Words in non-periodic branch groups

Elisabeth Fink

University of Oxford

May 28, 2013

Introduction

Introduction

A construction of a branch group G with

Introduction

A construction of a branch group G with

- no non-abelian free subgroups

Introduction

A construction of a branch group G with

- no non-abelian free subgroups
- exponential growth

Introduction

A construction of a branch group G with

- no non-abelian free subgroups
- exponential growth

For any $g, h \in G$ we construct $w_{g,h}(x, y) \in F(x, y)$ with $w_{g,h}(g, h) = 1 \in G$.

Rooted Trees

Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E

Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E
- distinguished root: r

Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E
- distinguished root: r
- distance function: $d(v, w)$, number of edges in unique path from v to w

Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E
- distinguished root: r
- distance function: $d(v, w)$, number of edges in unique path from v to w
- level n: $\Omega(n) = \{v \in V : d(v, r) = n\}$

Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E
- distinguished root: r
- distance function: $d(v, w)$, number of edges in unique path from v to w
- level n: $\Omega(n) = \{v \in V : d(v, r) = n\}$

$T_n = (V', E')$ with

- $V' = \{v \in V : d(v, r) \leq n\}$

Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E
- distinguished root: r
- distance function: $d(v, w)$, number of edges in unique path from v to w
- level n: $\Omega(n) = \{v \in V : d(v, r) = n\}$

$T_n = (V', E')$ with

- $V' = \{v \in V : d(v, r) \leq n\}$
- $E' = \{e \in E : e = e_{vw}, v, w \in V'\}$

Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E
- distinguished root: r
- distance function: $d(v, w)$, number of edges in unique path from v to w
- level n: $\Omega(n) = \{v \in V : d(v, r) = n\}$

$T_n = (V', E')$ with

- $V' = \{v \in V : d(v, r) \leq n\}$
- $E' = \{e \in E : e = e_{vw}, v, w \in V'\}$

T_v: subtree with root $v \in V$

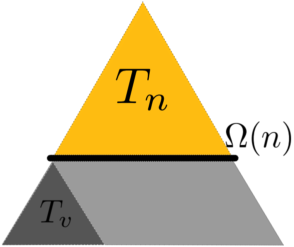
Rooted Trees

- Rooted tree: cyclefree graph with vertices V and edges E
- distinguished root: r
- distance function: $d(v, w)$, number of edges in unique path from v to w
- level n: $\Omega(n) = \{v \in V : d(v, r) = n\}$

$T_n = (V', E')$ with

- $V' = \{v \in V : d(v, r) \leq n\}$
- $E' = \{e \in E : e = e_{vw}, v, w \in V'\}$

T_v: subtree with root $v \in V$



Automorphisms acting on rooted trees

Automorphisms acting on rooted trees

- acting on vertices, preserve edge incidence and root

Automorphisms acting on rooted trees

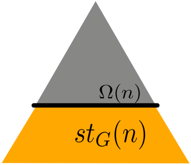
- acting on vertices, preserve edge incidence and root
- $\mathrm{st}_G(n) = \{g \in G : \text{g acts trivially on } T_n\}$

Automorphisms acting on rooted trees

- acting on vertices, preserve edge incidence and root
- $\mathrm{st}_G(n) = \{g \in G : \text{ g acts trivially on } T_n\}$
- $\mathrm{rst}_G(v) = \{g \in G : g \text{ acts trivially on } T \backslash T_v\}$

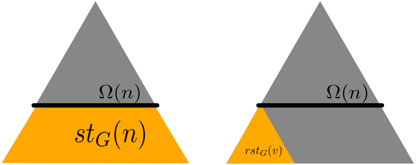
Automorphisms acting on rooted trees

- acting on vertices, preserve edge incidence and root
- $\mathrm{st}_G(n) = \{g \in G :$ g acts trivially on $T_n\}$
- $\mathrm{rst}_G(v) = \{g \in G : g$ acts trivially on $T \setminus T_v\}$



