Friends and relatives of $\mathrm{BS}(1,2)$

The role and importance of the many
variations and constructions based on this familiar group

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We use the commutator notation $[x, y] = x^{-1}y^{-1}xy$ and the notation $x^y = y^{-1}xy$ for conjugation. Notice that $[x, y] = x^{-1}x^y = y^{-x}y$. If G is a group, the commutator subgroup is denoted $[G, G]$. The factor group $G/[G, G] = H_1(G, \mathbb{Z})$ is the largest abelian quotient of G.

The **Baumslag-Solitar groups** are the groups of the form

$$\mathrm{BS(n, m)} = \langle \mathrm{s, x} \mid \mathrm{s^{-1}x^n s = x^m} \rangle$$

where $n, m \in \mathbb{Z}$. For convenience we will assume n, m are both positive integers.

Initially we will concentrate on $\mathrm{BS}(1, 2) = \langle \mathrm{s, x} \mid \mathrm{s^{-1}xs = x^2} \rangle$. Here are a number of equivalent ways to write this defining relation: $sx^2s^{-1} = x$, $xs = sx^2$, $x^{-1}s = sx^{-2}$, $s^{-1}x = x^2s^{-1}$, and $s^{-1}x^{-1} = x^{-2}s^{-1}$

An elementary solution to the word problem for $G = \mathrm{BS}(1,2)$.
Starting with any word w on x and s, the relations $x^{\pm 1}s = sx^{\pm 2}$, can be applied to move the letter s from right to left over $x^{\pm 1}$ symbols creating additional x's. Similarly an s^{-1} can be moved from left to right over $x^{\pm 1}$ symbols. So, freely reducing when possible and iterating one finds

$$w =_G s^i x^j s^{-k}$$

where $i \geq 0$ and $k \geq 0$ and $j \in \mathbb{Z}$.

In case $j = 2m$ is even and both $i > 0$ and $k > 0$ we can apply the relation $x = sx^2s^{-1}$ to deduce that $w =_G s^{i-1}x^m s^{-(k-1)}$ which has fewer s-symbols. This process is called **pinching** a pair of s-symbols, or an s-**pinch**. Repeatedly pinching one obtains $w =_G s^i x^j s^{-k}$ where either j is odd or at least one of i or k is 0 - in either case no further pinches are possible. If the right hand side of this equation is not the trivial word, then one can show $w \neq_G 1$. So the method described solves the word problem for G and also computes a unique normal form for w.

This algorithm removes inverse pairs of s-symbols by free reduction and during the pinching operation, but inverse pairs of s-symbols were never inserted. Also note that in the word sxs^{-1} an s-pinch is not possible and this word is not equal in G to any word with fewer s-symbols.

Here is the general situation for HNN-extensions.

Lemma (HNN, Novikov, Britton)

Let $G = \langle H, s \mid s^{-1}as = \phi(a), a \in A \rangle$ be an HNN-extension where H is a group with isomorphic subgroups $\phi : A \cong B$. Then

1. *(Higman-Neumann-Neumann) H is embedded in G.*
2. *(Novikov) If w is a word of H which involves s and if $w =_H u$ where u is s-free, then w can be transformed into U without inserting inverse pairs of s-symbols.*
3. *(Britton) If w is a word of H which involves s and if $w =_H u$ where u is s-free, then w contains a subword of the form $s^{-1}as$ or of the form $s\phi(a)s^{-1}$ with $a \in A$, that is, an s-**pinch**.*

Note that in $G = \mathrm{BS}(1,2)$ we have$w \in [G,G]$ if and only if $i = k$ in the above normal form. From this one can check $[G,G]$ is abelian and generated by the conjugates of $x = [s,x]$.

In fact the group $G = \mathrm{BS}(1,2)$ is a linear group over $\mathbb{Q}$ and one can easily check the map

$$x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad s \mapsto \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

is a homomorphism which embeds $\mathrm{BS}(1,2)$ as a subgroup of $GL(2,\mathbb{Q})$. Note that $s^i x^j s^{-k} \mapsto \begin{bmatrix} \frac{2^k}{2^i} & \frac{j}{2^i} \\ 0 & 1 \end{bmatrix}$. It follows that $\mathrm{BS}(1,2)$ is residually finite and hopfian and (again) has solvable word problem.

