Groups generalizing a Theorem of Benjamin Baumslag

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- RESIDUAL PROPERTIES

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- Let $\mathcal{X}$ be a class of groups. Then a group G is **residually** $\mathcal{X}$ if given any nontrivial element $g \in G$ there is a homomorphism $\phi : G \to H$ where H is a group in $\mathcal{X}$ such that $\phi(g) \neq 1$. A group G is **fully residually** $\mathcal{X}$ if given finitely many nontrivial elements $g_1, ..., g_n$ in G there is a homomorphism $\phi : G \to H$, where H is a group in $\mathcal{X}$, such that $\phi(g_i) \neq 1$ for all $i = 1, ..., n$.

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- If $\mathcal{X}$ is the class of nonabelian free groups then we have residually free and fully residually free respectively.

Benjamin Baumslag's Theorem

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- More generally

Theorem

If F is a free group $\overline{F}$ a copy of F, $U \in F$ and $\overline{U}$ its counterpart in $\overline{F}$ then the group $G = F \star_{U=\overline{U}} \overline{F}$ with U not a proper power in F is residually free. G is called a **Baumslag double**.

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- group G is **commutative transitive** or **CT** if commutativity is transitive on the set of nontrivial elements of G. That is if $[x, y] = 1$ and $[y, z] = 1$ for nontrivial elements $x, y, z \in G$ then $[x, z] = 1$.

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Theorem

Suppose G is residually free. Then the following are equivalent:
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Universal Freeness

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- A **universal sentence** in the language of group theory is a first order sentence using only universal quantifiers.
- The **universal theory** of a group G consists of all universal sentences true in G.
- All nonabelian free groups share the same universal theory.
- A group G is called **universally free** if it shares the same universal theory as the class of nonabelian free groups.

Equivalence of Fully Residual Freeness and Universal Freeness

- Gaglione and Spellman [GS] and independently Remeslennikov [Re] extended B.Baumslag's Theorem. This is really one of the cornerstones of the proof, by Kharlampovich and Myasnikov and independently by Sela, of the Tarksi problems

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The Class of $B\mathcal{X}$ Groups

- We want to consider classes of groups which satisfy Benjamin Baumslag's Theorem. For a class of groups, $\mathcal{X}$, we call this property $B\mathcal{X}$ and we show it is quite extensive. Motivated in part by work of Kharlampovich and Myasnikov who showed that much of the development of the structure theory for fully residually free groups can be extended to fully residually torsion-free hyperbolic groups.

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- **Definition**

  A class of groups $\mathcal{X}$ which are all CT satisfies $B\mathcal{X}$ if a group G is fully residually $\mathcal{X}$ if and only if G is residually $\mathcal{X}$ and CT.

The Class of $B\mathcal{X}$ Groups

- With this definition B. Baumslag's original theorem says that the class of nonabelian free groups $\mathcal{F}$ satisfies $B\mathcal{F}$.

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- With this definition B. Baumslag's original theorem says that the class of nonabelian free groups $\mathcal{F}$ satisfies $B\mathcal{F}$.
- we prove that a class of CT groups $\mathcal{X}$ satisfies $B\mathcal{X}$ under very mild conditions and hence the classes of groups for which this is true is quite extensive.

Some Background Information

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Lemma

Every CSA group is CT.

- The converse is not true. However, as we will show, in the presence of property $B\mathcal{X}$ CSA is equivalent to CT.

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- Recall that the infinite dihedral group has the presentation $D = < x, y; x^2 = y^2 = 1 >$. Then $xxyx^{-1} = yxyy^{-1} = yx = (xy)^{-1}$ and hence D is not CSA. We need this in considering classes satisfying $B\mathcal{X}$.

Lemma

If G is a group that contains the infinite dihedral group D then G is not CSA.

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- Two elements $a, b \in G$ are in **power relation** to each other if there exists an $x \in G \setminus \{1\}$ with $a = x^n, b = x^m$ for some $n, m \in \mathbb{Z}$.

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- A group G is **power commutative** if $[x, y^n] = 1$ implies that $[x, y] = 1$ whenever $y^n \neq 1$.
- Two elements $a, b \in G$ are in **power relation** to each other if there exists an $x \in G \setminus \{1\}$ with $a = x^n, b = x^m$ for some $n, m \in \mathbb{Z}$.
- G is **power transitive** or **PT** if this relation is transitive on nontrivial elements.

Some Background Informationy

- A group G is an **RG- group** or **Restricted-Gromov group** if for any $g, h \in G$ either the subgroup $< g, h >$ is cyclic or there exists a positive integer t with $g^t \neq 1, h^t \neq 1$ and $< g^t, h^t > = < g^t > \star < h^t >$. Note that torsion-free hyperbolic groups are RG-groups.

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- The following ideas are crucial in handling property $B\mathcal{X}$. A group G is **ALC** if every abelian subgroup is locally cyclic. This is of course the case in free groups. A group G is **ANC** if every abelian normal subgroup is contained in the center of G. Finally a group is **NID** if it does not contain a copy of the infinite dihedral group $\mathbb{Z}_2 \star \mathbb{Z}_2$. If G has only odd torsion then clearly G is NID.