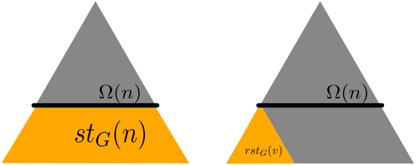
Automorphisms acting on rooted trees

- acting on vertices, preserve edge incidence and root
- $\mathrm{st}_G(n) = \{g \in G : \text{g acts trivially on } T_n\}$
- $\mathrm{rst}_G(v) = \{g \in G : g \text{ acts trivially on } T \setminus T_v\}$



Automorphisms acting on rooted trees

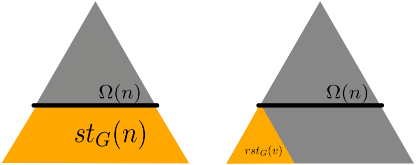
- acting on vertices, preserve edge incidence and root
- $\mathrm{st}_G(n) = \{g \in G : \text{ g acts trivially on } T_n\}$
- $\mathrm{rst}_G(v) = \{g \in G : g \text{ acts trivially on } T \setminus T_v\}$



- $\mathrm{rst}_G(n) = \prod_{v \in \Omega(n)} \mathrm{rst}_G(v)$

Automorphisms acting on rooted trees

- acting on vertices, preserve edge incidence and root
- $\mathrm{st}_G(n) = \{g \in G : \text{ g acts trivially on } T_n\}$
- $\mathrm{rst}_G(v) = \{g \in G : g \text{ acts trivially on } T \setminus T_v\}$



- $\mathrm{rst}_G(n) = \prod_{v \in \Omega(n)} \mathrm{rst}_G(v)$

Definition

A group G is a *branch group* if it acts transitively on each level of a tree and each $\mathrm{rst}_G(n)$ has finite index.

A specific construction

A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$,

A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i \rangle = A_i$

A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i \rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$

A specific construction

- Finite cyclic groups $\{A_i\}_{i \in \mathbb{N}}$, $\langle a_i \rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$

A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i\rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$

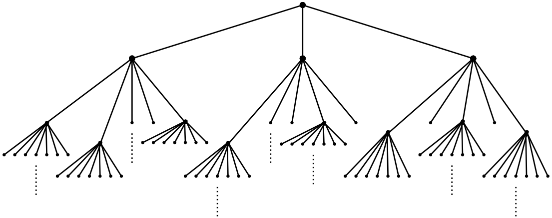
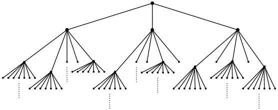


Figure: A tree with $p_0 = 3, p_1 = 5, p_2 = 7$.

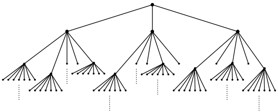
A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i\rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$



A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i\rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$

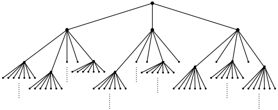


$G = \langle a, b\rangle$ with

- rooted automorphism a_0

A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i\rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$

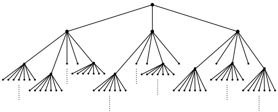


$G = \langle a, b\rangle$ with

- rooted automorphism a_0, cyclically permutes $\Omega(1)$

A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i\rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$

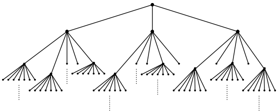


$G = \langle a, b\rangle$ with

- rooted automorphism a_0, cyclically permutes $\Omega(1)$
- spinal automorphism b

A specific construction

- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i\rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$

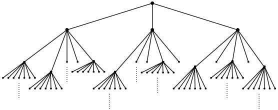


$G = \langle a, b\rangle$ with

- rooted automorphism a_0, cyclically permutes $\Omega(1)$
- spinal automorphism b recursively defined as $b_n = (b_{n+1}, a_{n+1}, 1, \ldots, 1)_1$

