Acylindrically hyperbolic groups

Denis Osin
Vanderbilt University

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Theorem (O., 2013)

$$\mathcal{B} = \mathcal{C}_{geom} = \mathcal{BF} = \mathcal{S} = \mathcal{HE} = \left\{ \begin{array}{c} \text{acylindrically} \\ \text{hyperbolic} \\ \text{groups} \end{array} \right\}$$

Group actions on hyperbolic spaces

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- $\Lambda(G)$ denotes the limit set of G on ∂S; for $g \in G$, $\Lambda(g) := \Lambda(\langle g \rangle)$.
- $g \in G$ is elliptic (resp., parabolic, loxodromic) if $|\Lambda(g)| = 0$ (resp., 1, 2).
- Loxodromic elements $f, g \in G$ are independent if $\Lambda(f) \cap \Lambda(g) = \emptyset$.

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3. *(Lineal)* $|\Lambda(G)| = 2 \Leftrightarrow$ *G contains loxodromic elements and for any loxodromic $g \in G$ we have $\Lambda(g) = \Lambda(G)$.*
4. *(Non-elementary)* $|\Lambda(G)| = \infty$. *Then G always contains loxodromic elements. This case breaks in two subcases:*
   a) *(Quasi-Parabolic) Any two loxodromic $g, h \in G$ are dependent.*
   b) *(General) G contains infinitely many independent loxodromic elements.*

Acylindrical actions

The action of G on S is acylindrical if for every $\varepsilon > 0$ there exist $R, N > 0$ such that for any two points $x, y \in S$ with $\mathrm{d}(x, y) \geq R$, there are at most N elements $g \in G$ satisfying

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Theorem (O., 2013)

Let G be a group acting acylindrically on a hyperbolic space. Then G satisfies exactly one of the following three conditions.

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Corollary (Bowditch)

Every element of G is either elliptic or hyperbolic.

Acylindrically hyperbolic groups

Theorem (O., 2013)

For any group G, the following conditions are equivalent.

- *(AH_1)* *There exists a generating set X of G such that the corresponding Cayley graph $\Gamma(G,X)$ is hyperbolic, $|\partial\Gamma(G,X)| > 2$, and the natural action of G on $\Gamma(G,X)$ is acylindrical.*
- *(AH_2)* *G admits a non-elementary acylindrical action on a hyperbolic space.*
- *(AH_3)* *G is not virtually cyclic and admits an action on a hyperbolic space such that at least one element of G is loxodromic and satisfies the Bestvina-Fujiwara WPD condition.*
- *(AH_4)* *G contains a proper infinite hyperbolically embedded subgroup.*

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Definition

A group G is acylindrically hyperbolic if it satisfies either of (AH$_1$)–(AH$_4$)

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- (Dahmani – Guirardel – O.) $\mathrm{Bir}(\mathbb{P}^2)$.
- (Hamenstädt) If G acts properly on a hyperbolic space of uniformly bounded geometry, then G is either virtually nilpotent or acylindrically hyperbolic.

3-manifold groups

Theorem (Minasyan – O., 2013)

Let M be a compact 3-manifold, $G \leq \pi_1(M)$. Then exactly one of the following holds.

(a) *G is acylindrically hyperbolic.*

(b) *G has an infinite cyclic normal subgroup Z and G/Z is acylindrically hyperbolic.*

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Corollary

Let M be a compact, orientable, irreducible 3-manifold. Then either $\pi_1(G)$ is acylindrically hyperbolic or M is Seifert fibered.

Groups acting on hyperbolic spaces

Theorem (Minasyan-O., 2013)

Suppose that G acts minimally on a simplicial tree T without fixed points on ∂T and there exist vertices u, v of T such that $\mathrm{Stab}_G(\{u, v\})$ is finite. Then G is acylindrically hyperbolic.

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$A \leq G$ is weakly malnormal if there is $g \in G$ such that $|A^g \cap A| < \infty$.

Corollary

(a) *Let $G = H *_A K$. Suppose that $H \neq A \neq K$ and A is weakly malnormal. Then G is virtually cyclic or acylindrically hyperbolic.*

(b) *Let $G = H *_{A^t = B}$. Suppose that $A \neq H \neq B$ and A is weakly malnormal. Then G is acylindrically hyperbolic.*

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Examples: 1-relator groups with $\geq$ 3-generators, $PGL(2, k[t])$, $Aut(\mathbb{C}^2)$, ...

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Non-examples

1. $SL_n(\mathbb{Z})$ for $n \geq 3$.
2. $BS(m,n) = \langle a, t \mid t^{-1}a^m t = a^n \rangle$.

Analytic properties

Theorem (Hamenstadt, Bestvina-Bromberg-Fujiwara, Hull-O.)

G is acylindrically hyperbolic $\Longrightarrow \dim(H_b^2(G, \ell^p(G))) = \infty$ *for any* $p \geq 1$.

Theorem (Hamenstadt, Bestvina-Bromberg-Fujiwara, Hull-O.)

G is acylindrically hyperbolic $\Longrightarrow$ $\dim(H_b^2(G, \ell^p(G))) = \infty$ for any $p \geq 1$.

G is inner amenable if $G \setminus \{1\}$ admits a finitely additive conjugacy invariant probability measure.

Theorem (Dahmani-Guirardel-Osin)

For any acylindrically hyperbolic group G, the following conditions are equivalent.

(a) *G has no nontrivial finite normal subgroups.*

(b) *G has infinite conjugacy classes.*

(c) *G is not inner amenable.*

(d) *The reduced C^*-algebra of G is simple.*

(e) *The reduced C^*-algebra of G has unique trace.*

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Theorem (Sisto)

For any finitely generated acylindrically hyperbolic group, the probability that the simple random walk arrives at a generalized loxodromic element in n steps is at least $1 - O(\varepsilon^n)$ for some $\varepsilon \in (0, 1)$.

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Theorem (Minasyan–O., 2008)

Let G be a finitely generated residually finite group with infinitely many ends. Then $Out(G)$ is residually finite.

Mapping class groups of 3-manifolds

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Theorem (Antolin–Minasyan–Sisto, 2013)

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Corollary

For any compact 3-manifold M, $Out(\pi_1(M))$ is residually finite.