Here is a list of properties of $G = \mathrm{BS}(1,2)$:

- one-relator
- solvable word problem
- ascending HNN-extension of cyclic group
- metabelian and hence solvable with derived group isomorphic to $\mathbb{Z}[\frac{1}{2}]$
- residually finite and hence hopfian
- linear over $\mathbb{Q}$
- cohomological dimension 2 and $H_1(G,\mathbb{Z}) = \mathbb{Z}$, $H_n(G,\mathbb{Z}) = 0, n \geq 2$
- rational growth function
- no regular language of length minimal normal forms
- not almost convex
- exponential Dehn function

Higman's non-hopfian group. Observe that for $\mathrm{BS}(1,2) = \langle \mathrm{s}, \mathrm{x} \mid \mathrm{s}^{-1}\mathrm{x}\mathrm{s} = \mathrm{x}^2 \rangle$ the map $s \mapsto s$ and $x \mapsto x^2$ defines an automorphism, in fact it is just conjugation by s. Now amalgamate two copies of this group to obtain

$$\begin{aligned} H &= \langle s_1, x \mid s_1^{-1} x s_1 = x^2 \rangle \underset{x=x}{\star} \langle s_2, x \mid s_2^{-1} x s_2 = x^2 \rangle \\ &= \langle x, s_1, s_2 \mid s_1^{-1} x s_1 = x^2, s_2^{-1} x s_2 = x^2 \rangle. \end{aligned}$$

This, group constructed by Graham Higman in 1951, was the first example of a finitely presented, non-hopf group. The map $\theta : H \to H$ defined by $s_i \mapsto s_i$ and $x \mapsto x^2$ is a surjective homomorphism from H onto itself (easy check). But $s_1 x s_1^{-1} s_2 x^{-1} s_2^{-1} \neq_H 1$, so θ is not injective and H is non-hopfian. Hence H is also not residually finite.

We return to the larger family of *Baumslag-Solitar groups*

$$\mathrm{BS}(\mathrm{n},\mathrm{m}) = \langle \mathrm{x},\mathrm{s} \mid \mathrm{s}^{-1}\mathrm{x}^{\mathrm{n}}\mathrm{s} = \mathrm{x}^{\mathrm{m}} \rangle$$

which were studied by Baumslag and Solitar in 1962. Among other things they famously showed that the one-relator group $\mathrm{BS}(2,3)$ **is non-hopfian**. This can be easily deduced from Britton's Lemma using the map defined by $x \to x^2$ and $s \to s$ which is a subjective homomorphism but not an isomorphism. The word $[x, s^{-1}xs]$ is a non-trivial element in the kernel.

The groups $\mathrm{BS}(1,\mathrm{m})$ are again metabelian and share most of the above listed properties of $\mathrm{BS}(1,2)$ The main result about $\mathrm{BS}(\mathrm{n},\mathrm{m})$ is that

- $\mathrm{BS}(\mathrm{n},\mathrm{m})$ is residually finite if and only if $|n| = |m|$ or $|n| = 1$ or $|m| = 1$.
- $\mathrm{BS}(\mathrm{n},\mathrm{m})$ is hopfian if and only if *m* and *n* have the same set of prime divisors.

We now return to the group $\mathrm{BS}(1,2)$ and use it as a building block for some other groups with interesting properties.

Here is a lemma and construction due to Graham Higman.

Lemma (Higman)

Suppose that x and y are two non-trivial elements in a group G which satisfy the relation $y^{-1}xy = x^2$. If both x and y have finite order, then the smallest prime divisor or the order on y is strictly less than the smallest prime divisor of the order of x.

Corollary

Suppose that x and y are two elements in a group which have the same finite order m and satisfy $y^{-1}xy = x^2$. Then $x = y = 1$. $\square$

Theorem (Higman)

The four generator, four relator group defined by

$$G = \langle a, b, c, d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2, a^{-1}da = d^2 \rangle$$

is infinite and torsion-free but has no proper subgroups of finite index and hence no proper finite quotient groups. $\square$

This group G is perfect and has a balanced presentation. It is built from cyclic groups using HNN-extensions and amalgamated free products along free subgroups. From this one can easily deduce the following.

Corollary (Miller, Dyer-Vasquez)

The group G has cohomological dimension 2 and is acyclic, that is, $H_n(G, \mathbb{Z}) = 0$ for all $n > 0$.

Next we recall a group constructed by Gilbert Baumslag. We know $\mathrm{BS}(1,2) = \langle \mathrm{a}, \mathrm{t} \mid \mathrm{t}^{-1}\mathrm{at} = \mathrm{a}^2 \rangle$ is torsion free and the elements a and t both have infinite order, so we can make them conjugate in the HNN-extension:

$$\begin{aligned} B &= \langle a, t, b \mid t^{-1}at = a^2, b^{-1}ab = t \rangle \\ &= \langle a, b \mid (b^{-1}ab)^{-1}a(b^{-1}ab) = a^2 \rangle = \langle a, b \mid a^{a^b} = a^2 \rangle \end{aligned}$$

Theorem (Baumslag)

The finite quotient groups of the one relator group $B = \langle a, b \mid a^{a^b} = a^2 \rangle$ are exactly the finite cyclic groups. But B is not cyclic and, moreover, B has non-abelian free subgroups and contains a copy of $\mathrm{BS}(1,2)$.