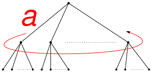
A specific construction

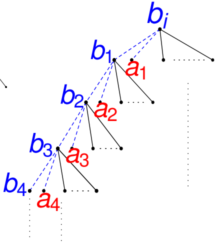
- Finite cyclic groups $\{A_i\}_{i\in\mathbb{N}}$, $\langle a_i\rangle = A_i$
- $|A_i| = p_i$, $(p_i, p_j) = 1$ for $i \neq j$
- defining sequence $\{p_i\}$



$G = \langle a, b\rangle$ with

- rooted automorphism a_0, cyclically permutes $\Omega(1)$
- spinal automorphism b recursively defined as $b_n = (b_{n+1}, a_{n+1}, 1, \ldots, 1)_1$





Basic properties of G

Basic properties of G

Proposition

- G acts on $\Omega(n)$ as $A_{n-1} \wr \cdots \wr A_0$

Basic properties of G

Proposition

- G acts on $\Omega(n)$ as $A_{n-1} \wr \cdots \wr A_0$
- $G^{ab} = C_{p_0} \times C_{\infty}$, indirectly implies G is not just infinite

Basic properties of G

Proposition

- G acts on $\Omega(n)$ as $A_{n-1} \wr \cdots \wr A_0$
- $G^{ab} = C_{p_0} \times C_\infty$, indirectly implies G is not just infinite
- $1 \neq N \triangleleft G$, then N is finitely generated

Basic properties of G

Proposition

- G acts on $\Omega(n)$ as $A_{n-1} \wr \cdots \wr A_0$
- $G^{ab} = C_{p_0} \times C_\infty$, indirectly implies G is not just infinite
- $1 \neq N \lhd G$, then N is finitely generated
- $N \lhd G$, N non-trivial, then G/N is soluble

Basic properties of G

Proposition

- G acts on $\Omega(n)$ as $A_{n-1} \wr \cdots \wr A_0$
- $G^{ab} = C_{p_0} \times C_\infty$, indirectly implies G is not just infinite
- $1 \neq N \lhd G$, then N is finitely generated
- $N \lhd G$, N non-trivial, then G/N is soluble
- G has infinite virtual first betti number

Basic properties of G

Proposition

- G acts on $\Omega(n)$ as $A_{n-1} \wr \cdots \wr A_0$
- $G^{ab} = C_{p_0} \times C_\infty$, indirectly implies G is not just infinite
- $1 \neq N \lhd G$, then N is finitely generated
- $N \lhd G$, N non-trivial, then G/N is soluble
- G has infinite virtual first betti number
- G is not large

Basic properties of G

Proposition

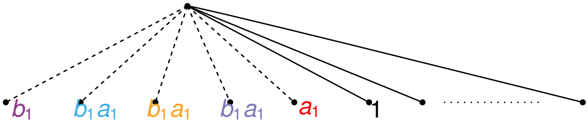
- G acts on $\Omega(n)$ as $A_{n-1} \wr \cdots \wr A_0$
- $G^{ab} = C_{p_0} \times C_\infty$, indirectly implies G is not just infinite
- $1 \neq N \lhd G$, then N is finitely generated
- $N \lhd G$, N non-trivial, then G/N is soluble
- G has infinite virtual first betti number
- G is not large
- $G^{(n+1)} \leq \mathrm{rst}_G(n)$

An Example

$$g_1 = (ab)^4 a^{-4} = (b_1, b_1 a_1, b_1 a_1, b_1 a_1, a_1, 1, \ldots, 1)_1,$$

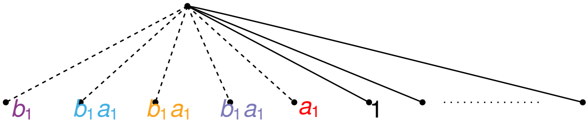
An Example

$$g_1 = (ab)^4 a^{-4} = (b_1, b_1 a_1, b_1 a_1, b_1 a_1, a_1, 1, \ldots, 1)_1,$$



An Example

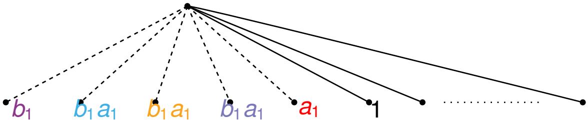
$$g_1 = (ab)^4 a^{-4} = (b_1, b_1 a_1, b_1 a_1, b_1 a_1, a_1, 1, \ldots, 1)_1,$$