The Basic $B\mathcal{X}$ Theorem

- We show that if $\mathcal{X}$ is any class where the nonabelian groups are CSA groups then it satisfies $B\mathcal{X}$

Theorem

Let $\mathcal{X}$ be a class of groups such that each nonabelian $H \in \mathcal{X}$ is CSA. Let G be a nonabelian and residually $\mathcal{X}$ group. Then the following are equivalent

(1) G is fully residually $\mathcal{X}$

(2) G is CSA

(3) G is CT

Therefore the class $\mathcal{X}$ has the property $B\mathcal{X}$.

The Basic $B\mathcal{X}$ Theorem

- To prove this we need the following lemmas.

Lemma

CSA implies ANC, that is, if G is a CSA group then G is ANC. Hence if a class of groups satisfies CSA then it also satisfies ANC.

Lemma

Let $\mathcal{X}$ be a class of groups such that each nonabelian $H \in \mathcal{X}$ is CSA. Let G be nonabelian and residually $\mathcal{X}$. Let A be an abelian normal subgroup of G. Then A is contained in the center of G. In particular if G is CT then A must be trivial.

Classes of Groups Satisfying $B\mathcal{X}$

Theorem

Each of the following classes satisfies $B\mathcal{X}$:

(1) The class of nonabelian free groups.

(2) The class of noncyclic torsion-free hyperbolic groups (see [FR]).

(3) The class of noncyclic one-relator groups with only odd torsion (see [FR]).

(4) The class of cocompact Fuchsian groups with only odd torsion.

(5) The class of noncyclic groups acting freely on Λ-trees where Λ is an ordered abelian group (see [H]).

(6) The class of noncylic free products of cyclics with only odd torsion.

Classes of Groups Satisfying $B\mathcal{X}$

Theorem

(7) The class of noncyclic torsion-free RG-groups (see[FMgrRR] and [AgrRR]).

(8) The class of conjugacy pinched one-relator groups of the following form

$$G = < F, t; tut^{-1} = v >$$

where F is a free group of rank $n \geq 1$ and u, v are nontrivial elements of F that are not proper powers in F and for which $< u > \cap x < v > x^{-1} = \{1\}$ for all $x \in F$.

The theorem follows from the fact that each of these classes has the property that each nonabelian group in them is CSA.

Classes of $B\mathcal{X}$ Groups

- Since CSA always implies CT we have the following corollary.

Corollary

Let $\mathcal{X}$ be a class of CSA groups. Then if G is a nonabelian residually $\mathcal{X}$ group then CT is equivalent to CSA.

Classes of $B\mathcal{X}$ Groups

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- Commutative transitivity (CT) has been shown to be equivalent to many other properties (see[AgrRR]) under the additional condition that abelian subgroups are locally cyclic (ALC) Hence we get the corollary.

Classes of $B\mathcal{X}$ Groups

Corollary

Let $\mathcal{X}$ be a class of groups such that each nonabelian $H \in \mathcal{X}$ is CSA. Let $\mathcal{Y}$ be the subclass of $\mathcal{X}$ consisting of those groups in $\mathcal{X}$ which are ALC. Let G be a nonabelian residually $\mathcal{Y}$ group which is ALC and has trivial center. Then the following are equivalent.

(1) G is fully residually $\mathcal{Y}$.
(2) G is CSA.
(3) G is CT.
(4) G is PC.
(5) G is PT.

Some Closure Properties

- Here we show that classes of groups satisfying $B\mathcal{X}$ are closed under certain amalgam constructions. This follows from the fact that CT and hence CSA is preserved under such constructions.

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- **Theorem**

  Let $\mathcal{X}$ be a class of CSA groups closed under free products with malnormal amalgamated subgroups, in particular under free products. Let G_1 and G_2 be nonabelian, residually $\mathcal{X}$ and CSA. Then $G_1 \star G_2$ and $G_1 \star_A G_2$ where A is malnormal in G_1 and G_2 are both residually $\mathcal{X}$ and CSA.

Some Closure Properties

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- **Corollary**

  Let $\mathcal{X}$ be a class of nonabelian CSA groups. Hence $\mathcal{X}$ satisfies $B\mathcal{X}$. Let G_1 and G_2 be fully residually $\mathcal{X}$ groups and $G = G_1 \star G_2$. Then G is fully residually $\mathcal{X}$.

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- **Corollary**

  Let $\mathcal{X}$ be a class of nonabelian CSA groups. Hence $\mathcal{X}$ satisfies $B\mathcal{X}$. Consider the class of groups which are free products $G_1 \star G_2$ of groups from $\mathcal{X}$. Then this class also satisfies $B\mathcal{X}$.

Some Closure Properties

Corollary

Let $\mathcal{X}$ be a class of nonabelian CSA groups. Hence $\mathcal{X}$ satisfies $B\mathcal{X}$.Consider the class of groups which have the form $G_1 \star_A G_2$ where G_1, G_2 are groups from $\mathcal{X}$ and A is malnormal in G_1 and G_2. Then this class also satisfies $B\mathcal{X}$.

