Digraph groups and their applications

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October 6, 2009

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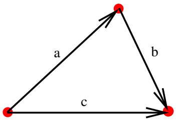

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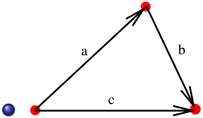

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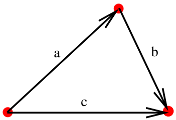
- 

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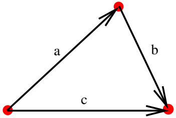

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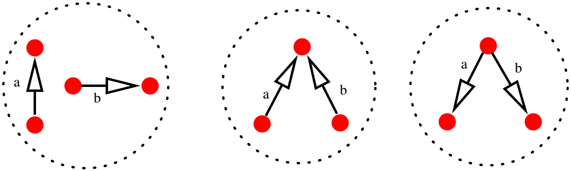

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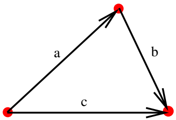
- 

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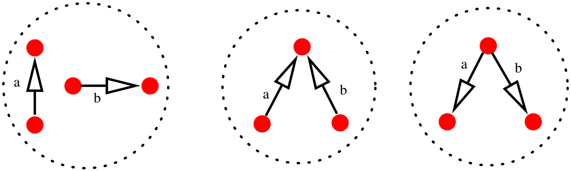
- 

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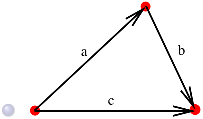

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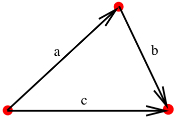
  - 

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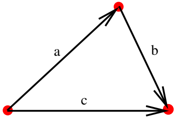
    - 

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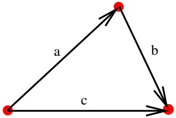
- 

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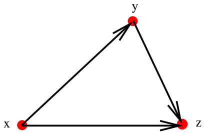

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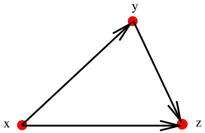
- 

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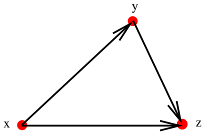

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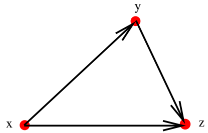

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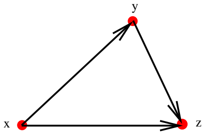

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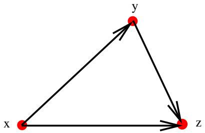

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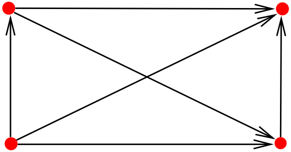

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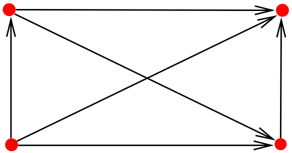

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(J. Milnor, Introduction to Algebraic K-theory thm. 10.1) For $n \geq 3$ the group $St_n(\mathbb{Z})$ is a central extension of the form

$$C_n \rightarrowtail St_n(\mathbb{Z}) \twoheadrightarrow SL_n(\mathbb{Z}),$$

where C_2 is a cyclic group of order 2, generated by the symbol $(x_{12}x_{21}^{-1}x_{12})^4$. For $n = 2$ there is an exact sequence

$$R \rightarrowtail F_2 \twoheadrightarrow SL_2(\mathbb{Z}),$$

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$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

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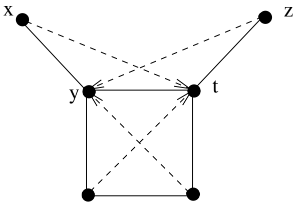

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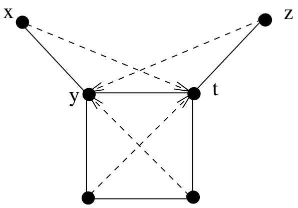

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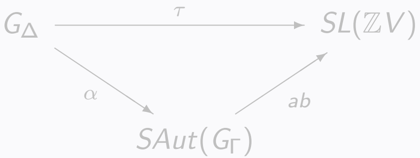

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Let G be a group generated by a set V and assume that V is "independent" modulo the commutator subgroup i.e. a natural homomorphism $G/G' \to \mathbb{Z}V$ is bijective. Suppose $V = \cup V_i$ is a partition of V into commutative subsets and denote $G_i = gp\langle V_i \rangle$. Then the stabilizer S in $Aut(G)$ of the set $\{G_i\}$ is mapped injectively under abelianization.

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Let G_Γ be a graph group. Suppose $V = \cup V_i$ is a partition of V into commutative subsets and denote $G_i = gp\langle V_i \rangle$. Then the stabilizer S in $Aut(G)$ of the set $\{G_i\}$ is mapped injectively under abelianization.

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Let G be a group generated by a set V and assume that V is "independent" modulo the commutator subgroup i.e. a natural homomorphism $G/G' \to \mathbb{Z}V$ is bijective. Suppose $V = \cup V_i$ is a partition of V into commutative subsets and denote $G_i = gp\langle V_i \rangle$. Then the stabilizer S in $Aut(G)$ of the set $\{G_i\}$ is mapped injectively under abelianization.

Lemma

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Applications to the property T for automorphism group of a graph group

Theorem

There is a graph Γ with $|V| = 6$ such that $Aut(G_\Gamma)$ has positive first Betti number, i.e. it has a subgroup which maps onto $\mathbb{Z}$. Consequently $Aut(G_\Gamma)$ does not have property T.

Applications to the abelianization for graph groups

Theorem

For any finite graph Γ the image of the abelianization map

$$\alpha : Aut\,(G_\Gamma) \to GL\,(\mathbb{Z}V) \tag{3}$$

is an arithmetic subgroup of $GL\,(\mathbb{R}V)$.

- Last problem - prove something interesting about digraph groups!

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