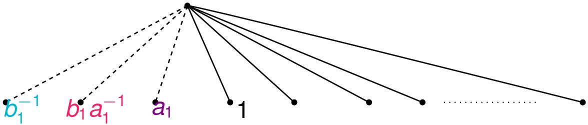
$$g_2 = [b, b^a] = (b_1^{-1}, b_1 a_1^{-1}, a_1, 1, \ldots, 1)_1,$$

An Example

$$g_1 = (ab)^4 a^{-4} = (b_1, b_1 a_1, b_1 a_1, b_1 a_1, a_1, 1, \dots, 1)_1,$$

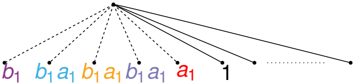


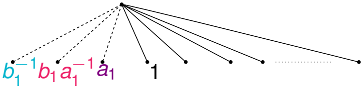
$$g_2 = [b, b^a] = \left(b_1^{-1}, b_1 a_1^{-1}, a_1, 1, \dots, 1\right)_1,$$



An Example

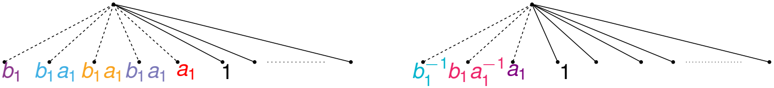
$g_1 = (ab)^4 a^{-4}$, $g_2 = [b, b^a]$





An Example

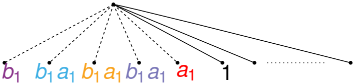
$$g_1 = (ab)^4 a^{-4}, \qquad g_2 = [b, b^a]$$

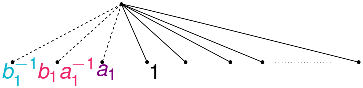


$$c_1 = [g_1, g_2] = \left(1, \left(a_2, b_2^{-1}, a_2^{-1}, 1, \ldots, b_2\right), \left(1, b_2^{-1}, a_2^{-1} b_2, a_2, 1, \ldots, 1\right), 1, \ldots, 1\right)_2$$

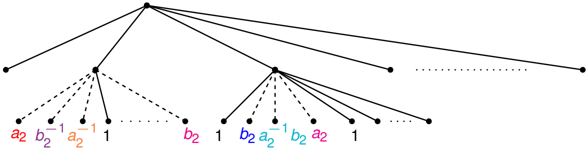
An Example

$$g_1 = (ab)^4 a^{-4}, \qquad g_2 = [b, b^a]$$



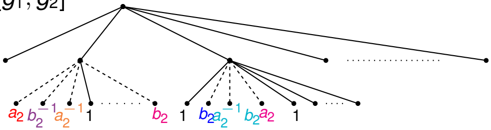


$$c_1 = [g_1, g_2] = \left(1, \left(a_2, b_2^{-1}, a_2^{-1}, 1, \dots, b_2\right), \left(1, b_2^{-1}, a_2^{-1} b_2, a_2, 1, \dots, 1\right), 1, \dots, 1\right)_2$$