The Relationship To Universally $\mathcal{X}$

- We now consider the equivalence with universally $\mathcal{X}$ groups. We say that a group G is **universally** $\mathcal{X}$ if it satisfies the universal theory of a countable nonabelian group from $\mathcal{X}$. Recall that if $\mathcal{X}$ is the class of free groups we have the following equivalences

Theorem

Suppose G is residually free. Then the following are equivalent:
(1) G is fully residually free,
(2) G is commutative transitive,
(3) G is universally free.

The Big Powers Condition

- Our Basic $B\mathcal{X}$ theorem shows the equivalence of (1) and (2) for any class $\mathcal{X}$ of CSA groups. To prove an equivalence with (3) we need the **big powers condition**. This was introduced originally by G.Baumslag in [GB].

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- **Definition**

  Let G be a group and $u = (u_1, ..., u_k)$ be a sequence of nontrivial elements of G. Then

  (1) u is **generic** if neighboring elements in u do not commute, that is $[u_1, u_{i+1}] \neq 1$ for every $i \in \{1, ..., k\}$.

  (2) u is **independent** if there exists an $n = n(u) \in \mathbb{N}$ such that for any $\alpha_1, ..., \alpha_k \geq n$ we have $u_1^{\alpha_1} \cdots u_k^{\alpha_k} \neq 1$.

  (3) A group satisfies the **big powers condition** or **BP** if every generic sequence in G is independent. We call such groups BP-groups.

The Big Powers Condition

- G. Baumlsag proved that free groups are BP-groups [GB] while Olshansky [O] showed that torsion-free hyperbolic groups are BP-groups. For BP groups the following results are known.

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- **Lemma (KMS)**

  A subgroup of a BP-group is itself a BP-group.

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- **Lemma (KMS)**

  A subgroup of a BP-group is itself a BP-group.

- **Lemma (O)**

  Every torsion-free hyperbolic group is a BP-group

  A stronger version of this lemma for relatively hyperbolic groups is given in [KM].

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- **Lemma**

  A free product of CSA BP-groups is also a BP-group

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- **Lemma**

  Let $G = F_1 \underset{u=v}{\star} F_2$ *where* F_1, F_2 *are finitely generated free groups and* u, v *are nontrivial elements of* F_1, F_2 *respectively with not both proper powers. Then* G *is a CSA BP-group.*

Universally $\mathcal{X}$ Groups

We now consider a class of groups $\mathcal{Z}$ in which each finitely generated nonabelian group G in $\mathcal{Z}$ is CSA and BP. Reinterpreting a result in [BMR 1] and [BMR 2] (see also [KM]) we obtain the following using the same proof utilizing the BP condition.

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Theorem

Let $\mathcal{Z}$ be a class of finitely presented groups such that each nonabelian $H \in \mathcal{Z}$ is CSA and BP. Let $H \in \mathcal{Z}$ and G a nonabelian group. Then the following are equivalent.

(1) G is fully residually H,

(2) G is universally equivalent to H.

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Theorem

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(1) G is fully residually $\mathcal{Z}$,
(2) G is CSA,
(3) G is CT,
(4) G is universally $\mathcal{Z}$.

Universally $\mathcal{X}$ Groups

- Equivalences (1),(2),(3) are from the basic $B\mathcal{X}$ Theorem and the fact that $\mathcal{Z}$ consists of CSA groups while the equivalence with (4) follows from the BP condition. As before, in the case of ALC groups there are additional equivalences.

Universally $\mathcal{X}$ Groups

- Equivalences (1),(2),(3) are from the basic $B\mathcal{X}$ Theorem and the fact that $\mathcal{Z}$ consists of CSA groups while the equivalence with (4) follows from the BP condition. As before, in the case of ALC groups there are additional equivalences.

- **Corollary**

Let $\mathcal{C}$ be a class of finitely presented groups such that each nonabelian group in $\mathcal{C}$ is CSA, ALC and BP. Let G be a nonabelian residually $\mathcal{C}$ group which is ALC and has a trivial center. Then the following are equivalent.

(1) G is fully residually $\mathcal{C}$,
(2) G is CSA,
(3) G is CT,
(4) G is PC,
(5) G is PT.

Universally $\mathcal{X}$ Groups

- As with classes of CSA groups, the subclasses of these which are also BP are quite extensive.

Universally $\mathcal{X}$ Groups

- As with classes of CSA groups, the subclasses of these which are also BP are quite extensive.

- **Theorem**

  The following classes of groups consist of groups that are both CSA and BP. Hence the equivalences in th above theorem hold for any group G which is residually in any of these classes.

  (1) The class of nonabelian finitely generated free groups,

  (2) The class of noncyclic torsion-free hyperbolic groups,

  (3) The class of noncyclic cyclically pinched one-relator groups $F_1 \underset{u=v}{\star} F_2$ with not both u, v proper powers in their respective finitely generated free groups (This is a subclass of (2)).

  (4) The class of free products $G_1 \star G_2$ where both G_1 and G_2 are nonabelian, finitely presented and satisfy CSA and BP.