Let N be the group with generators $\ldots, a_{-1}, a_0, a_, a_2, \ldots$ and relations

$$a_{i+1}^{-1} a_i a_{i+1} = a_i^2 \text{ for } i \in \mathbb{Z}.$$

Clearly the shift map $a_i \mapsto a_{i+1}$ defines an automorphism of N and the HNN-extension $\langle N, b \mid b^{-1}a_i b = a_{i+1} \rangle$ is isomorphic to Baumslag's B.

Unsolvability of the word problem

A result of major importance is the following:

Theorem (Novikov 1955, Boone 1957)

There exists a finitely presented group with unsolvable word problem.

These proofs were independent and are quite different, but interestingly they both involve versions of Higman's non-hopf group. That is, both constructions contain subgroups with presentations of the form

$$\langle x, s_1, \ldots, s_M \mid x s_b = s_b x^2, b = 1, \ldots, M \rangle.$$

We are going to describe Boone's construction (as modified by Britton, Boone, Collins and Miller) and try to indicate the crucial role these subgroups play in proof.

The construction of Boone's group begins with a Turing machine T having an unsolvable halting problem. The Turing machine has symbols s_b and certain states q_i. Following Emil Post, an instantaneous description of T corresponds to a **special word** of the form

$$s_{j_1} \dots s_{j_m} q_i s_{j_{m+1}} \dots s_{j_n}$$

in a certain semigroup $\Gamma(T)$. The relations of $\Gamma(T)$ provide semigroup transformations on special words which exactly corresponds to steps in the computation in T.

The result is a finitely presented semigroup $\Gamma(T)$ of the form

$$\Gamma(T) = \langle q, q_0, \dots, q_N, s_0, \dots, s_M \mid F_i q_{i_1} G_i = H_i q_{i_2} K_i \ (i \in I) \ \rangle$$

where the F_i, G_i, H_i, K_i are positive s-words and $q_{i_j} \in \{q, q_0, \dots, q_N\}$. Here an *s-word* is a word on the symbols $s_0, \dots, s_M$. Post shows that the turing machine T started at a special word Xq_iY halts in state q if and only if $Xq_iY = q$ in $\Gamma(T)$.

Theorem (Post, Markov)

The problem of deciding for an arbitrary pair of positive s-words X, Y whether or not $Xq_iY = q$ in the semigroup $\Gamma(T)$ is recursively unsolvable.

Next one wants to somehow encode the Post semigroup construction into a finitely presented group. But the presence of inverses is a serious difficulty, and the semigroup $\Gamma(T)$ doesn't even embed in a group.

We use $X \equiv Y$ to mean the words X and Y are identical (letter by letter). If $X \equiv s_{b_1}^{e_1} \cdots s_{b_m}^{e_m}$ is an s-word, we define $X^{\#} \equiv s_{b_1}^{-e_1} \cdots s_{b_m}^{-e_m}$. Note that $X^{\#}$ is not the same as X^{-1}. Also, if X and Y are s-words, then $(X^{\#})^{\#} \equiv X$ and $(XY)^{\#} = X^{\#}Y^{\#}$.

Boone's group $\mathcal{B} = \mathcal{B}(T)$ is then the finitely presented group depending on $\Gamma(T)$ described as follows:

generators: $q, q_0, \ldots, q_N, s_0, \ldots, s_M, r_i \ (i \in I), x, t, k$;

relations: for all $i \in I$ and all $b = 0, \ldots, M$,

$$
\begin{array}{ll}
x s_b = s_b x^2 & \Big]\Delta_1 \quad \Big]\Delta_2 \quad \Big]\Delta_3 \\
r_i s_b = s_b x r_i x & \Delta_2,\ \Delta_3 \\
r_i^{-1} F_i^{\#} q_{i_1} G_i r_i = H_i^{\#} q_{i_2} K_i & \Delta_2,\ \Delta_3 \\
t r_i = r_i t & \Delta_3 \\
t x = x t & \Delta_3 \\
k r_i = r_i k & \\
k x = x k & \\
k(q^{-1} t q) = (q^{-1} t q) k &
\end{array}
$$

The subsets $\Delta_1 \subset \Delta_2 \subset \Delta_3$ of the relations each define a presentation of a group $\mathcal{B}_i$ generated by the symbols appearing in the Δ_i.