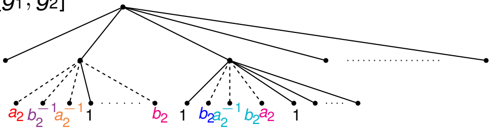
An Example

$c_1 = [g_1, g_2]$



An Example

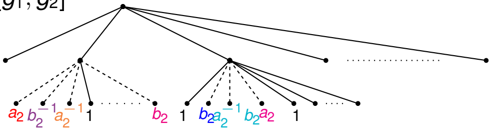
$$c_1 = [g_1, g_2]$$



$$c_1^{g_1^4} = [g_1, g_2]^{g_1^4}$$

An Example

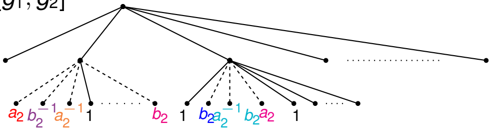
$c_1 = [g_1, g_2]$



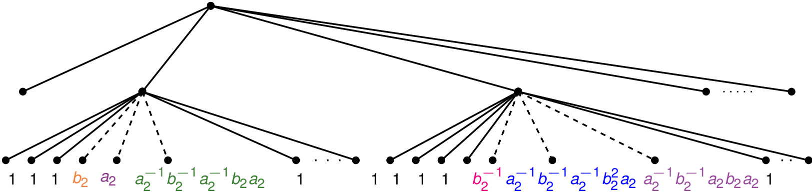
$$c_1^{g_1^4} = [g_1, g_2]^{g_1^4} = \Big(1, \big(1, 1, 1, b_2, a_2, b_2^{-1}, a_2^{-1} b_2^{-1} a_2^{-1} b_2 a_2, 1, \ldots, 1\big), \big(1, 1, 1, 1, b_2^{-1}, a_2^{-1} b_2^{-1} a_2^{-1} b_2^2 a_2, a_2^{-1} b_2^{-1} a_2 b_2 a_2, 1, \ldots, 1\big), 1, \ldots, 1\Big)_2 .$$

An Example

$$c_1 = [g_1, g_2]$$

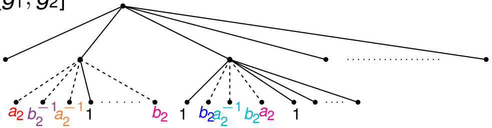


$$c_1^{g_1^4} = [g_1, g_2]^{g_1^4} = \Big(1, \big(1, 1, 1, b_2, a_2, b_2^{-1}, a_2^{-1} b_2^{-1} a_2^{-1} b_2 a_2, 1, \ldots, 1\big), \big(1, 1, 1, 1, b_2^{-1}, a_2^{-1} b_2^{-1} a_2^{-1} b_2^2 a_2, a_2^{-1} b_2^{-1} a_2 b_2 a_2, 1, \ldots, 1\big), 1, \ldots, 1\Big)_2 .$$

