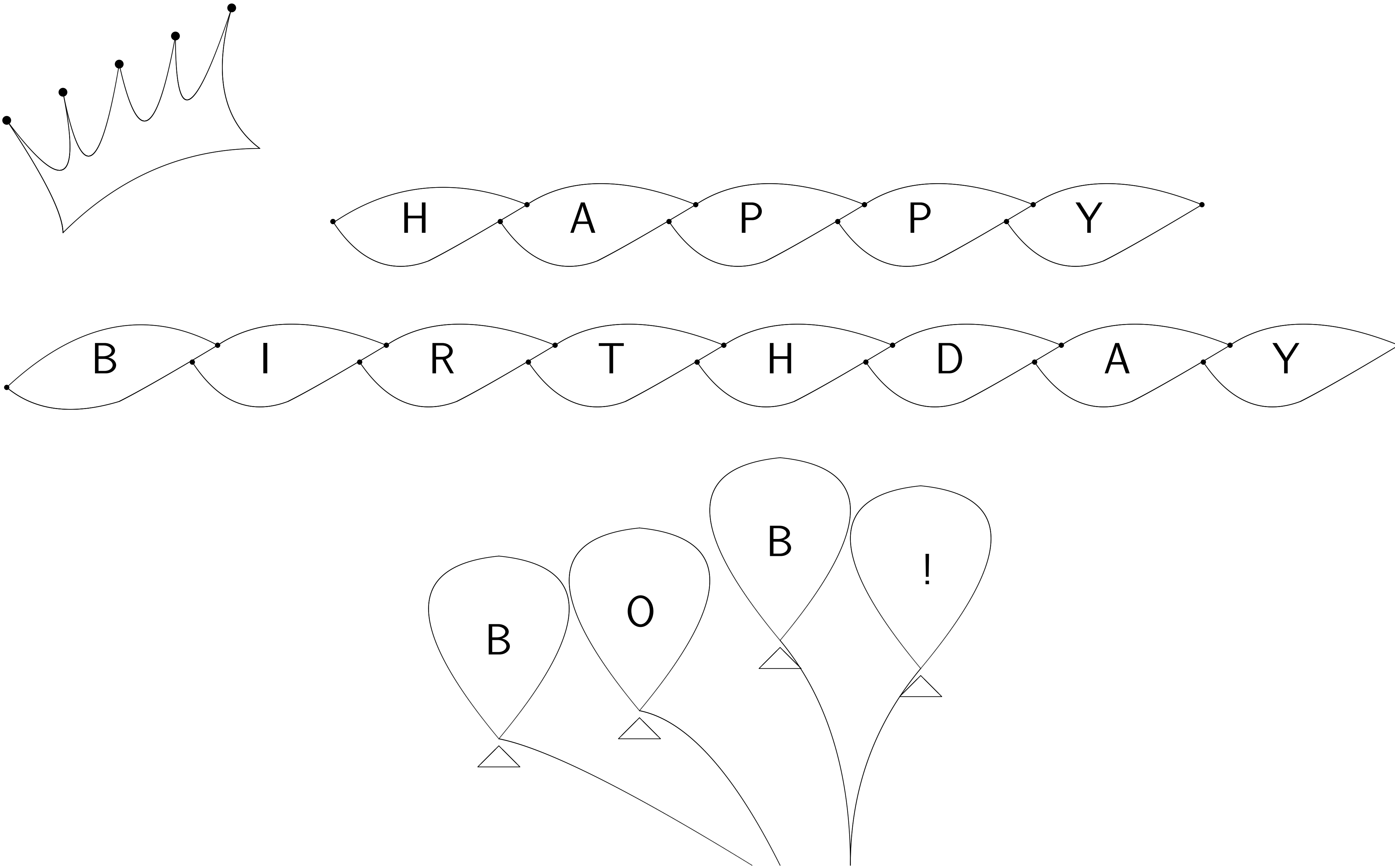
We find an appropriate action of G on a CAT(0) metric using work of Brady&McCammond (2000).

- G as above has a presentation with relators all of length 3.
- We define a metric on the presentation complex K to make each triangular 2-cell either equilateral Euclidean of side length 1, or isosceles right-angled Euclidean, with two sides of length 1; K is locally CAT(0).
- G acts naturally by isometries on both K and its universal cover $\tilde{K}$ (the Cayley complex); the action on $\tilde{K}$ is proper, continuous and co-compact.
- $\tilde{K}$ is locally CAT(0), and simply connected, so CAT(0).

Application of Lafforgue's result now gives us Baum-Connes.

Further questions

- Can we solve the word problem in linear time in Artin groups of large or at least extra-large type?
- Is there a good solution to the conjugacy problem in these groups?
- Do these groups have a good biautomatic structure?
- Can we extend the proof of rapid decay to cover more Artin groups? (only previously known for braid groups and RAAGs.) or indeed other classes of groups?
- How much does our proof of rapid decay share with Drutu and Sapir's proof for relatively hyperbolic groups.
- Can we find appropriate actions to deduce Baum-Connes for further Artin groups?


HAPPY
BIRTHDAY
BOB!