Definition

A word Σ is special *if $\Sigma \equiv X^{\#} q_j Y$ where X and Y are positive s-words and $q_j \in \{q, q_0, \ldots, q_N\}$.*

The main technical result linking the word problem in $\mathcal{B}(T)$ as presented above to the word problem in Post's semigroup $\Gamma(T)$ is the following:

Lemma (Boone's Lemma)

If $\Sigma \equiv X^{\#} q_j Y$ is a special word in $\mathcal{B}(T)$, then the following are equivalent:

1. $k(\Sigma^{-1} t \Sigma) = (\Sigma^{-1} t \Sigma) k$ *in* $\mathcal{B}(T)$
2. $\Sigma \equiv X^{\#} q Y =_{\mathcal{B}_2} L q R$ *where* L, R *are words on* $\{r_i \ (i \in I), x\}$
3. $X q_j Y = q$ *in* $\Gamma(T)$

Perhaps the most difficult implications in Boone's Lemma is (2) $\Rightarrow$ (3). With the aid of Britton's Lemma the other implications are more straight forward. So supposing (2), one rewrites it as

$$L^{-1}\Sigma R^{-1} \equiv L^{-1}X^{\#}qYR^{-1} =_{\mathcal{B}_2} q$$

where L, R are words on $\{r_i \ (i \in I), x\}$. By Britton's Lemma there is a sequence of r_i-pinches eventually yielding q. One needs to show that the corresponding sequence of rewrite moves in the semigroup can be carried out. A pinch looks like, for instance,

$$r_i^{-1}x^{\alpha}X^{\#}q_jYx^{\beta}r_i \quad = \quad r_i^{-1}(x^{\alpha}X^{\#}F_i^{\#-1})F_i^{\#}q_{i_1}G_i(G_i^{-1}Yx^{\beta})r_i$$

So we must have $G_i^{-1}Yx^{\beta}$ is equal in $\mathcal{B}_1$ to a word in s_bx. To correspond to a semigroup move $G_i^{-1}Y$ must be a **positive word** in s_b's after free reduction.

Lemma

Suppose that U and V are positive words in the s_b-symbols and that $U^{-1}V$ is freely reduced as written, that is, the last symbol of U^{-1} is not the inverse of the first symbol of V. If the word $U^{-1}Vx^{\beta}$ is equal in $\mathcal{B}_1$ to a word in the elements $s_b x$, then U must be empty. Similarly, if $U^{-1}Vx^{\beta}$ is equal to a word in the elements $s_b x^{-1}$, then U must be empty.

Proof: Note that the s_b-symbols freely generate a free subgroup which is a retract of $\mathcal{B}_1$. Suppose that U is not the empty word. If we write out $U^{-1}Vx^{\beta}$ in detail it has the form

$$U^{-1}Vx^{\beta} \equiv s_{b_1}^{-1} \cdots s_{b_\lambda}^{-1} s_{c_1} \cdots s_{c_\rho} x^{\beta}.$$

Assume this is equal in $\mathcal{B}_1$ to a word in the $s_b x$ which must have the same retraction onto the free group on the s_b. So we must have

$$U^{-1}Vx^{\beta} \equiv s_{b_1}^{-1} \cdots s_{b_\lambda}^{-1} s_{c_1} \cdots s_{c_\rho} x^{\beta} =_{\mathcal{B}_1} x^{-1} s_{b_1}^{-1} \cdots x^{-1} s_{b_\lambda}^{-1} s_{c_1} x \cdots s_{c_\rho} x.$$

Equivalently this can be expressed as

$$x^{-\beta} s_{c_\rho}^{-1} \cdots s_{c_1}^{-1} s_{b_\lambda} \cdots s_{b_1}\ x^{-1} s_{b_1}^{-1} \cdots x^{-1} s_{b_\lambda}^{-1} s_{c_1} x \cdots s_{c_\rho} x =_{\mathcal{B}_1} 1.$$

Now $\mathcal{B}_1$ is an HNN extension with stable letters the s_b-symbols so by Britton's Lemma there must be an s_b-pinch. But by the assumptions on free reductions, **the only place such a pinch could occur is at $s_{b_1} x^{-1} s_{b_1}^{-1}$. But this is not a pinch** since the relevant relation is $x = s_{b_1} x^2 s_{b_1}^{-1}$ and x^{-1} does not lie in the subgroup generated by x^2. So we have a contradiction, proving the claim. The proof for equality to words in $s_b x^{-1}$ is very similar with x^{-1} in place of x. $\square$