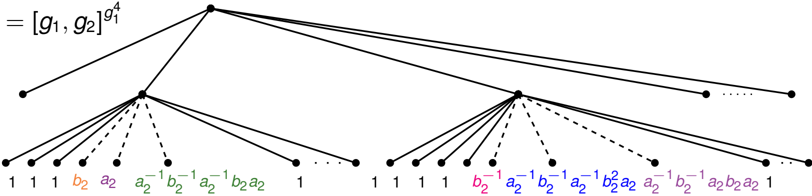


An Example

$c_1 = [g_1, g_2]$

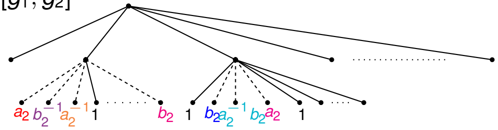


$c_1^{g_1^4} = [g_1, g_2]^{g_1^4}$

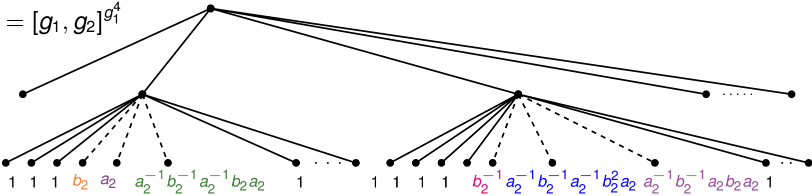


An Example

$c_1 = [g_1, g_2]$



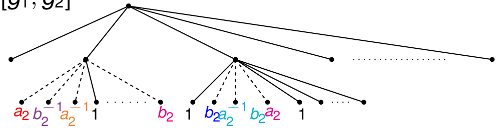
$c_1^{g_1^4} = [g_1, g_2]^{g_1^4}$



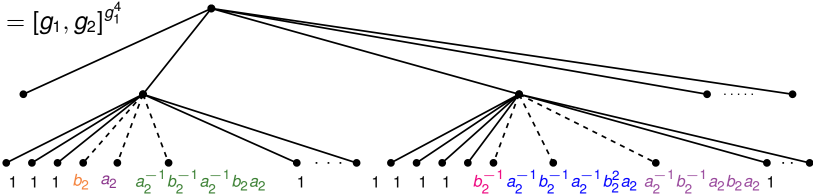
Comparing pictures: $\left[c_1, c_1^{g_1^4}\right] = 1$

An Example

$c_1 = [g_1, g_2]$



$c_1^{g_1^4} = [g_1, g_2]^{g_1^4}$



Comparing pictures: $\left[c_1, c_1^{g_1^4}\right] = 1$

$w_{g_1,g_2}(x, y) = \left[[x, y], [x, y]^{x^4}\right]$ with $w_{g_1,g_2}(g_1, g_2) = 1$

Lemmata

Definition

Let $g \in G, q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1} b^q g$.

Lemmata

Definition

Let $g \in G, q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1}b^q g$. Number of spines of g:

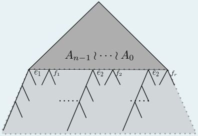
$$\zeta(g) = \min \left\{ s_g : g = a^k \cdot \prod_{i=1}^{s_g} a^{-k_i} b^{q_i} a^{k_i} \right\}$$

Lemmata

Definition

Let $g \in G, q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1}b^q g$. Number of spines of g:

$$\zeta(g) = \min \left\{ s_g : g = a^k \cdot \prod_{i=1}^{s_g} a^{-k_i} b^{q_i} a^{k_i} \right\}$$

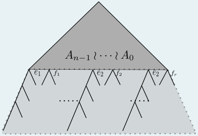


Lemmata

Definition

Let $g \in G$, $q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1} b^q g$. Number of spines of g:

$$\zeta(g) = \min \left\{ s_g : g = a^k \cdot \prod_{i=1}^{s_g} a^{-k_i} b^{q_i} a^{k_i} \right\}$$



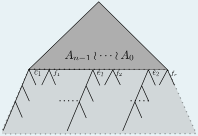
$$c_0 = g, c_1 = [g_1, g_2],$$

Lemmata

Definition

Let $g \in G, q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1}b^q g$. Number of spines of g:

$$\zeta(g) = \min \left\{ s_g : g = a^k \cdot \prod_{i=1}^{s_g} a^{-k_i} b^{q_i} a^{k_i} \right\}$$



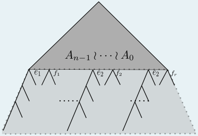
$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Lemmata

Definition

Let $g \in G, q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1} b^q g$. Number of spines of g:

$$\zeta(g) = \min \left\{ s_g : g = a^k \cdot \prod_{i=1}^{s_g} a^{-k_i} b^{q_i} a^{k_i} \right\}$$



$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Lemma

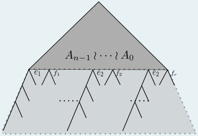
- $\zeta(c_i) \leq 5^i (\zeta(g_1) + \zeta(g_2)), i \geq 0$

Lemmata

Definition

Let $g \in G, q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1} b^q g$. Number of spines of g:

$$\zeta(g) = \min \left\{ s_g : g = a^k \cdot \prod_{i=1}^{s_g} a^{-k_i} b^{q_i} a^{k_i} \right\}$$



$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Lemma

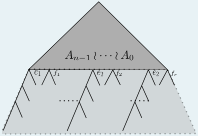
- $\zeta(c_i) \leq 5^i (\zeta(g_1) + \zeta(g_2)), i \geq 0$
- $c_i \in \mathrm{rst}_G(i-1) \leq \mathrm{st}_G(i-1), i \geq 2$

Lemmata

Definition

Let $g \in G$, $q \in \mathbb{Z}$. A *spine* s is a g-conjugate of a power of the generator b: $s = g^{-1} b^q g$. Number of spines of g:

$$\zeta(g) = \min \left\{ s_g : g = a^k \cdot \prod_{i=1}^{s_g} a^{-k_i} b^{q_i} a^{k_i} \right\}$$



$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Lemma

- $\zeta(c_i) \leq 5^i (\zeta(g_1) + \zeta(g_2)), i \geq 0$
- $c_i \in \mathrm{rst}_G(i-1) \leq \mathrm{st}_G(i-1), i \geq 2$

c_i stabilises level i!

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Proposition

c_i has the form $c_i = \left(d_{i,1}, \ldots, d_{i,m_i}\right)_i$ with $d_{i,j}$ one of the following

1. $d_{i,j} = b_i^t, t \in \mathbb{Z}$

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Proposition

c_i has the form $c_i = \left(d_{i,1}, \ldots, d_{i,m_i}\right)_i$ with $d_{i,j}$ one of the following

1. $d_{i,j} = b_i^t, t \in \mathbb{Z}$
2. $d_{i,j} = a^q b, q \neq k \cdot p_i, b \in B \cap \mathrm{st}_G(n)$

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Proposition

c_i has the form $c_i = \left(d_{i,1}, \ldots, d_{i,m_i}\right)_i$ with $d_{i,j}$ one of the following

1. $d_{i,j} = b_i^t, t \in \mathbb{Z}$
2. $d_{i,j} = a^q b, q \neq k \cdot p_i, b \in B \cap \mathrm{st}_G(n)$
3. recursive: $d_{i,j} = (d_{i+1,t_1}, \ldots, d_{i+1,t_m})_{i+1}$

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Proposition

c_i has the form $c_i = \left(d_{i,1}, \ldots, d_{i,m_i}\right)_i$ with $d_{i,j}$ one of the following

1. $d_{i,j} = b_i^t, t \in \mathbb{Z}$
2. $d_{i,j} = a^q b, q \neq k \cdot p_i, b \in B \cap \mathrm{st}_G(n)$
3. recursive: $d_{i,j} = (d_{i+1,t_1}, \ldots, d_{i+1,t_m})_{i+1}$
4. $d_{i,j} = 1$

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Proposition

c_i has the form $c_i = \left(d_{i,1}, \ldots, d_{i,m_i}\right)_i$ with $d_{i,j}$ one of the following

1. $d_{i,j} = b_i^t, t \in \mathbb{Z}$
2. $d_{i,j} = a^q b, q \neq k \cdot p_i, b \in B \cap \mathrm{st}_G(n)$
3. recursive: $d_{i,j} = (d_{i+1,t_1}, \ldots, d_{i+1,t_m})_{i+1}$
4. $d_{i,j} = 1$

$\exists n \in \mathbb{N}$ with 3 only occurs for $d_{i,j}$ with $i \leq n$.

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Proposition

c_i has the form $c_i = \left(d_{i,1}, \ldots, d_{i,m_i}\right)_i$ with $d_{i,j}$ one of the following

1. $d_{i,j} = b_i^t, t \in \mathbb{Z}$
2. $d_{i,j} = a^q b, q \neq k \cdot p_i, b \in B \cap \mathrm{st}_G(n)$
3. recursive: $d_{i,j} = (d_{i+1,t_1}, \ldots, d_{i+1,t_m})_{i+1}$
4. $d_{i,j} = 1$

$\exists n \in \mathbb{N}$ with 3 only occurs for $d_{i,j}$ with $i \leq n$.

Theorem

If $p_i \geq (25p_{i-1})^{3 \prod_{k=0}^{i-1} p_k}$, then for each $g, h \in G$ there exists a word $w_{g,h}(x, y) \in F(x, y)$ with $w_{g,h}(g, h) = 1 \in G$.

Theorem

$$c_0 = g, c_1 = [g_1, g_2], c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], i \geq 2$$

Proposition

c_i has the form $c_i = \left(d_{i,1}, \ldots, d_{i,m_i}\right)_i$ with $d_{i,j}$ one of the following

1. $d_{i,j} = b_i^t, t \in \mathbb{Z}$
2. $d_{i,j} = a^q b, q \neq k \cdot p_i, b \in B \cap \mathrm{st}_G(n)$
3. recursive: $d_{i,j} = (d_{i+1,t_1}, \ldots, d_{i+1,t_m})_{i+1}$
4. $d_{i,j} = 1$

$\exists n \in \mathbb{N}$ with 3 only occurs for $d_{i,j}$ with $i \leq n$.

Theorem

If $p_i \geq (25p_{i-1})^{3\prod_{k=0}^{i-1} p_k}$, then for each $g, h \in G$ there exists a word $w_{g,h}(x, y) \in F(x, y)$ with $w_{g,h}(g, h) = 1 \in G$. Hence G has no non-abelian free subgroups.

Growth

Growth

Proposition

G does not have polynomial growth.

Growth

Proposition

G does not have polynomial growth.

Self-Similarities

Growth

Proposition

G does not have polynomial growth.

Self-Similarities

Find copies of G_1 in G:

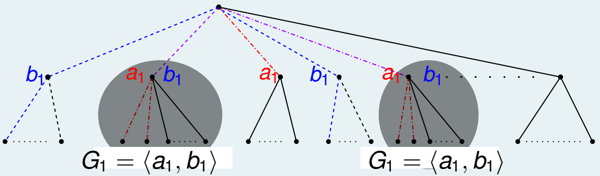
Growth

Proposition

G does not have polynomial growth.

Self-Similarities

Find copies of G_1 in G:



Growth

Growth

Lemma

$\gamma_G\left(\prod_{k=1}^{i} p_k^2 n\right) \geq \gamma_{G_i}(n)^{\prod_{k=1}^{i} p_k/3}$.

Growth

Lemma

$$\gamma_G\left(\prod_{k=1}^{i} p_k^2 n\right) \geq \gamma_{G_i}(n)^{\prod_{k=1}^{i} p_k/3}.$$

Proposition

$$\gamma_{B_i}(2p_i) \geq 2^{p_i - 1} \cdot (p_i - 1)^{\frac{p_i}{2} - 2}.$$

Growth

Lemma

$\gamma_G\left(\prod_{k=1}^{i} p_k^2 n\right) \geq \gamma_{G_i}(n)^{\prod_{k=1}^{i} p_k/3}$.

Proposition

$\gamma_{B_i}(2p_i) \geq 2^{p_i-1} \cdot (p_i-1)^{\frac{p_i}{2}-2}$.

Theorem

If $\{p_i\}$ satisfies $\log(p_i-1) \geq 5 \cdot \left(\frac{47}{5}\right)^i \cdot \prod_{k=0}^{i} p_k$ for all sufficiently large i, then G has exponential growth.

Further Questions

- Is G amenable?

Further Questions

- Is G amenable?
- Growth in all cases?

Further Questions

- Is G amenable?
- Growth in all cases?
  Similar examples are amenable for **slow** growth (Brieussel)

Further Questions

- Is G amenable?
- Growth in all cases?
  Similar examples are amenable for **slow** growth (Brieussel)
- Does G contain a free semigroup?

Thank you!